

**Advanced Dynamics**  
**Prof. Anirvan Dasgupta**  
**Department of Mechanical Engineering**  
**Indian Institute of Technology - Kharagpur**

**Module No # 10**  
**Lecture No # 50**  
**Hamilton's Principle and Lagrange's Equation of Motion – III**

In this lecture we are going to continue with Lagrange's equation of motion, and later discuss the Hamilton-Ostrogradski principle.

(Refer Slide Time: 00:20)

## Overview

- Hamilton's principle for dynamical paths
- Lagrange's equation of motion
- Hamilton-Ostrogradski principle

Hamilton-Ostrogradski principle is extension of Hamilton's principle where we also consider externally applied forces or non-conservative forces (for which we do not have a potential).

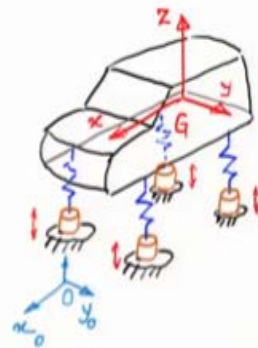
(Refer Slide Time: 00:51)

## Example 5:

### Vehicle with ground excitation

Generalized coordinate:  $(x_g, y_g, z_g, \underbrace{\psi, \theta, \phi}_{\text{Yaw, Pitch, Roll}})$

Lagrange's equation of motion:  $\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \quad i = 1, \dots, 6$



We first consider a vehicle with ground excitation as shown above.

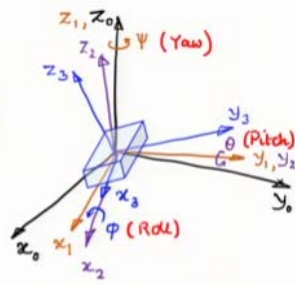

(Refer Slide Time: 02:45)

## Angular velocity vector

Recall

Angular velocity (body frame):  $\vec{\omega}^3 = \dot{\psi} \hat{z}_0^3 + \dot{\theta} \hat{y}_1^3 + \dot{\phi} \hat{x}_2^3$

$$\hat{z}_0^3 = \begin{Bmatrix} -s\theta \\ c\theta s\phi \\ c\theta c\phi \end{Bmatrix} \quad \hat{y}_1^3 = \begin{Bmatrix} 0 \\ c\phi \\ -s\phi \end{Bmatrix} \quad \hat{x}_2^3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \vec{\omega}^3 = \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta s\phi & c\phi & 0 \\ c\theta c\phi & -s\phi & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix}$$



For the orientation part recall the discussion in our previous lecture where we derived the rotation matrix for the orientation of a frame with a given a yaw pitch and roll. Also recall the angular velocity vector written in the body frame in terms of yaw rate, pitch rate and roll rate, as shown in the slide above.

(Refer Slide Time: 05:12)

### Translational KE

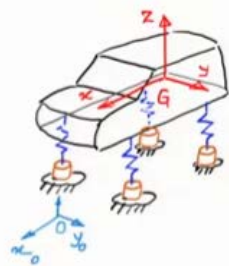
$$T_T = \frac{1}{2} m (\dot{x}_a^2 + \dot{y}_a^2 + \dot{z}_a^2)$$

### Rotational KE

$$T_R = \frac{1}{2} \vec{\omega} \cdot I_a \vec{\omega}$$

$$I_a = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \quad (x-y-z \text{ principal body-fixed frame})$$

$\vec{\omega}$  parametrized by yaw-pitch-roll coordinates (body-fixed frame)



The structure of the translational and rotational kinetic energy expressions are presented in the slide above.

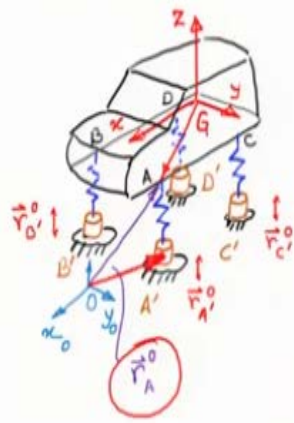
(Refer Slide Time: 06:42)

Front:  $\vec{r}_A^0 = \vec{r}_G + {}^0R_3 \vec{r}_{GA}^3$        $\vec{r}_B^0 = \vec{r}_G + {}^0R_3 \vec{r}_{GB}^3$   
Rear:  $\vec{r}_C^0 = \vec{r}_G + {}^0R_3 \vec{r}_{GC}^3$        $\vec{r}_D^0 = \vec{r}_G + {}^0R_3 \vec{r}_{GD}^3$

Potential energy of springs + gravity

$$V = \frac{1}{2} (\vec{r}_A^0 - \vec{r}_{A'}^0)^T [K] (\vec{r}_A^0 - \vec{r}_{A'}^0) + \frac{1}{2} (\vec{r}_B^0 - \vec{r}_{B'}^0)^T [K] (\vec{r}_B^0 - \vec{r}_{B'}^0)$$

$$+ \frac{1}{2} (\vec{r}_C^0 - \vec{r}_{C'}^0)^T [K] (\vec{r}_C^0 - \vec{r}_{C'}^0) + \frac{1}{2} (\vec{r}_D^0 - \vec{r}_{D'}^0)^T [K] (\vec{r}_D^0 - \vec{r}_{D'}^0) + mg z_G$$



Next, we write down the potential energy contribution as presented in the above slide.

(Refer Slide Time: 11:47)

$\mathcal{L} = \frac{1}{2} m (\dot{x}_G^2 + \dot{y}_G^2 + \dot{z}_G^2) + \frac{1}{2} \vec{\omega} \cdot I_G \vec{\omega}$

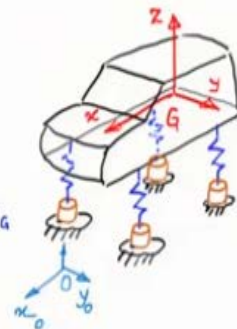
$- \frac{1}{2} (\vec{r}_A^0 - \vec{r}_{A'}^0)^T [K] (\vec{r}_A^0 - \vec{r}_{A'}^0) - \frac{1}{2} (\vec{r}_B^0 - \vec{r}_{B'}^0)^T [K] (\vec{r}_B^0 - \vec{r}_{B'}^0)$

$- \frac{1}{2} (\vec{r}_C^0 - \vec{r}_{C'}^0)^T [K] (\vec{r}_C^0 - \vec{r}_{C'}^0) - \frac{1}{2} (\vec{r}_D^0 - \vec{r}_{D'}^0)^T [K] (\vec{r}_D^0 - \vec{r}_{D'}^0) - mg z_G$

$= \mathcal{L} (x_G, y_G, z_G, \psi, \theta, \phi, \dot{x}_G, \dot{y}_G, \dot{z}_G, \dot{\psi}, \dot{\theta}, \dot{\phi}, t)$

Inputs:  $\vec{r}_{A'}^0(t), \vec{r}_{B'}^0(t), \vec{r}_{C'}^0(t), \vec{r}_{D'}^0(t)$

Non-linear coupled dynamics!



Finally we have the Lagrangian as kinetic minus potential energy as shown above. This will lead us to highly complex coupled second order ordinary differential equations which will be non-linear and extremely difficult to solve analytically. Therefore we have to take recourse to numerical methods for solution.

(Refer Slide Time: 14:11)

## Newton's 2<sup>nd</sup> law and Lagrange's equation

- Newton's law: local statement - balance of inertia and external forces
- Newtonian approach: internal forces are part of the solution
- Hamilton's principal: extremization of action over a path
- Lagrange's equation: local statement with global implication
- Lagrangian approach: internal forces and FBD are not involved

Let us now look back and compare the Lagrangian approach with the Newtonian approach, as shown in the slide above. It may be noted that both the approaches lead us to the same equation of motion for a given mechanical system.

(Refer Slide Time: 18:43)

## External and non-conservative forces

### Hamilton-Ostrogradski principle

- External and non-conservative forces included through virtual work

$$\int_{t_0}^{t_1} (\delta \mathcal{L} + \delta W) dt = 0$$

Hamilton-Ostrogradski principle

$$\delta W = \vec{F} \cdot \delta \vec{r}$$

$$\text{where } \vec{F} = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$$

{ Virtual work done by forces/moments  
acting on particles/bodies (physical coordinates)

- Not a variational principle



Next we consider systems with external forces or non-conservative forces. Till now we were looking at mechanical systems with potential forces. We now consider an approach to deal with non-potential external forces or non-conservative forces. For such systems, we have what is

known as the Hamilton-Ostrogradski principle, or sometime known as extended Hamilton's principle. This is presented in the slide above.

(Refer Slide Time: 22:45)

## External and non-conservative forces

Equation of motion: Hamilton-Ostrogradski principle (mechanical system)

Virtual work:  $\delta W = \vec{F} \cdot \delta \vec{r}$

- Time frozen
- Forces frozen
- Virtual displacement

(Refer Slide Time: 26:54)

## External and non-conservative forces

Equation of motion: Hamilton-Ostrogradski principle (mechanical system)

$$\int_{t_0}^{t_1} (\delta \mathcal{L} + \delta W) dt = 0 \quad \delta \mathcal{L} = \delta T - \delta V, \quad \delta W = \vec{F} \cdot \delta \vec{r}$$

Given  $\vec{r} = \vec{r}(\vec{q}, t) \Rightarrow \delta \vec{r} = \sum_{j=1}^n \frac{\partial \vec{r}}{\partial q_j} \delta q_j$

$$\Rightarrow \delta W = \vec{F} \cdot \left( \sum_{j=1}^n \frac{\partial \vec{r}}{\partial q_j} \delta q_j \right) = \sum_{i=1}^m F_i \left( \sum_{j=1}^n \frac{\partial r_i}{\partial q_j} \delta q_j \right)$$

$$= \sum_{j=1}^n \underbrace{\left( \sum_{i=1}^m F_i \frac{\partial r_i}{\partial q_j} \right)}_{Q_j} \delta q_j = \sum_{j=1}^n Q_j \delta q_j$$

Virtual work by generalized forces  $\Rightarrow \delta W = \vec{Q} \cdot \delta \vec{q}$   
Generalized force vector

We first need to understand virtual work. The concept is as discussed in the 2 slides above.



(Refer Slide Time: 31:44)

## External and non-conservative forces

Equation of motion: Hamilton-Ostrogradski principle (mechanical system)

$$\int_{t_0}^{t_1} (\delta \mathcal{L} + \delta W) dt = 0 \quad \delta \mathcal{L} = \delta T - \delta V, \quad \delta W = \vec{F} \cdot \delta \vec{r} = \vec{Q} \cdot \delta \vec{q}$$
$$\Rightarrow \int_{t_0}^{t_1} \left( \left[ -\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) + \frac{\partial \mathcal{L}}{\partial \vec{q}} \right] \cdot \delta \vec{q} + \vec{Q} \cdot \delta \vec{q} \right) dt = 0$$
$$\Rightarrow \int_{t_0}^{t_1} \left[ -\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) + \frac{\partial \mathcal{L}}{\partial \vec{q}} + \vec{Q} \right] \cdot \delta \vec{q} dt = 0$$
$$\Rightarrow \boxed{\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = \vec{Q}}$$



With the concept of virtual work, we can now use the Hamilton-Ostrogradski principle to obtain the equation of motion as presented in the slide above.

(Refer Slide Time: 33:05)

## Summary

- Hamilton's principle for dynamical paths
- Lagrange's equation of motion
- Hamilton-Ostrogradski principle