Advanced Dynamics
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Module No # 01
Lecture No # 05
Relative Motion – III

We will continue our discussions on relative motion in rotating frames.

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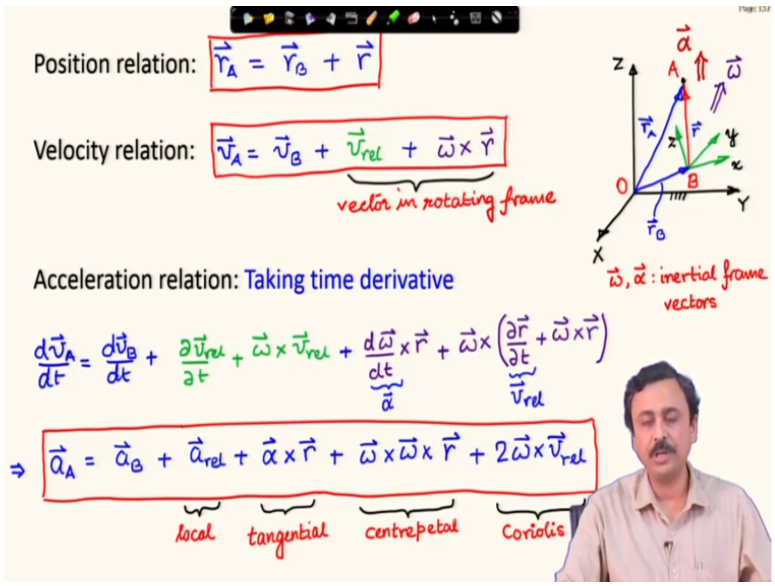

We first recapitulate the discussions in the previous lecture. The rotating frame xyz represented by the green frame in the figure has angular velocity ω and angular acceleration α. And XYZ is a fixed frame fixed with respect to distant stars. Now the position relation is

$$\boxed{\vec{r}_A = \vec{r}_B + \vec{r}}$$

We obtain the velocity and acceleration relations using the prescription described earlier as

$$\boxed{\vec{v}_A = \vec{v}_B + \underbrace{\vec{v}_{rel} + \vec{\omega}\times\vec{r}}_{\text{vector in rotating frame}}}$$

$$\boxed{\vec{a}_A = \vec{a}_B + \underbrace{\vec{a}_{rel}}_{\text{local}} + \underbrace{\vec{\alpha}\times\vec{r}}_{\text{tangential}} + \underbrace{\vec{\omega}\times\vec{\omega}\times\vec{r}}_{\text{centrepetal}} + \underbrace{2\vec{\omega}\times\vec{v}_{rel}}_{\text{Coriolis}}}$$

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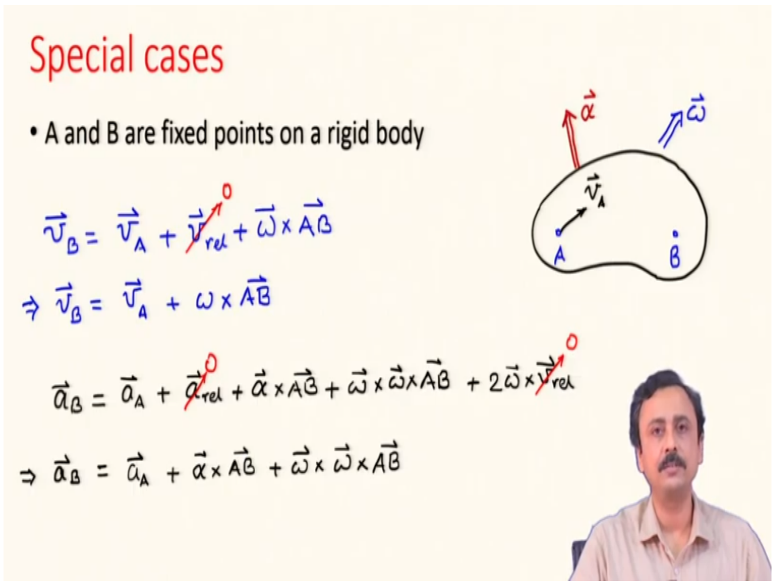

Now let us look at some special cases of the above general formulae that we have derived. Suppose A and B are 2 fixed points on a rigid body. What happens to our general velocity and acceleration relations?

Since both A and B are fixed on the rigid body, just like you and another person both are sitting in the same auditorium, $v_{rel} = 0$. Therefore, we have

$$\vec{v}_B = \vec{v}_A + \omega \times \vec{AB}$$

In the acceleration relation, we see that, because A and B are fixed on the rigid body, $v_{rel} = 0$, and $a_{rel} = 0$. Hence, we obtain

$$\vec{a}_B = \vec{a}_A + \vec{\alpha}\times\vec{AB} + \vec{\omega}\times\vec{\omega}\times\vec{AB}$$

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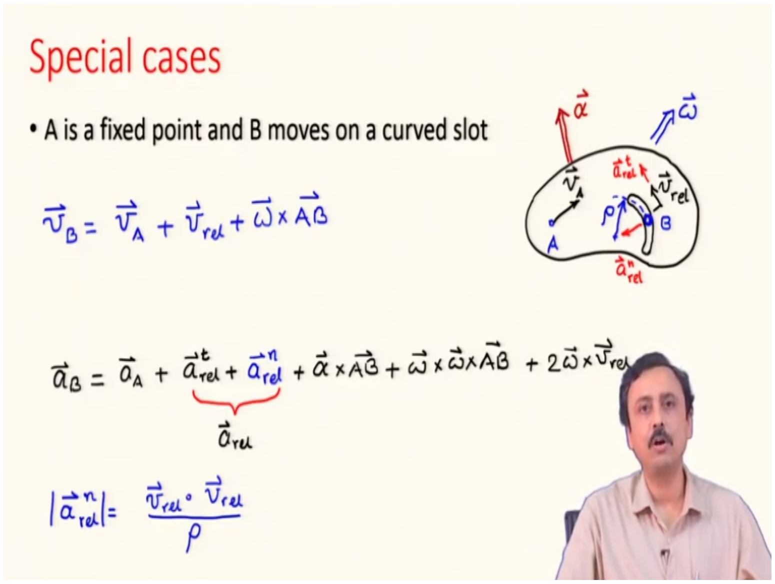

If B moves on a curved path of a certain radius of curvature as shown above the velocity relation is given by

$$\vec{v}_B = \vec{v}_A + \vec{v}_{rel} + \vec{\omega} \times \vec{AB}$$

Where v_{rel} is tangential to the path as measured by observed A in his/her local frame. Since B is moving on a curved path, a_{rel} of B has 2 components: the tangential relative acceleration (tangent to the path), and the normal relative acceleration (normal to the path). Therefore, we can write

$$\vec{a}_B = \vec{a}_A + \underbrace{\vec{a}^{\,t}_{rel} + \vec{a}^{\,n}_{rel}}_{\vec{a}_{rel}} + \vec{\alpha} \times \vec{AB} + \vec{\omega} \times \vec{\omega} \times \vec{AB} + 2\vec{\omega} \times \vec{v}_{rel}$$

$$|\vec{a}^{\,n}_{rel}| = \frac{\vec{v}_{rel} \cdot \vec{v}_{rel}}{\rho}$$

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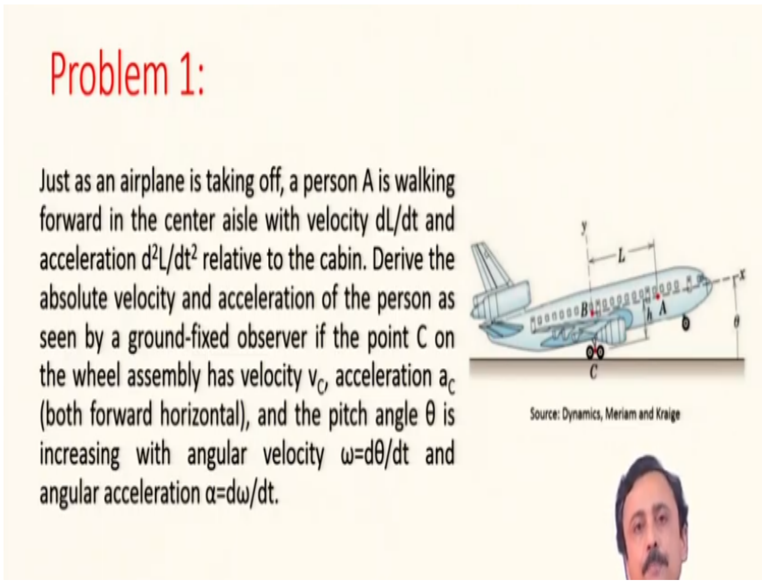

Let us understand with some examples. This problem is taken from the book of Meriam and Kraige.

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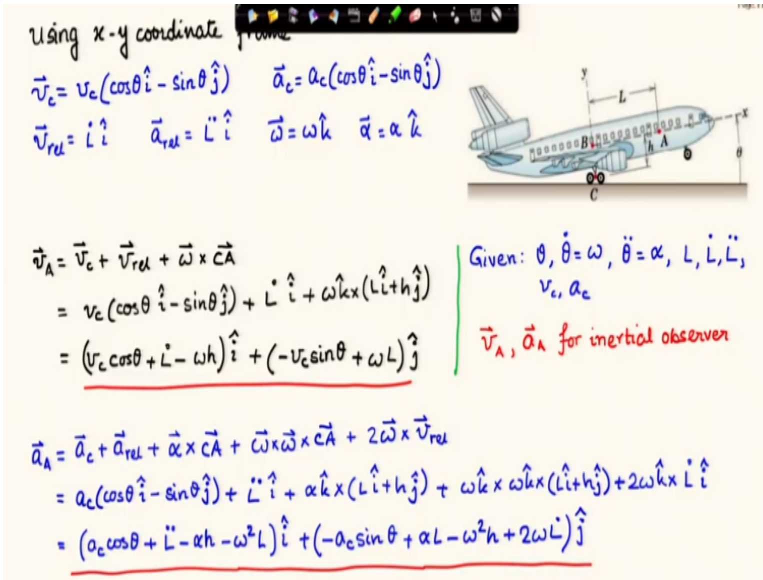

We will use the x-y frame that is given in the figure. I have written out all the given parameters that are specified in the problem we have to find out v_A and a_A as seen by inertial observer. We have represented all vectors in the x-y frame. Just plug in all the expressions and simplify to obtain

$$
\begin{aligned}
\vec{v}_A &= \vec{v}_c + \vec{v}_{rel} + \vec{\omega} \times \vec{CA} \\
&= v_c(\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) + \dot{L}\,\hat{i} + \omega\hat{k} \times (L\hat{i} + h\hat{j}) \\
&= (v_c\cos\theta + \dot{L} - \omega h)\hat{i} + (-v_c\sin\theta + \omega L)\hat{j}
\end{aligned}
$$

and

$$
\begin{aligned}
\vec{a}_A &= \vec{a}_c + \vec{a}_{rel} + \vec{\alpha} \times \vec{CA} + \vec{\omega} \times \vec{\omega} \times \vec{CA} + 2\vec{\omega} \times \vec{v}_{rel} \\
&= a_c(\cos\theta\,\hat{i} - \sin\theta\,\hat{j}) + \ddot{L}\,\hat{i} + \alpha\hat{k} \times (L\hat{i} + h\hat{j}) + \omega\hat{k} \times \omega\hat{k} \times (L\hat{i} + h\hat{j}) + 2\omega\hat{k} \times \dot{L}\,\hat{i} \\
&= (a_c\cos\theta + \ddot{L} - \alpha h - \omega^2 L)\hat{i} + (-a_c\sin\theta + \alpha L - \omega^2 h + 2\omega\dot{L})\hat{j}
\end{aligned}
$$

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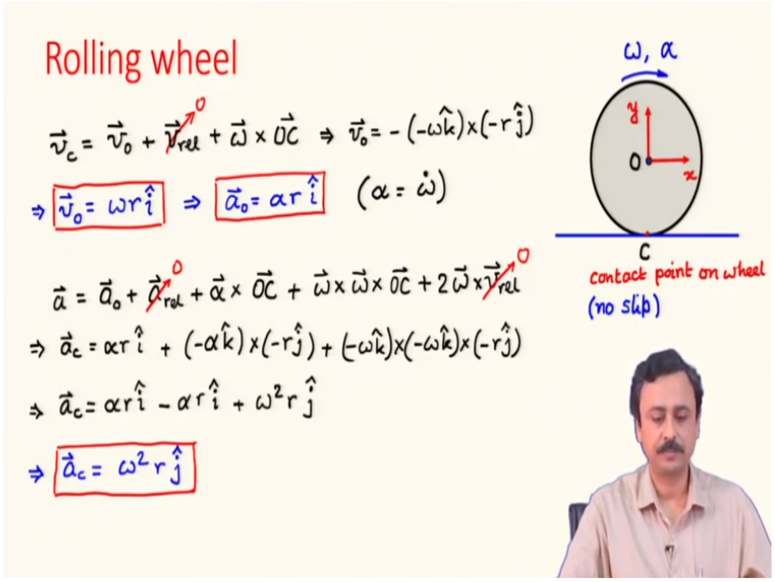

Consider the example of a wheel of radius r rolling on flat horizontal ground. The point C of the wheel is the point of contact, and I am assuming that this wheel is rolling without slipping. We consider a frame x-y I at the geometric center of the wheel and translating with it with the x-axis is always horizontal. Thus, this is a non-rotating frame.

Writing out the velocity relation of point C with respect to O, we have

$$\vec{v}_c = \vec{v}_0 + \cancel{\vec{v}_{rel}}^{\,0} + \vec{\omega}\times\vec{OC} \Rightarrow \vec{v}_0 = -(-\omega\hat{k})\times(-r\hat{j})$$

$$\Rightarrow \boxed{\vec{v}_0 = \omega r\hat{i}} \Rightarrow \boxed{\vec{a}_0 = \alpha r\hat{i}} \quad (\alpha = \dot{\omega})$$

Similarly, the acceleration relation is obtained as

$$\vec{a} = \vec{a}_0 + \cancel{\vec{a}_{rel}}^{\,0} + \vec{\alpha}\times\vec{OC} + \vec{\omega}\times\vec{\omega}\times\vec{OC} + 2\vec{\omega}\times\cancel{\vec{v}_{rel}}^{\,0}$$

$$\Rightarrow \vec{a}_c = \alpha r\hat{i} + (-\alpha\hat{k})\times(-r\hat{j}) + (-\omega\hat{k})\times(-\omega\hat{k})\times(-r\hat{j})$$

$$\Rightarrow \vec{a}_c = \alpha r\hat{i} - \alpha r\hat{i} + \omega^2 r\hat{j}$$

$$\Rightarrow \boxed{\vec{a}_c = \omega^2 r\hat{j}}$$

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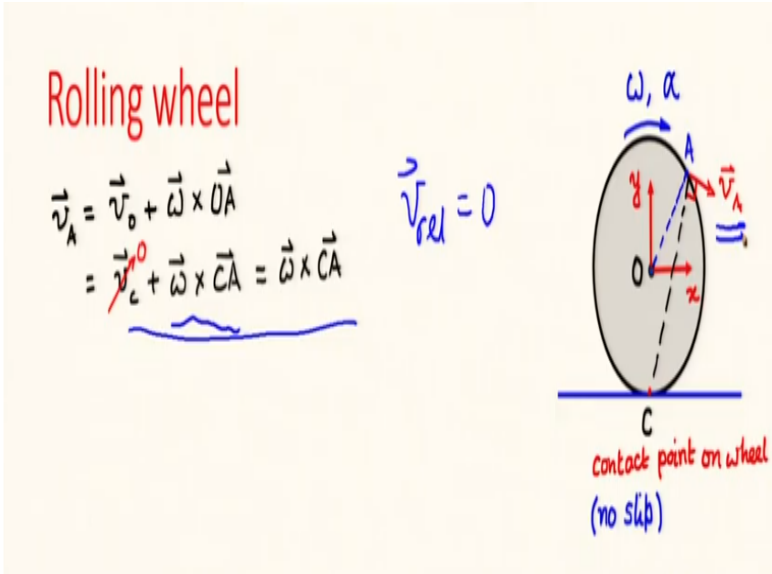

Now you can extend this to study acceleration or velocity of various other points on the wheel.

For example velocity of A can be written in two ways as

$$\vec{v}_A = \vec{v}_O + \vec{\omega} \times \vec{OA}$$
$$= \cancel{\vec{v}_c}^{\,0} + \vec{\omega} \times \vec{CA} = \vec{\omega} \times \vec{CA}$$

The second expression leads to a simpler result which says that v_A is perpendicular to CA.
The acceleration of A can be written as

$$\vec{a}_A = \vec{a}_c + \vec{\alpha} \times \vec{CA} + \vec{\omega} \times \vec{\omega} \times \vec{CA}$$

where $\vec{a}_c = \omega^2 r \hat{j}$

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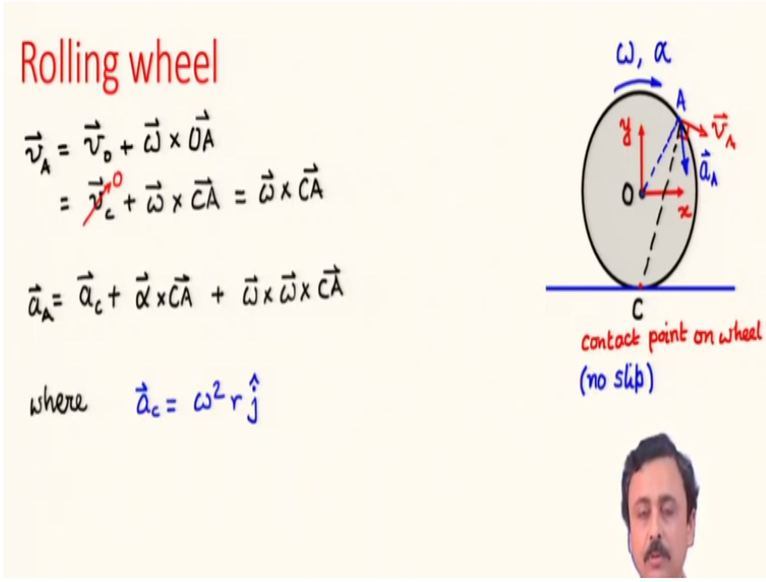

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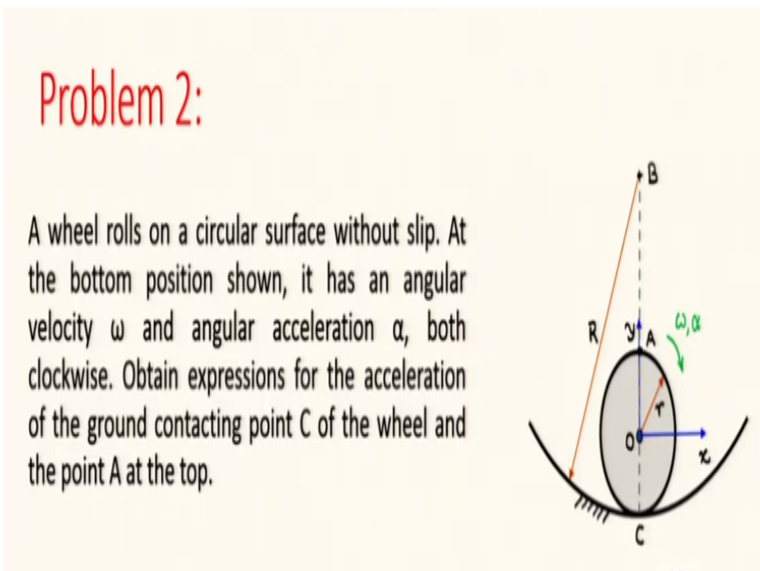

Let us extend this problem. Here the wheel is rolling on a circular track in a plane. It has angular velocity ω and angular acceleration α both clock wise. C is the point on the wheel which is currently in contact with the ground and A is the topmost point of the wheel at this instant.

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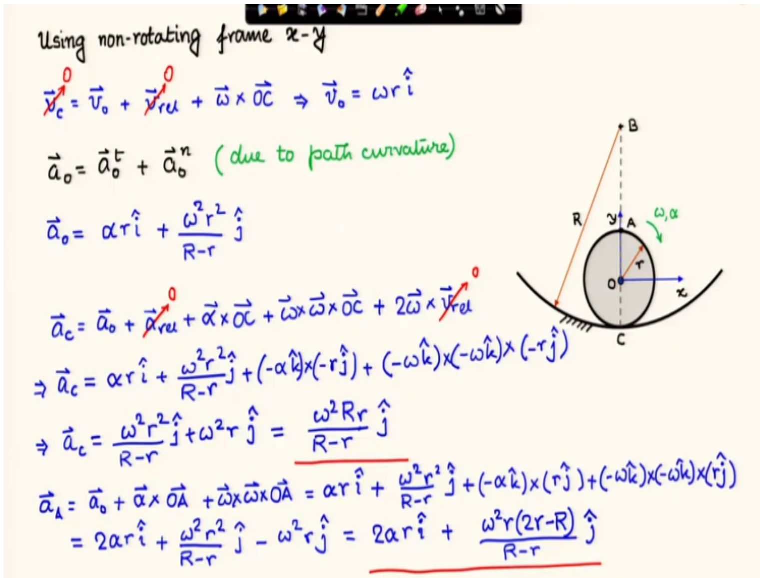

I take the non-rotating x-y frame and represent all vectors using it. From velocity relation

$$\cancelto{0}{\vec{v}_c} = \vec{v}_o + \cancelto{0}{\vec{v}_{rel}} + \vec{\omega} \times \vec{OC} \Rightarrow \vec{v}_o = \omega r \hat{i}$$

For acceleration, we note that the center point O is moving on a circular path of radius of curvature (R-r). Therefore the acceleration of point O will have 2 components tangential and normal given by

$$\vec{a}_o = \vec{a}_o^t + \vec{a}_o^n$$

$$\vec{a}_o = \alpha r \hat{i} + \frac{\omega^2 r^2}{R-r} \hat{j}$$

Plugging in all the expressions in the acceleration of C, we obtain

$$\vec{a}_c = \vec{a}_o + \cancelto{0}{\vec{\alpha}_{rel}} + \vec{\alpha} \times \vec{OC} + \vec{\omega} \times \vec{\omega} \times \vec{OC} + 2\vec{\omega} \times \cancelto{0}{\vec{v}_{rel}}$$

$$\Rightarrow \vec{a}_c = \alpha r \hat{i} + \frac{\omega^2 r^2}{R-r} \hat{j} + (-\alpha \hat{k}) \times (-r\hat{j}) + (-\omega \hat{k}) \times (-\omega \hat{k}) \times (-r\hat{j})$$

$$\Rightarrow \vec{a}_c = \frac{\omega^2 r^2}{R-r} \hat{j} + \omega^2 r \hat{j} = \underline{\frac{\omega^2 R r}{R-r} \hat{j}}$$

Also, acceleration of A is obtained as

$$\vec{a}_A = \vec{a}_O + \vec{\alpha}\times\vec{OA} + \vec{\omega}\times\vec{\omega}\times\vec{OA} = \alpha r\hat{i} + \frac{\omega^2 r^2}{R-r}\hat{j} + (-\alpha\hat{k})\times(r\hat{j}) + (-\omega\hat{k})\times(-\omega\hat{k})\times(r\hat{j})$$

$$= 2\alpha r\hat{i} + \frac{\omega^2 r^2}{R-r}\hat{j} - \omega^2 r\hat{j} = \underline{2\alpha r\hat{i} + \frac{\omega^2 r(2r-R)}{R-r}\hat{j}}$$

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Coordinate system x-y fixed to the eagle

$$\vec{v}_m = \vec{v}_e + \underbrace{\vec{v}_{rel} + \vec{\omega}\times\vec{x}}_{\vec{v}_{m/e}}$$

$$\Rightarrow v_m(c\theta\,\hat{i} + s\theta\,\hat{j}) = v_e\hat{i} + \dot{x}\hat{i} + (-\dot{\theta}\hat{k})\times(x\hat{i})$$

$$= (v_e + \dot{x})\hat{i} - \dot{\theta}x\hat{j} \quad (x:\ x\text{-coordinate of mouse in } x\text{-}y)$$

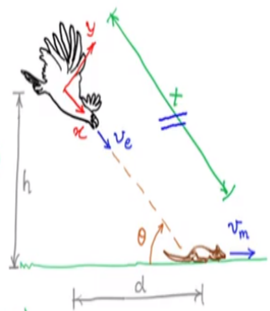
$$\Rightarrow \dot{x} = v_m c\theta - v_e$$

Integrating over time

$$\cancelto{0}{x(T)} - x(0) = \int_0^T v_m c\theta\, dt - v_e T \quad (T:\ \text{time to catch})$$

$$\Rightarrow \underline{\frac{-\sqrt{h^2+d^2} + v_e T}{v_m} = \int_0^T c\theta\, dt}$$

$$x(0) = \sqrt{h^2 + d^2}$$



Finally we have a problem of this eagle which is swooping down on a mouse moving on the flat ground as shown. The velocity vector of the eagle is always directed towards the mouse. We have to calculate the time taken by the eagle to catch the mouse if it starts initially from the height h when the mouse was at a horizontal distance d ahead of the eagle.

In order to solve this problem we fix the coordinate system on the eagle. I fix the coordinate system xyz on the eagle.

Now we can write

$$\vec{v}_m = \vec{v}_e + \underbrace{\vec{v}_{rel} + \vec{\omega}\times\vec{X}}_{\vec{v}_{m/e}}$$

$$\Rightarrow v_m\left(c\theta\,\hat{i} + s\theta\,\hat{j}\right) = v_e\hat{i} + \dot{X}\,\hat{i} + \left(-\dot{\theta}\,\hat{k}\right)\times\left(X\,\hat{i}\right)$$

$$= \left(v_e + \dot{X}\right)\hat{i} - \dot{\theta}X\,\hat{j} \qquad (X: \text{x-coordinate of mouse in x-y})$$

The x component gives

$$\dot{X} = v_m c\theta - v_e$$

Integrating over time

$$\cancelto{0}{X(T)} - X(0) = \int_0^T v_m c\theta\, dt - v_e T$$

$$\Rightarrow \frac{-\sqrt{h^2+d^2} + v_e T}{v_m} = \int_0^T c\theta\, dt$$

where T is the time to catch the mouse. Here the integral is not known. However, it can be determined from another perspective.

Therefore direction of increasing theta which is so theta dot is positive in the clockwise sense but this is in the sense of minus k cap. Remember xy is shown like this therefore the z axis is coming out of the plane of the screen therefore k cap is coming out of the plane of the screen and with this theta if this is omega of the frame then it is negative k cap correction. Therefore my omega vector is theta dot is its magnitude but now directed in minus k cap direction.

Therefore my omega which I have used here and x vector is x i cap the X which is this location times i cap. Once you do this rearrangement then on the right hand side I have this along the i cap direction I have ve + X dot and along the j cap I have minus theta dot x coordinate. Now comparing the 2 sides I can obtain say x this is from i cap equation i cap and here is this i cap ve velocity of the eagle plus X dot must be vm cosine theta therefore x dot is vm cosine theta – v.

If I integrate this equation so this gives me a relation between the velocities so relative velocity is X dot remember. If I integrate this equation then I expect to find out X now why do I want to find out this because once this eagle has caught the mouse X has gone to 0. Therefore if at time T eagle has got the mouse then X at time T at time equal to T must be 0 that is what I have written here.

I integrate this equation over 0 when the eagle just saw the mouse to T when it has caught the mouse. When it has caught the mouse X at T is 0. And this is at the start what, is this X? That i can find out in terms of h and d on the right hand side I have this integral ve is the constant. Therefore this integration is easy to do but I have this cosine sitting inside the integral and cosine of theta sitting inside the integral and the variation of the theta is not known to me.

Let me keep it as it is I have used this x0 remember this x0 is this length x at time t = 0 when the eagle has just seen the mouse and that must be equal to underfoot of x square + d square. Once you do this algebra you will get this integral of cosine theta over 0 to t which is not known as yet. But one side the left hand side is completely known let us see if we can find out this integral from another approach that is what we are going to do next.

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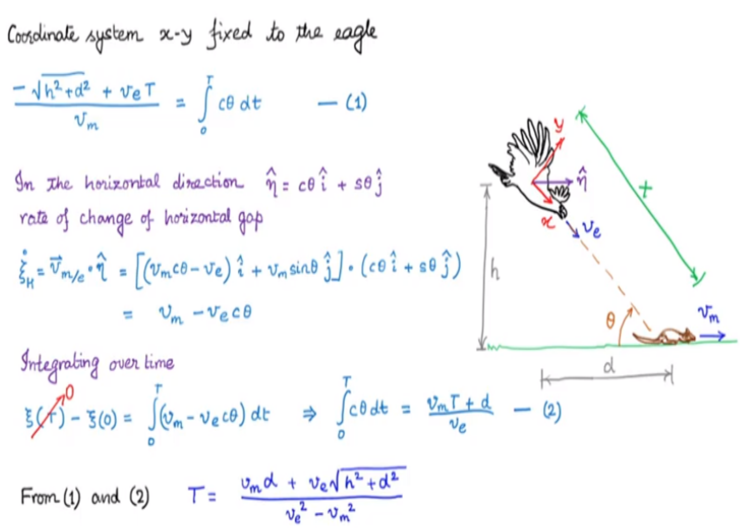

This is the first equation we have just now derived now let us look at this horizontal direction I have shown this unit vector in horizontal direction eta cap this is the unit vector in the horizontal direction which I can always represent in i cap j cap in the xy z coordinates this is the representation cosine theta i cap + sin theta j cap. And I will try to find out what is the horizontal gap. So right now at time t = 0 the horizontal gap is d but as the eagle flies somewhere here and the mouse goes here this is the horizontal gap is going to change is going to reduce.

And when the eagle has caught the mouse the horizontal gap will also vanish so with that idea I find out this rate of change of horizontal gap which is Psi h dot. I represent that horizontal gap as Psi h and Psi h dot is the rate of change of the horizontal gap which must be equal to this relative velocity of the mouse with respect to the eagle which we have just now discussed dot eta cap this is the relative velocity of the mouse with respect to the eagle dot eta cap will give me the rate of change of horizontal gap.

And if I write that now explicitly remember this is our expression of vm with respect to v mouse with respect to the eagle which is nothing but velocity of the mouse minus velocity of the eagle. And this was represented in terms of x dot and omega cross so v rel and omega cross r omega cross x vector. So that was in the as represented in the rotating frame but v mouse with respect to the eagle is v mouse – v eagle.

And that is what I have written out here once I take the dot product it simplifies to vm – ve cosine theta once again you see cosine theta is coming and that is Psi h dot therefore if I integrate over time from 0 to T the horizontal gap at T must vanish because it has caught the mouse and Psi at 0 is at the starting horizontal gap which is d and that is equal to the integral you can see here vm – cosine theta integrated over time.

Vm is the constant ve is the constant therefore those things will come out of the integration only thing that I am left with is integral of cosine theta dt integrated from 0 to t. Now you compare equation 1 and 2 you have this integral sitting on one side and both therefore they must be equal and from there I can solve for the time to catch the mouse. So this was an interesting example application of kinematics.

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Summary

- Relative motion relations rotating (non-inertial) observer

- Special cases

To summarize we have looked at relative motion in rotating frames and we have discussed some examples an interesting example where that of the eagle catching the mouse. With that I will close this lecture.