

Advanced Dynamics
Prof. Anirvan Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology - Kharagpur

Module No # 10
Lecture No # 49
Hamilton's Principle and Lagrange's Equation of Motion - II

We will continue with the topic of Lagrange's equation Hamilton's principle to application of Hamilton's principle and deriving Lagrange's equation of motion.

(Refer Slide Time: 00:21)

Overview

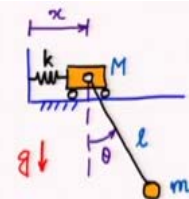
- Hamilton's principle for dynamical paths
- Lagrange's equation of motion

We consider an example as shown in the slide below, and derive the equations of motion.

(Refer Slide Time: 00:26)

Example 2:

Equation of motion of a simple pendulum on a flexibly anchored cart



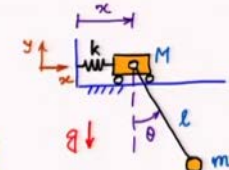
Generalized coordinate: (θ, x)

Lagrange's equations of motion:

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$$
$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$



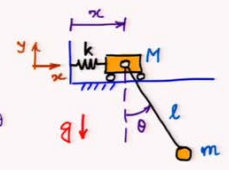
(Refer Slide Time: 01:57)



$\mathcal{L} = T - V$
 $T = \frac{1}{2} m v_m^2 + \frac{1}{2} M \dot{x}^2$
 $\vec{r}_m = x \hat{i} + l(s\theta \hat{i} - c\theta \hat{j}) \Rightarrow \vec{v}_m = (\dot{x} + l\dot{\theta}c\theta) \hat{i} + l\dot{\theta}s\theta \hat{j}$
 $\Rightarrow v_m^2 = (\dot{x} + l\dot{\theta}c\theta)^2 + l^2\dot{\theta}^2s^2\theta = \dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}c\theta$
 $\Rightarrow T = \frac{1}{2} (m+M) \dot{x}^2 + \frac{1}{2} m (l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}c\theta)$
 $V = \frac{1}{2} kx^2 - mglc\theta \quad (x\text{-axis as datum})$
 $\Rightarrow \mathcal{L} = \frac{1}{2} (m+M) \dot{x}^2 + \frac{1}{2} m (l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}c\theta) - \frac{1}{2} kx^2 + mglc\theta$

The Lagrangian of the system is derived in the slide above.

(Refer Slide Time: 05:11)



$\mathcal{L} = \frac{1}{2} (m+M) \dot{x}^2 + \frac{1}{2} m (l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}c\theta) - \frac{1}{2} kx^2 + mglc\theta$
 $\frac{\partial \mathcal{L}}{\partial \theta} = ml^2\ddot{\theta} + ml\dot{x}c\theta \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = ml^2\ddot{\theta} + ml\ddot{x}c\theta - ml\dot{x}\dot{\theta}s\theta$
 $\frac{\partial \mathcal{L}}{\partial \theta} = -ml\dot{x}\dot{\theta}s\theta - mgl s\theta$
 Equation of motion θ : $ml^2\ddot{\theta} + mlc\theta\ddot{x} + mgl s\theta = 0 \Rightarrow \ddot{\theta} + \frac{1}{l}c\theta\ddot{x} + \frac{g}{l}s\theta = 0$
 $\frac{\partial \mathcal{L}}{\partial x} = (m+M)\ddot{x} + ml\dot{\theta}c\theta \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = (m+M)\ddot{x} + ml\ddot{\theta}c\theta - ml\dot{\theta}^2s\theta$
 $\frac{\partial \mathcal{L}}{\partial x} = -kx$
 Equation of motion x : $(m+M)\ddot{x} + mlc\theta\ddot{\theta} - ml\dot{\theta}^2s\theta + kx = 0$

The equations of motion of the pendulum on a spring anchored cart is presented in the slide above.

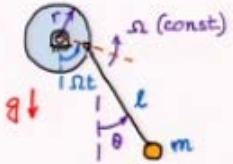

(Refer Slide Time: 08:28)

Example 3:

Equation of motion of a simple pendulum on a uniformly rotating disc

Generalized coordinate: θ

Lagrange's equation of motion: $\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$

We move to the next example of a simple pendulum on a uniformly rotating disk as shown above.

(Refer Slide Time: 09:37)

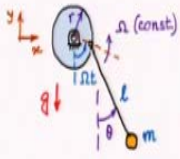

$$\mathcal{L} = \frac{1}{2} m \left[r^2 \Omega^2 + l^2 \dot{\theta}^2 + 2 r l \Omega \dot{\theta} \cos(\theta - \Omega t) \right] + m g (r \cos \Omega t + l \cos \theta)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m l^2 \dot{\theta} + m r l \Omega \cos(\theta - \Omega t)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = m l^2 \ddot{\theta} - m r l \Omega (\dot{\theta} - \Omega) \sin(\theta - \Omega t)$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = -m r l \Omega \dot{\theta} \sin(\theta - \Omega t) - m g l \sin \theta$$

Equation of motion: $m l^2 \ddot{\theta} + m r l \Omega^2 \sin(\theta - \Omega t) + m g l \sin \theta = 0$

$$\Rightarrow \ddot{\theta} + \frac{r}{l} \Omega^2 \sin(\theta - \Omega t) + \frac{g}{l} \sin \theta = 0$$



The equation of motion of the system is derived in the slide above.

(Refer Slide Time: 13:21)

$$\mathcal{L} = \frac{1}{2} m (r^2 \dot{\theta}^2 + r^2 \Omega^2 \sin^2 \theta) + mgr \cos \theta$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = mr^2 \dot{\theta} \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}} \right) = mr^2 \ddot{\theta}$$

$$\frac{\partial \mathcal{L}}{\partial \theta} = mr^2 \Omega^2 \sin \theta \cos \theta - mgr \sin \theta$$

Equation of motion: $mr^2 \ddot{\theta} - mr^2 \Omega^2 \sin \theta \cos \theta + mgr \sin \theta = 0$

$$\Rightarrow \ddot{\theta} - \Omega^2 \sin \theta \cos \theta + \frac{g}{r} \sin \theta = 0$$

We move to the next example of motion of a bead sliding on a uniformly rotating circular hoop as shown above. The equation of motion of the bead is presented.

(Refer Slide Time: 17:24)

Summary

- Hamilton's principle for dynamical paths
- Lagrange's equation of motion

We have considered some examples to demonstrate the application of Lagrange's equation of motion to mechanical systems.