

**Advanced Dynamics**  
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**Module No # 10**  
**Lecture No # 48**  
**Hamilton's Principle and Lagrange's Equation of Motion – I**

In this lecture I am going to start discussions on the formulation of dynamics as formulated by Lagrange.

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## Overview

- Hamilton's principle for dynamical paths
- Lagrange's equation of motion

We will start with the Hamilton's principle which is a very important principle, and is the starting point.

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## Hamilton's principle

- Evolution of a dynamical system in configuration space
- At  $t=t_0$  the system configuration is completely known (say  $\mathbf{q}_0$ )
- At  $t=t_1$  the system configuration is completely known (say  $\mathbf{q}_1$ )

**Question:** Can we determine the path taken by the particle in moving from  $\mathbf{q}_0$  to  $\mathbf{q}_1$ ?

**Answer:** Yes! Using Hamilton's principle.

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## Hamilton's principle



**Hamilton's principle:** A system will take that path which extremizes (minimizes/maximizes) the action.

**Action:**  $\mathcal{A}[\vec{q}_r(t)] = \int_{t_1}^{t_2} \mathcal{L}(\vec{q}_r, \dot{\vec{q}}_r, t) dt$  (functional: function of a function)  
 $\Rightarrow$   $\mathcal{L}(\vec{q}_r, \dot{\vec{q}}_r, t) : \text{Lagrangian}$   $\underline{q_r(t)}$

Hamilton's principle is stated above.

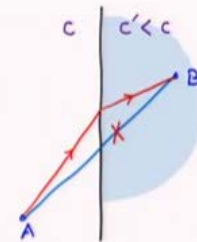
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## Structure of Lagrangian

Lagrangian is different for different systems

**Light:**  $\mathcal{L} = 1$   $\mathcal{A} = \int_{t_1}^{t_2} dt$  (Fermat's principle)

- Light travels on the minimum time path
- Homogeneous and isotropic media: minimum time path = minimum distance path



The idea of Hamilton's principle as an extremizing principle for dynamical systems is motivated by Fermat's principle for light.

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## Structure of Lagrangian

Lagrangian is different for different systems

Mechanical systems:

$$\mathcal{L}(\vec{q}, \dot{\vec{q}}, t) = T - V$$

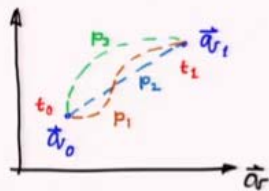
$T$ : Kinetic energy

$V$ : Potential energy

For a mechanical system, Lagrangian is equal to the kinetic energy minus the potential energy.

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## Extremizing a functional



$$\mathcal{A}_1 = \mathcal{A}[P_1]$$

$$\mathcal{A}_2 = \mathcal{A}[P_2]$$

$$\mathcal{A}_3 = \mathcal{A}[P_3]$$

$\vdots$   
 $\infty$

$$\text{Extremum } [\mathcal{A}] = \mathcal{A}_i \quad \text{s.t. } \mathcal{A}_i \leq \mathcal{A}_j \quad \forall j \quad (\text{minimum})$$

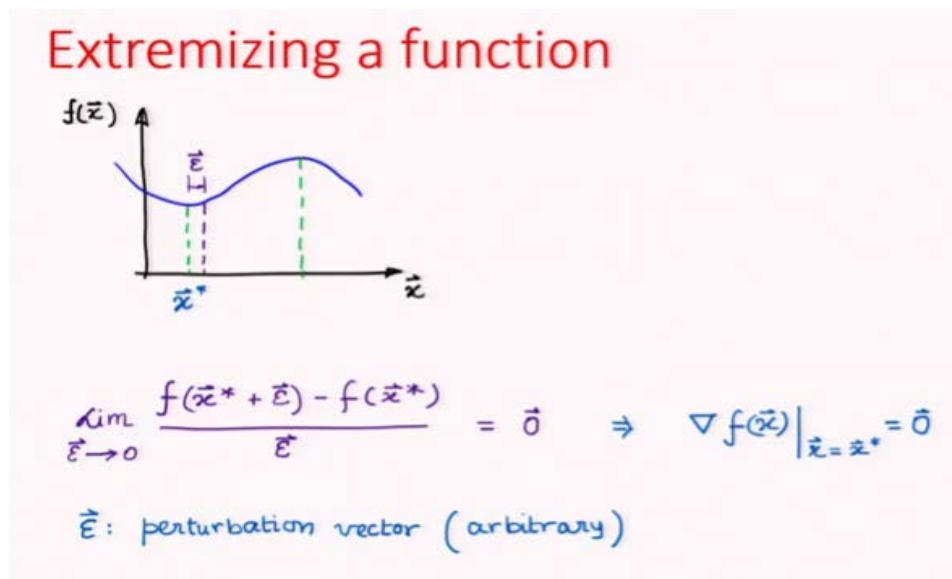
$$\mathcal{A}_i \geq \mathcal{A}_j \quad \forall j \quad (\text{maximum})$$

Impractical approach!



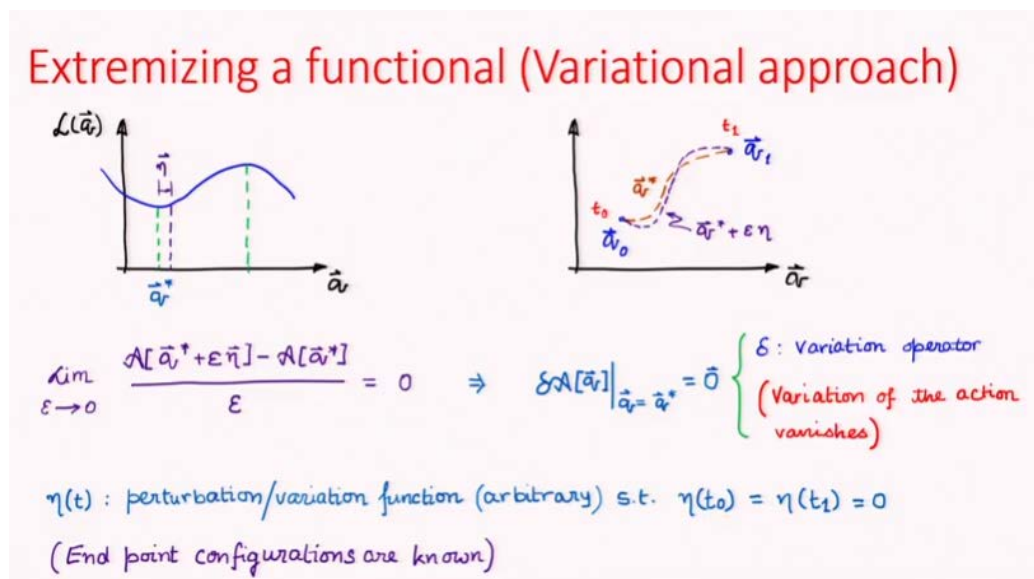
Now we discuss a little bit about how we extremize a functional, as shown above.

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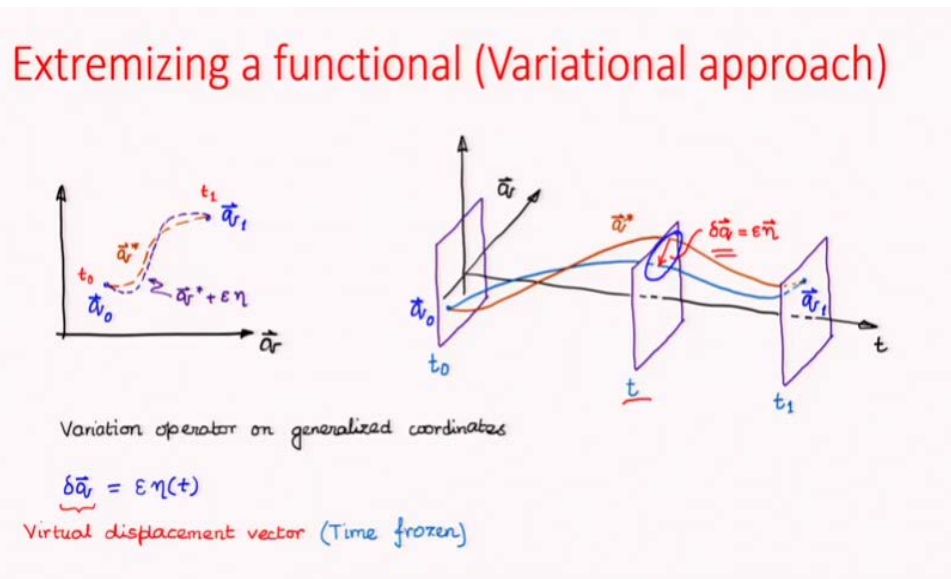
Let us go back a little bit and look at how we extremize a function. This is discussed in the slide above.

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Now we come back to our problem of extremizing a functional, as discussed above.

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We can also visualize variation in the configuration space as shown above.

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## Hamilton's principle (Variational principle)

**Hamilton's principle:** A system will take that path for which the variation of the action vanishes

$$\Rightarrow \delta \mathcal{A} = 0$$

**Mechanical system:** Action is minimized

We come back to Hamilton's principle once again. It says the system will take that path in the configuration space for which the variation of the action vanishes.

In the following 2 slides, we discuss the derivation of Lagrange's equation of motion.

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### Lagrange's equation

$$\mathcal{A} = \int_{t_0}^{t_1} \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) dt$$
$$\delta \mathcal{A} = 0 \Rightarrow \int_{t_0}^{t_1} \delta \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) dt = 0$$


- $\delta$  is similar to  $d$  (total derivative operator)  
but does not derivate time
- $\delta \frac{d}{dt} = \frac{d}{dt} \delta$  (commutes with time derivative)

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### Lagrange's equation

$$\int_{t_0}^{t_1} \delta \mathcal{L}(\vec{q}, \dot{\vec{q}}, t) dt = 0$$
$$\Rightarrow \int_{t_0}^{t_1} \left[ -\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) + \frac{\partial \mathcal{L}}{\partial \vec{q}} \right] \cdot \delta \vec{q} dt = 0$$
$$\Rightarrow \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \right) - \frac{\partial \mathcal{L}}{\partial \vec{q}} = \vec{0} \quad (\delta \vec{q} \text{ is arbitrary variation})$$

- Euler-Lagrange equation
- In context of dynamical systems: Lagrange's equation of motion
- Number of equations:  $F = n$





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**Example 1:**

Equation of motion of a simple pendulum

Generalized coordinate:  $\theta$

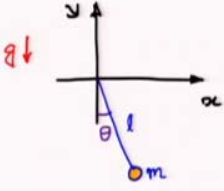
Lagrange's Equation of motion:  $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) - \frac{\partial \mathcal{L}}{\partial \theta} = 0$

$\mathcal{L} = T - V$      $T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$      $V = -mgl\cos\theta$     ( $x$ -axis as datum)

$\Rightarrow \mathcal{L} = \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$      $\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = ml^2\dot{\theta}$      $\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$      $\frac{\partial \mathcal{L}}{\partial \theta} = -mgl\sin\theta$

Equation of motion:

$\Rightarrow ml^2\ddot{\theta} + mgl\sin\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\sin\theta = 0$



As an example, we will look at the pendulum example equation of motion of a simple pendulum, as presented in the slide above.

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**Summary**

- Hamilton's principle for dynamical paths
- Lagrange's equation of motion