

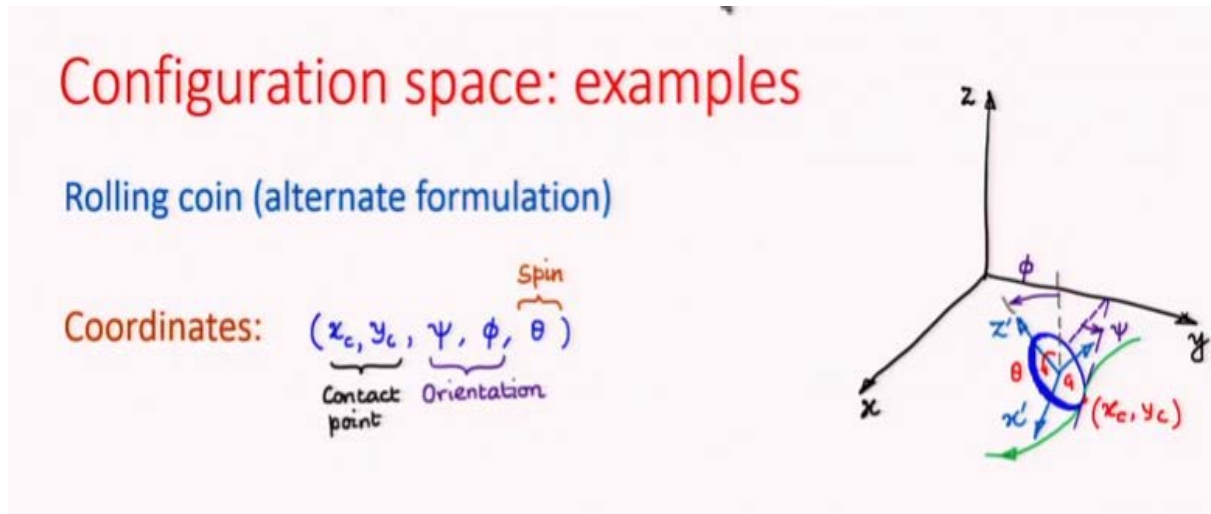
Advanced Dynamics
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Module No # 10

Lecture No # 47

Introduction to Analytical Dynamics: Generalized Coordinates - II

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We are going to continue our discussion on generalized coordinates constraints generalized and the configuration that we had introduced in the last lecture we are going to look at these concepts once again and with examples. Just to recapitulate, the configuration space is the space with minimum number of coordinates in which a single point completely fixes the configuration of a dynamical system. The coordinates that are used in the configuration space are known as the generalized coordinates.

We revisit the rolling coin example and present an alternate formulation as discussed below.

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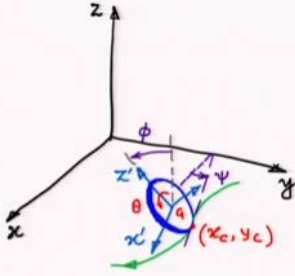

Configuration space: examples

Rolling coin (alternate formulation)

Coordinates: $(x_c, y_c, \underbrace{\psi, \phi, \theta}_{\text{Orientation}}, \underbrace{\theta}_{\text{Spin}})$

Constraints:

- No-slip constraint $\Rightarrow \begin{cases} \dot{x}_c - r\dot{\theta} \cos \psi = 0 \\ \dot{y}_c - r\dot{\theta} \sin \psi = 0 \end{cases} \left. \vphantom{\begin{matrix} \dot{x}_c \\ \dot{y}_c \end{matrix}} \right\} \begin{array}{l} \text{cannot be} \\ \text{integrated} \end{array}$
- Coordinate transformation $\begin{Bmatrix} x_q \\ y_q \\ z_q \end{Bmatrix} = \begin{Bmatrix} x_c + r \sin \psi \sin \phi \\ y_c - r \cos \psi \sin \phi \\ r \cos \phi \end{Bmatrix}$

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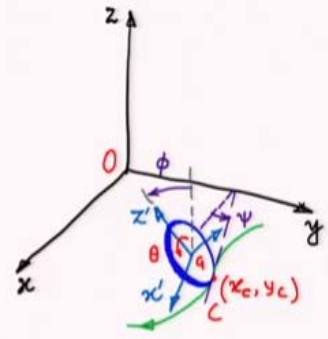

Configuration space: examples

Rolling coin (alternate formulation)

$$\dot{x}_q - r\dot{\phi} \cos \psi - r(\dot{\theta} + \dot{\psi} \sin \phi) \cos \psi = 0$$

$$\dot{y}_q + r\dot{\phi} \sin \psi - r(\dot{\theta} + \dot{\psi} \sin \phi) \sin \psi = 0$$

$$\vec{OQ} = \vec{OC} + \vec{CQ}$$

$$\vec{r}_q = \vec{r}_c + \vec{c}_q$$



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Configuration space: examples

Rolling coin (alternate formulation)

$$\dot{x}_a - r\dot{\phi} \cos\psi - r(\dot{\theta} + \dot{\psi} \sin\phi) \cos\psi = 0$$

$$\dot{y}_a + r\dot{\phi} \sin\psi - r(\dot{\theta} + \dot{\psi} \sin\phi) \sin\psi = 0$$

$$\vec{r}_a = \vec{r}_c + \vec{c}_a$$

$$\vec{c}_a^{xyz} = {}^{xyz}R_{z'y'z'} \begin{Bmatrix} 0 \\ 0 \\ r \end{Bmatrix}$$

$$\Rightarrow x_a \hat{i} + y_a \hat{j} + z_a \hat{k} = x_c \hat{i} + y_c \hat{j} + R \begin{Bmatrix} 0 \\ 0 \\ r \end{Bmatrix} \quad \left(R = \begin{bmatrix} c\psi & -s\psi c\phi & s\psi s\phi \\ s\psi & c\psi c\phi & -c\psi s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \right)$$

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Configuration space: examples

Rolling coin (alternate formulation)

$$\dot{x}_a - r\dot{\phi} \cos\psi - r(\dot{\theta} + \dot{\psi} \sin\phi) \cos\psi = 0$$

$$\dot{y}_a + r\dot{\phi} \sin\psi - r(\dot{\theta} + \dot{\psi} \sin\phi) \sin\psi = 0$$

$$\vec{r}_a = \vec{r}_c + \vec{c}_a$$

$$\Rightarrow x_a \hat{i} + y_a \hat{j} + z_a \hat{k} = x_c \hat{i} + y_c \hat{j} + R \begin{Bmatrix} 0 \\ 0 \\ r \end{Bmatrix} \quad \left(R = \begin{bmatrix} c\psi & -s\psi c\phi & s\psi s\phi \\ s\psi & c\psi c\phi & -c\psi s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \right)$$

$$\Rightarrow \begin{Bmatrix} x_a \\ y_a \\ z_a \end{Bmatrix} = \begin{Bmatrix} x_c + r s\psi s\phi \\ y_c - r c\psi s\phi \\ r c\phi \end{Bmatrix}$$

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Configuration space: examples

Rolling coin (alternate formulation)

$$\ddot{x}_a - r\dot{\phi}c\psi s\phi - r(\ddot{\theta} + \dot{\psi}s\phi)c\psi = 0$$

$$\ddot{y}_a + r\dot{\phi}c\phi c\psi - r(\ddot{\theta} + \dot{\psi}s\phi)s\psi = 0$$

$$\begin{Bmatrix} \ddot{x}_a \\ \ddot{y}_a \\ \ddot{z}_a \end{Bmatrix} = \begin{Bmatrix} \ddot{x}_c + r\dot{\psi}c\psi s\phi + r\dot{\phi}s\psi c\phi \\ \ddot{y}_c + r\dot{\psi}s\psi s\phi - r\dot{\phi}c\psi c\phi \\ -r\dot{\phi}s\phi \end{Bmatrix}$$

Substituting in constraints

$$\begin{aligned} \Rightarrow \ddot{x}_c - r\ddot{\theta}c\psi &= 0 \\ \ddot{y}_c - r\ddot{\theta}s\psi &= 0 \end{aligned}$$

The 2 new constraint equations in our new choice of coordinates are presented in the above slide.

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Configuration space: examples

Rolling coin (alternate formulation)

Coordinates: $(\underbrace{x_c, y_c}_{\text{Contact point}}, \underbrace{\psi, \phi}_{\text{Orientation}}, \underbrace{\theta}_{\text{Spin}})$

Constraints:

- Always in ground contact (considered)
- No-slip constraint

$$\Rightarrow \begin{aligned} \ddot{x}_c - r\ddot{\theta}c\psi &= 0 \\ \ddot{y}_c - r\ddot{\theta}s\psi &= 0 \end{aligned} \left. \vphantom{\begin{aligned} \ddot{x}_c - r\ddot{\theta}c\psi &= 0 \\ \ddot{y}_c - r\ddot{\theta}s\psi &= 0 \end{aligned}} \right\} \begin{array}{l} \text{cannot be} \\ \text{integrated} \end{array}$$

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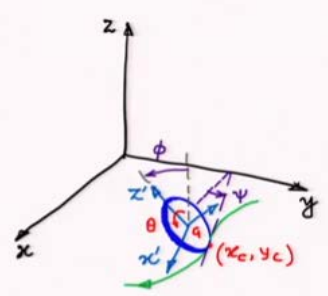
Configuration space: examples

Rolling coin (alternate formulation)

Coordinates: $(x_c, y_c, \psi, \phi, \theta)$
 Contact point: (x_c, y_c)
 Orientation: ψ, ϕ, θ
 Spin: θ

Constraints:

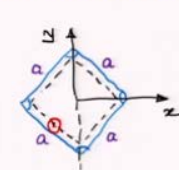
- No-slip constraint $\Rightarrow \begin{cases} \dot{x}_c - r\dot{\theta}\cos\psi = 0 \\ \dot{y}_c - r\dot{\theta}\sin\psi = 0 \end{cases} \left. \vphantom{\begin{matrix} \dot{x}_c \\ \dot{y}_c \end{matrix}} \right\} \begin{array}{l} \text{cannot be} \\ \text{integrated} \end{array}$
- Coordinate transformation $\begin{Bmatrix} x_q \\ y_q \\ z_q \end{Bmatrix} = \begin{Bmatrix} x_c + r\sin\psi\sin\phi \\ y_c - r\cos\psi\sin\phi \\ r\cos\phi \end{Bmatrix}$



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Coordinates and configuration space

- Physical coordinates (useful for position, velocity, acceleration)
 (r_1, r_2, \dots, r_m)



Carrom board coins (diameter d)

$(r_1, r_2) = (x, y)$

$C(r_1, r_2) = |x| + |y| - \frac{(a-d)}{\sqrt{2}} \leq 0$

Constraints:

- Holonomic: explicitly in terms of physical coordinates
- Non-holonomic: in terms of rates of physical coordinates (non-integrable)

(a) $C(r_1, \dots, r_m, \dot{r}_1, \dots, \dot{r}_m, t)$

(b) $C(r_1, \dots, r_m) \leq 0$

Now we move into a more detailed discussions on coordinates and configuration space. Various types of constraints are discussed in the slide above.

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Coordinates and configuration space

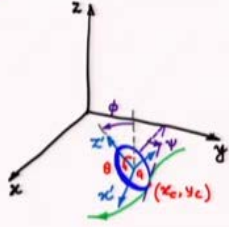
- Generalized coordinates (q_1, q_2, \dots, q_n)

$$r_1 = r_1(q_1, q_2, \dots, q_n, t)$$

$$\vdots$$

$$r_m = r_m(q_1, q_2, \dots, q_n, t)$$
- Configuration space: defined using the generalized coordinates
- Degree of freedom F : dimension of the configuration space = no. of physical coordinates – no. of holonomic constraints

$$F = m - (m - n) = n$$



$m = 6$ $m - n = 1$
 $F = 5$
 $\vec{q} = (x_c, y_c, \psi, \phi, \theta)$

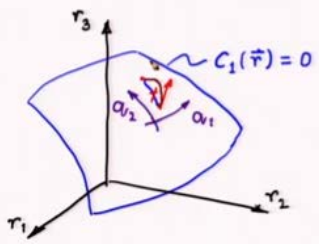
It is important to realize that holonomic constraints decide the degree of freedom of a system, while non-holonomic constraints do not.

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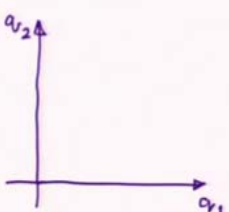
Geometric picture of coordinates


- Physical coordinates (r_1, r_2, \dots, r_m)
- Generalized coordinates (q_1, q_2, \dots, q_n)
 $(n \leq m)$

$$\left. \begin{array}{l} r_1 = r_1(q_1, q_2, \dots, q_n, t) \\ \vdots \\ r_m = r_m(q_1, q_2, \dots, q_n, t) \end{array} \right\}$$



$C_1(\vec{r}) = 0$





We picturize the physical coordinates and generalized coordinates as shown above.

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Summary

- Configuration space
- Generalized coordinates
- Constraints

The summary is presented above.