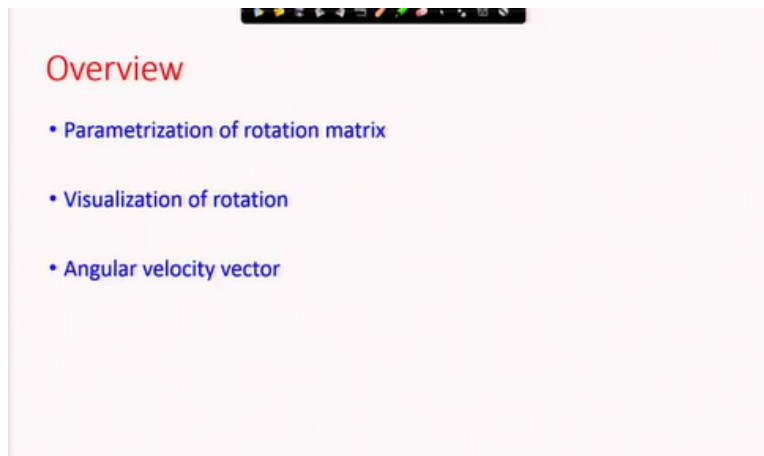


**Advanced Dynamics**  
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**Lecture - 45**  
**Kinematics of Rotation - V**

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In this lecture we are going to continue our discussions on Kinematics of rotation. We were discussing parameterization of the rotation matrix and visualization of rotation. Here we will also look at the representation of the angular velocity vector. And we are going to represent the angular velocity vector in terms of the parameters, the rotation parameter rates.

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## Parametrization of rotation

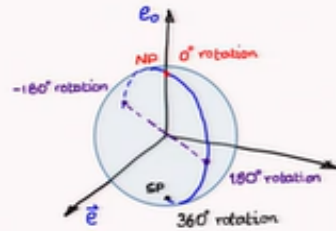
### Euler parameters (unit quaternion)

- Uses axis-angle representation:  $(\hat{n}, \theta)$

$$e_0 = \cos \frac{\theta}{2} \quad \vec{e} = [e_1 \ e_2 \ e_3] = \sin \frac{\theta}{2} [n_x \ n_y \ n_z]$$

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

- Zero rotation:  $\theta=0 \Rightarrow [e] = [1 \ 0 \ 0 \ 0]$  (NP)
- 360 deg rotation:  $[e] = [-1 \ 0 \ 0 \ 0]$  (anti-podal point of NP)
- Anti-podal points are identified
- Any point on the sphere represents a rotation (orientation)
- Changing orientation is represented by a path on the sphere



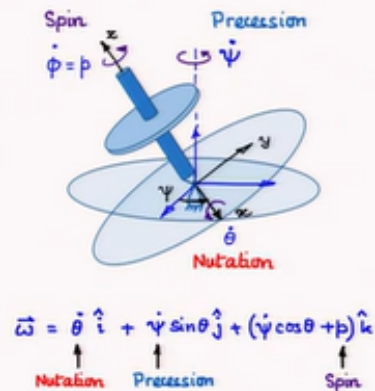
The above slide presents the Euler parameters for representing orientation of a rigid body. This is also known as the unit quaternion.

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## Angular velocity vector

### Yaw-pitch-roll parameter rates

- Angular velocity is not the time derivative of any vector quantity
- One representation for axisymmetric bodied was discussed for tops
- Representation of angular velocity in terms of time rates of yaw-pitch-roll
- Body-fixed frame



$$\vec{\omega} = \underbrace{\dot{\theta}}_{\text{Nutation}} \hat{i} + \underbrace{\dot{\psi} \sin \theta}_{\text{Precession}} \hat{j} + \underbrace{(\dot{\psi} \cos \theta + \dot{\phi})}_{\text{Spin}} \hat{k}$$

We have already looked at this yaw pitch roll parametrization. Now we are going to use these parameters to represent angular velocity. Remember that angular velocity vector is not a derivative of any other vector. Here, we will represent it in terms of derivatives of the yaw, pitch roll parameters.

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### Angular velocity vector

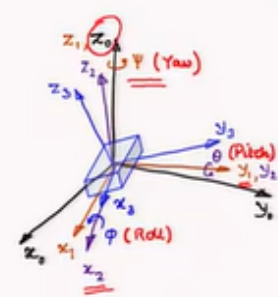
Recall  ${}^0R_3 = R_{z_0, \psi} R_{y_1, \theta} R_{x_2, \phi}$

$$R_{z_0, \psi} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{y_1, \theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$R_{x_2, \phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

$${}^0R_3 = \begin{bmatrix} c\psi c\theta & -s\psi c\theta + c\psi s\theta s\phi & s\psi s\theta + c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\theta + s\psi s\theta s\phi & -c\psi s\theta + s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$

Angular velocity (body frame):  $\vec{\omega}^3 = \dot{\psi} \hat{z}_0^3 + \dot{\theta} \hat{y}_1^3 + \dot{\phi} \hat{x}_2^3$



First recall the rotation matrix representing the orientation of a body in terms of the yaw pitch and roll coordinates as presented in the slide above.

The following 2 slides discuss the representation of the angular velocity in terms of the yaw rate, pitch rate and roll rate.

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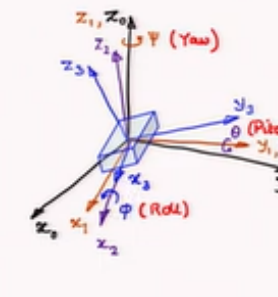
### Angular velocity vector

$$\vec{\omega}^3 = \dot{\psi} \hat{z}_0^3 + \dot{\theta} \hat{y}_1^3 + \dot{\phi} \hat{x}_2^3$$

$$R_{z_0, \psi} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\hat{y}_1^0 \quad \hat{z}_1^0 (\hat{z}_0^0)$

$${}^0R_3 = \begin{bmatrix} c\psi c\theta & -s\psi c\theta + c\psi s\theta s\phi & s\psi s\theta + c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\theta + s\psi s\theta s\phi & -c\psi s\theta + s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$

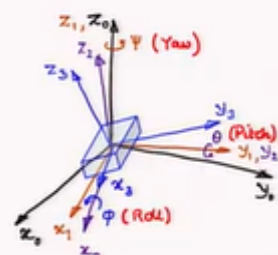
$$\hat{z}_0^3 = {}^3R_0 \hat{z}_0^0 = \begin{bmatrix} -s\theta \\ c\theta s\phi \\ c\theta c\phi \end{bmatrix} \quad \hat{y}_1^3 = {}^3R_0 \hat{y}_1^0 = \begin{bmatrix} 0 \\ c\phi \\ -s\phi \end{bmatrix} \quad \hat{x}_2^3 = \hat{x}_2^0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$


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### Angular velocity vector

$$\vec{\omega}^3 = \dot{\psi} \hat{z}_0^3 + \dot{\theta} \hat{y}_1^3 + \dot{\phi} \hat{z}_2^3$$

$$\hat{z}_0^3 = \begin{Bmatrix} -s\theta \\ c\theta s\phi \\ c\theta c\phi \end{Bmatrix} \quad \hat{y}_1^3 = \begin{Bmatrix} 0 \\ c\phi \\ -s\phi \end{Bmatrix} \quad \hat{z}_2^3 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\Rightarrow \vec{\omega}^3 = \begin{bmatrix} -s\theta & 0 & 1 \\ c\theta s\phi & c\phi & 0 \\ c\theta c\phi & -s\phi & 0 \end{bmatrix} \begin{Bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{Bmatrix}$$


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### Angular velocity vector

General approach

Recall  ${}^0R_3 = \begin{bmatrix} \hat{x}_3^0 & \hat{y}_3^0 & \hat{z}_3^0 \end{bmatrix} = {}^0R_3(\psi, \theta, \phi)$

${}^0\dot{R}_3 = \begin{bmatrix} \dot{\hat{x}}_3^0 & \dot{\hat{y}}_3^0 & \dot{\hat{z}}_3^0 \end{bmatrix} = {}^0R_3(\psi, \theta, \phi, \dot{\psi}, \dot{\theta}, \dot{\phi})$

$\frac{d\hat{x}_3^0}{dt} = \vec{\omega} \times \hat{x}_3^0 \quad \frac{d\hat{y}_3^0}{dt} = \vec{\omega} \times \hat{y}_3^0 \quad \frac{d\hat{z}_3^0}{dt} = \vec{\omega} \times \hat{z}_3^0$

$\Rightarrow {}^0\dot{R}_3 = \vec{\omega} \times \begin{bmatrix} \hat{x}_3^0 & \hat{y}_3^0 & \hat{z}_3^0 \end{bmatrix} \quad {}^0\dot{R}_3({}^0R_3)^T = [\Omega] \Rightarrow [\Omega] = {}^0\dot{R}_3 {}^0R_3^T$

$\Rightarrow \boxed{{}^0\dot{R}_3 = [\Omega] {}^0R_3}$  Rodrigues' form  $[\Omega] = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix}$

$\Rightarrow \boxed{[\Omega] = {}^0\dot{R}_3 {}^0R_3^T}$

We can have a general approach as presented in the slide above.

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## Summary

- Parametrization of rotation matrix
- Visualization of rotation
- Angular velocity vector

The discussions are summarized in the above slide.