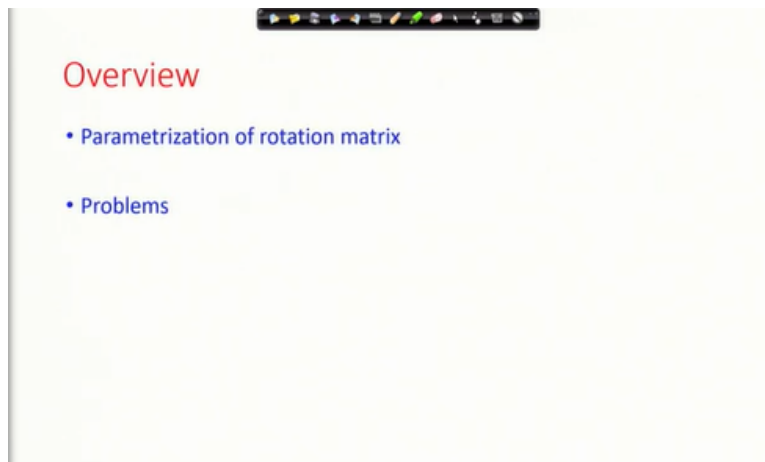


**Advanced Dynamics**  
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**Lecture - 44**  
**Kinematics of Rotation - IV**

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In this lecture we will continue our discussions on kinematics of rotation. We will start by looking at parameterization of rotation matrix. We have looked at parametrization of rotation using yaw pitch roll. We are going to look at a few more of these parameterizations.

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## Parametrization of rotation

- Euler angles: 3-1-3, 3-2-1
- Yaw-pitch-roll
- Euler axis-angle
- Euler parameters (unit quaternions)



Various approaches to parametrizing rotation is presented in the slide above.

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## Parametrization of rotation

**Euler angles: 3-1-3**

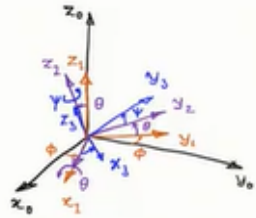

- Rotate  $\phi$  about  $z_0 \rightarrow {}^0R_1$
- Rotate  $\theta$  about  $x_1 \rightarrow {}^1R_2$
- Rotate  $\psi$  about  $z_2 \rightarrow {}^2R_3$

$${}^0R_1 = \begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi & c\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^2R_3 = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^1R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}$$

Net transformation:  ${}^0R_3 = {}^0R_1 {}^1R_2 {}^2R_3$

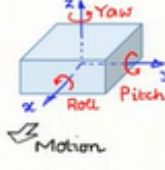
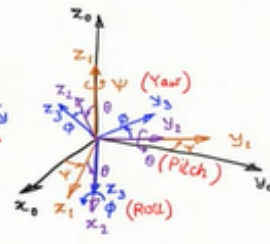
The construction of the rotation matrix using the Euler angles 3 1 3 is presented above.

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## Parametrization of rotation

### Yaw-pitch-roll

- Rotate  $\psi$  about  $z_0 \rightarrow {}^0R_1$  (Yaw)
- Rotate  $\theta$  about  $y_1 \rightarrow {}^1R_2$  (Pitch)
- Rotate  $\phi$  about  $x_2 \rightarrow {}^2R_3$  (Roll)

$${}^0R_1 = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^1R_2 = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$${}^2R_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

Net transformation:  ${}^0R_3 = {}^0R_1 {}^1R_2 {}^2R_3$

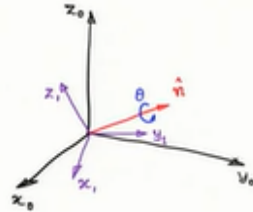
Yaw pitch roll parametrization is presented in the above slide.

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## Parametrization of rotation

### Euler axis-angle: Euler rotation theorem

- Any motion of a rigid body with a point of the body (or its extension) fixed can be expressed as a single rotation about a fixed axis.



Now we come to the Euler axis angle parameterization. For this, we first look at the Euler rotation theorem as presented above. The Euler rotation theorem says any motion of a rigid body

with a point of the body or its extension fixed can be expressed as a single rotation about a fixed axis.

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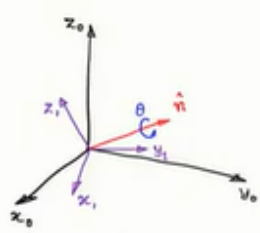
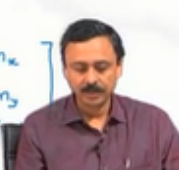
### Parametrization of rotation

**Euler axis-angle: Euler rotation theorem**

- Rotate  $\theta$  about unit vector  $\hat{n} = (n_x, n_y, n_z)$   
s.t.  $n_x^2 + n_y^2 + n_z^2 = 1$

$${}^0R_1 = I + [\hat{n}] \sin \theta + [\hat{n}]^2 (1 - \cos \theta)$$

where  $[\hat{n}] = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$  (Rodrigues' form)

$$[\hat{n}]^2 = \begin{bmatrix} -n_y^2 - n_z^2 & n_y n_x & n_z n_x \\ n_x n_y & -n_x^2 - n_z^2 & n_z n_y \\ n_x n_z & n_y n_z & -n_x^2 - n_y^2 \end{bmatrix} = \begin{bmatrix} n_x^2 - 1 & n_y n_x & n_z n_x \\ n_x n_y & n_y^2 - 1 & n_z n_y \\ n_x n_z & n_y n_z & n_z^2 - 1 \end{bmatrix}$$



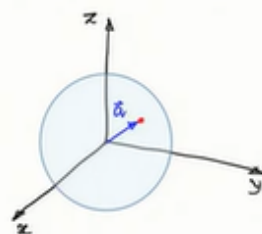

The above slide gives the representation of the rotation matrix in terms of rotation about an axis.

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### Parametrization of rotation

**Euler axis-angle: visualization**

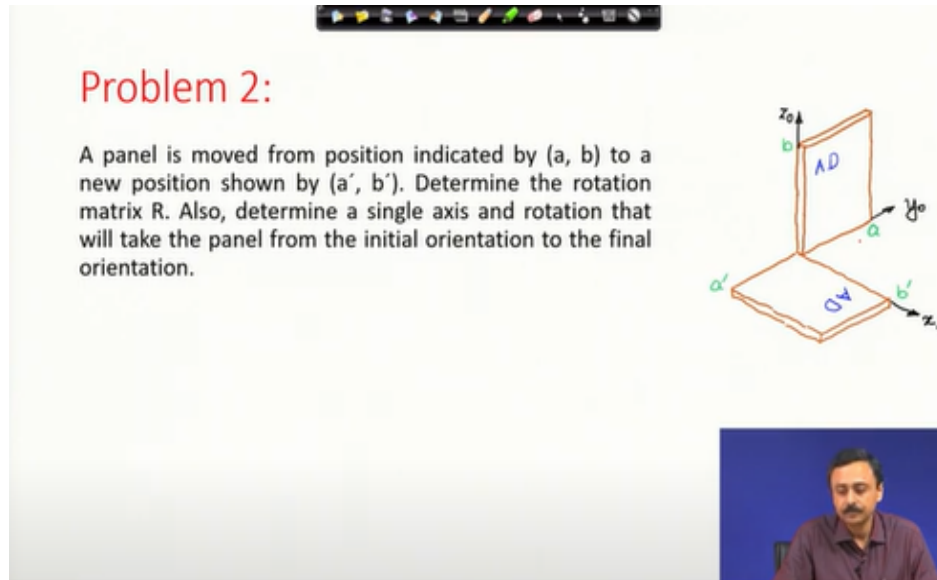
- Solid sphere of radius  $\pi$  in 3D Cartesian space
- Orientation:  $\vec{q} = \theta \hat{n}$  (rad)
- Anti-podal points are identified:  $\pi \hat{n} \sim -\pi \hat{n}$
- Any point inside or on the boundary represents a rotation (orientation of the body)
- Changing orientation is represented by a path within the sphere
- A path reaching the boundary can enter the sphere from the anti-podal point

Now let us look at the visualization of Euler axis angle as presented above. The orientation of a rigid body can be thought of as a point inside a solid sphere of radius  $\pi$  in three dimensional Cartesian space. The antipodal (diametrically opposite) points of this sphere are identified (represent the same orientation). Change in orientation are paths traversed inside the sphere.

We consider the following problem.

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**Problem 2:**

A panel is moved from position indicated by  $(a, b)$  to a new position shown by  $(a', b')$ . Determine the rotation matrix  $R$ . Also, determine a single axis and rotation that will take the panel from the initial orientation to the final orientation.

The solution is presented in the slide below.

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Euler axis-angle

$${}^0R_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = I + [\hat{n}] \sin \theta + [\hat{n}]^2 (1 - \cos \theta)$$

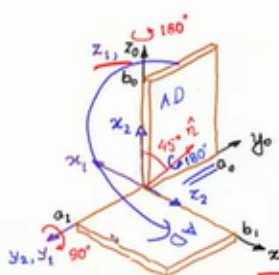

$$= \begin{bmatrix} n_x^2 \sin \theta + \cos \theta & n_x n_y \sin \theta - n_x \cos \theta & n_x n_z \sin \theta + n_y \cos \theta \\ n_x n_y \sin \theta + n_x \cos \theta & n_y^2 \sin \theta + \cos \theta & n_y n_z \sin \theta - n_x \cos \theta \\ n_x n_z \sin \theta - n_y \cos \theta & n_y n_z \sin \theta + n_x \cos \theta & n_z^2 \sin \theta + \cos \theta \end{bmatrix}$$

(where  $\sin \theta = 1 - \cos \theta$ )

Trace( ${}^0R_2$ ) =  $\sin \theta + 3 \cos \theta = -1 \Rightarrow \cos \theta = -1 \Rightarrow \theta = 180^\circ$

$$n_x^2 (2) + (-1) = 0 \Rightarrow n_x = \pm \frac{1}{\sqrt{2}} \quad \hat{n} = \begin{Bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{Bmatrix} \quad \theta = \pi \text{ rad}$$


$$n_y^2 (2) + (-1) = -1 \Rightarrow n_y = 0$$

$$n_z = \pm \frac{1}{\sqrt{2}}$$



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Summary

- Parametrization of rotation matrix
- Problems



The above slide summarizes the discussions.