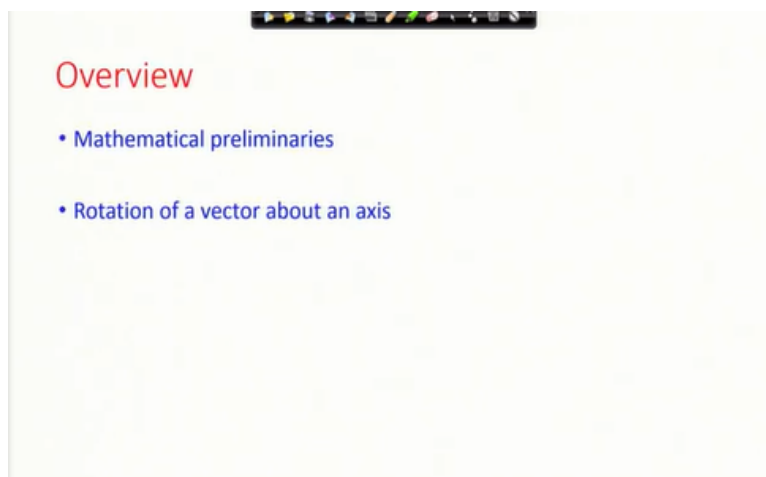


Advanced Dynamics
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Lecture - 43
Kinematics of Rotation - III

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We are going to continue our discussions on the rotation kinematics of rotation. And in this lecture, I am going to talk about few mathematical preliminaries that will be required to understand our further developments. And I am going to discuss the situation of rotation of a vector about an axis.

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Mathematical preliminaries
Dot and cross products as matrix products

$$\hat{n} \cdot \vec{a} = \begin{bmatrix} n_x & n_y & n_z \end{bmatrix} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = \hat{n}^T \vec{a} = \vec{a}^T \hat{n}$$
$$\hat{n} \times \vec{a} = \underbrace{\begin{bmatrix} 0 & -n_z & n_y \\ n_x & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}}_{\text{(Rodrigues' form)}} \begin{Bmatrix} a_x \\ a_y \\ a_z \end{Bmatrix} = [\hat{n}] \vec{a}$$

First, I will develop the dot and cross product the standard vector products as matrix products. This is shown in the slide above.

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Mathematical preliminaries

Orthogonal decomposition of a vector

- Given a vector and a direction (\vec{a}, \hat{n})

$$\left. \begin{aligned} \vec{a}_{\parallel} &= (\vec{a} \cdot \hat{n}) \hat{n} \\ \vec{a}_{\perp} &= \vec{a} - \vec{a}_{\parallel} \end{aligned} \right\} \Rightarrow \vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$$

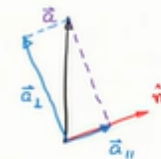
$$\vec{a}_{\perp} = \vec{a} - (\vec{a} \cdot \hat{n}) \hat{n} = -[\hat{n}]^2 \vec{a}$$

From vector algebra

$$\hat{n} \times (\hat{n} \times \vec{a}) = (\hat{n} \cdot \vec{a}) \hat{n} - (\hat{n} \cdot \hat{n}) \vec{a} = (\vec{a} \cdot \hat{n}) \hat{n} - \vec{a}$$

$$\Rightarrow [\hat{n}]^2 \vec{a} = (\vec{a} \cdot \hat{n}) \hat{n} - \vec{a}$$

$$\Rightarrow (\vec{a} \cdot \hat{n}) \hat{n} = (\mathbf{I} + [\hat{n}]^2) \vec{a}$$



$$\left. \begin{aligned} \vec{a}_{\parallel} &= (\mathbf{I} + [\hat{n}]^2) \vec{a} \\ \vec{a}_{\perp} &= -[\hat{n}]^2 \vec{a} \end{aligned} \right\}$$

Now we are going to look at orthogonal decomposition of a vector. This is presented in the above slide.

Mathematical preliminaries

Cayley-Hamilton Theorem

Every square matrix satisfies its characteristic equation

$$[\hat{n}] = \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix}$$

Characteristic equation : $\det(\lambda \mathbf{I} - [\hat{n}]) = 0$

$$\Rightarrow \lambda^3 + \lambda = 0$$

C-H Theorem \Rightarrow $[\hat{n}]^3 + [\hat{n}] = [0]$

The above slide presents the Cayley-Hamilton theorem.

Next, we determine the rotation matrix involved when we give a rotation about an arbitrary direction. This is presented in the next two slides.

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Rotation of a vector about an axis

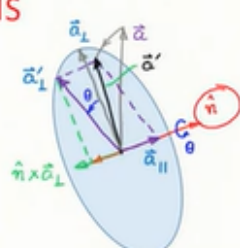

- Given a vector and rotation: $[\vec{a}, (\hat{n}, \theta)]$

$$\vec{a} = \vec{a}_{||} + \vec{a}_{\perp} \quad \begin{cases} \vec{a}_{||} = (1 + [\hat{n}]^0) \vec{a} \\ \vec{a}_{\perp} = -[\hat{n}]^2 \vec{a} \end{cases}$$

Upon rotation $|\vec{a}'_{\perp}| = |\vec{a}_{\perp}|$

$$\Rightarrow \vec{a}'_{\perp} = |\vec{a}_{\perp}| \left(\cos \theta \frac{\vec{a}_{\perp}}{|\vec{a}_{\perp}|} + \sin \theta \frac{\hat{n} \times \vec{a}_{\perp}}{|\hat{n} \times \vec{a}_{\perp}|} \right)$$

$|\hat{n} \times \vec{a}_{\perp}| = |\vec{a}_{\perp}| \sin \theta$

$$\Rightarrow \vec{a}'_{\perp} = \cos \theta \vec{a}_{\perp} + \sin \theta \hat{n} \times \vec{a}_{\perp} \quad (|\hat{n} \times \vec{a}_{\perp}| = |\vec{a}_{\perp}| \sin \theta)$$



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Rotation of a vector about an axis

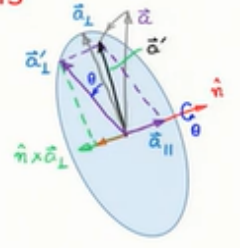
- Given a vector and rotation: $[\vec{a}, (\hat{n}, \theta)]$

$$\vec{a} = \vec{a}_{||} + \vec{a}_{\perp} \quad \begin{cases} \vec{a}_{||} = (1 + [\hat{n}]^0) \vec{a} \\ \vec{a}_{\perp} = -[\hat{n}]^2 \vec{a} \end{cases}$$

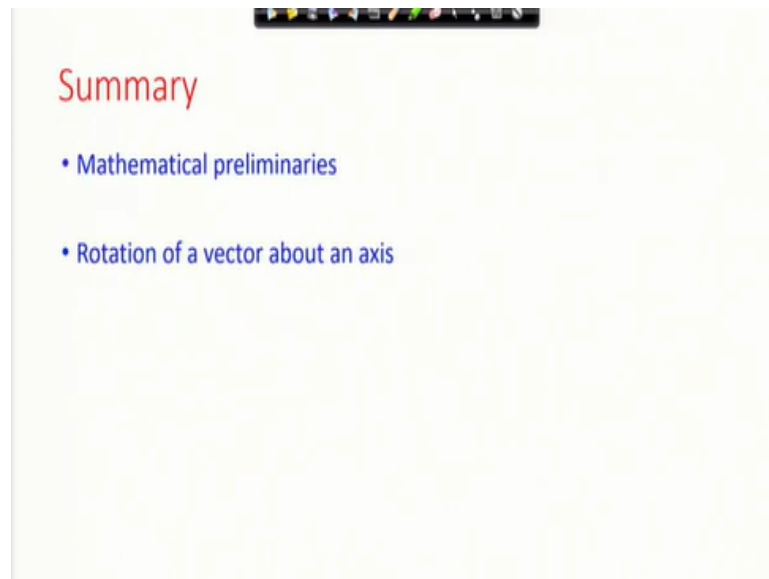
$$\vec{a}'_{\perp} = \cos \theta \vec{a}_{\perp} + \sin \theta \hat{n} \times \vec{a}_{\perp}$$

$$\vec{a}' = \vec{a}_{||} + \vec{a}'_{\perp} = (1 + [\hat{n}]^0) \vec{a} - \cos \theta [\hat{n}]^2 \vec{a} + \sin \theta \underbrace{(-[\hat{n}]^3 \vec{a})}_{= [\hat{n}] \vec{a} \text{ (C-H Theorem)}}$$

$$\Rightarrow \boxed{\vec{a}' = [1 + \sin \theta [\hat{n}] + (1 - \cos \theta) [\hat{n}]^2] \vec{a}}$$

$$\Rightarrow \vec{a}' = [R_{\hat{n}, \theta}] \vec{a}$$


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The above slide presents the summary.