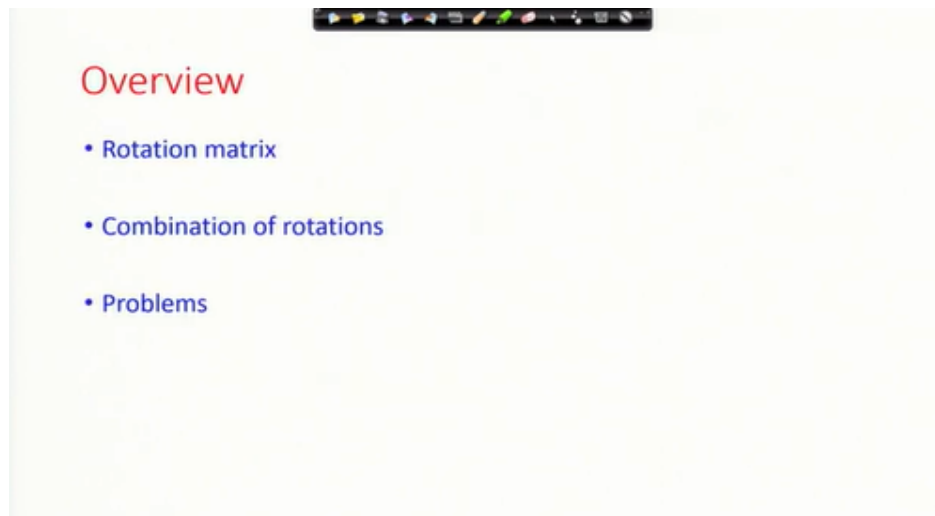


Advanced Dynamics
Prof. Anirvan Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 42
Kinematics of Rotation - II

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We are going to continue our discussions on Kinematics of rotation. In this lecture I am going to talk about the rotation matrix once again and will recapitulate what we have discussed. Then I will look at the combination of rotations, the previous lecture we had looked at single rotations and represented the rotation matrix. Here we are going to look at the combination of rotations and look at problems.

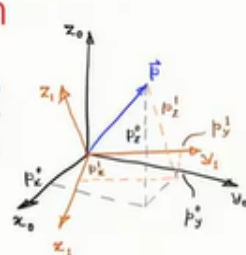
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The rotation matrix: interpretation

$${}^0R_1 = \begin{bmatrix} \hat{i}' \cdot \hat{i} & \hat{j}' \cdot \hat{i} & \hat{k}' \cdot \hat{i} \\ \hat{i}' \cdot \hat{j} & \hat{j}' \cdot \hat{j} & \hat{k}' \cdot \hat{j} \\ \hat{i}' \cdot \hat{k} & \hat{j}' \cdot \hat{k} & \hat{k}' \cdot \hat{k} \end{bmatrix} = [\hat{z}_1^0 \ \hat{y}_1^0 \ \hat{x}_1^0] = \begin{Bmatrix} (\hat{z}_1^0)^T \\ (\hat{y}_1^0)^T \\ (\hat{x}_1^0)^T \end{Bmatrix}$$

$$\hat{z}_1^0 = {}^0R_1 \hat{z}_1^1 \quad \hat{y}_1^0 = {}^0R_1 \hat{y}_1^1 \quad \dots$$

$${}^1R_0 = \begin{bmatrix} \hat{i} \cdot \hat{i}' & \hat{j} \cdot \hat{i}' & \hat{k} \cdot \hat{i}' \\ \hat{i} \cdot \hat{j}' & \hat{j} \cdot \hat{j}' & \hat{k} \cdot \hat{j}' \\ \hat{i} \cdot \hat{k}' & \hat{j} \cdot \hat{k}' & \hat{k} \cdot \hat{k}' \end{bmatrix} = [\hat{z}_0^1 \ \hat{y}_0^1 \ \hat{x}_0^1] = \begin{Bmatrix} (\hat{z}_0^1)^T \\ (\hat{y}_0^1)^T \\ (\hat{x}_0^1)^T \end{Bmatrix}$$

$${}^0R_1 {}^0R_1^T = {}^0R_1 {}^1R_0 = {}^1R_0 {}^0R_1 = I \quad \left[(\hat{z}_0^1)^T \hat{z}_0^1 = 1, (\hat{y}_0^1)^T \hat{z}_0^1 = 0 \right]$$


The above slide recapitulates the discussions in the previous lecture.

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Combination of rotation: case I

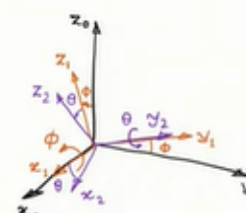
Successive rotations about new frame

$${}^0R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \quad {}^1R_2 = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

Net transformation: ${}^0R_2 = {}^0R_1 {}^1R_2 = \begin{bmatrix} c\theta & 0 & s\theta \\ s\theta s\phi & c\phi & -c\theta s\phi \\ -s\theta c\phi & s\phi & c\theta c\phi \end{bmatrix}$

Check: $\det({}^0R_2) = 1, \quad {}^0R_2 {}^2R_0 = {}^0R_2 ({}^0R_2)^T = I$

Handwritten calculation for determinant: $c^2\theta + (s\theta s\phi)^2 + (-s\theta c\phi)^2 = 1$



Now let us look at successive rotations. This is presented in the slide above for successive rotation about the new frame axis.

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Combination of rotation: case I

Yaw-pitch-roll

$${}^0R_3 = R_{z_0, \psi} R_{y_1, \theta} R_{x_2, \phi}$$

$$R_{z_0, \psi} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{y_1, \theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$R_{x_2, \phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

$${}^0R_3 = \begin{bmatrix} c\psi c\theta & -s\psi c\theta + c\psi s\theta s\phi & s\psi s\theta + c\psi s\theta c\phi & \hat{z}_0^3 \\ s\psi c\theta & c\psi c\theta + s\psi s\theta s\phi & -c\psi s\theta + s\psi s\theta c\phi & \hat{y}_0^3 \\ -s\theta & c\theta s\phi & c\theta c\phi & \hat{z}_0^2 \end{bmatrix}$$

We look at an example of a yaw pitch roll, combination of rotation as shown in the slide above. This is a very useful combination of rotation for studying rigid body dynamics.

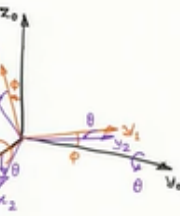
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Combination of rotation: case II

Successive rotations about old frame

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix} \quad R_2 = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

Net transformation: ${}^0R_2 = R_2 R_1 = \begin{bmatrix} c\theta & s\theta s\phi & s\theta c\phi \\ 0 & c\phi & -s\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$



(premultiplication of successive rotation matrices)

Check: $\det({}^0R_1) = 1$, ${}^0R_2 {}^2R_0 = {}^0R_2 ({}^0R_2)^T = I$

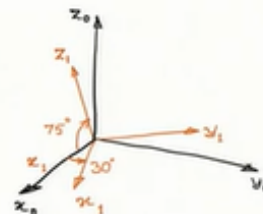
Now we look at a second case of combination of rotation in which successive rotations are given about the old frame axes. This is presented in the above slide.

We consider the following problem.

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Problem 1:

In the figure shown, the axis x_1 lies in the x_0 - y_0 plane and makes an angle 30 deg with x_0 , while z_1 makes 75 deg angle with x_0 . Determine the rotation matrix 0R_1 .



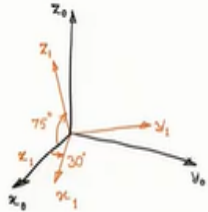
The detailed solution is provided below.

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From figure

$${}^0R_1 = \begin{bmatrix} c30^\circ & R_{12} & c75^\circ \\ s30^\circ & R_{22} & R_{23} \\ 0 & R_{32} & R_{33} \end{bmatrix} = [\hat{z}_1 \quad \hat{y}_1 \quad \hat{z}_1]$$

- $\hat{z}_1 \cdot \hat{z}_1 = 0 \Rightarrow c30^\circ c75^\circ + s30^\circ R_{23} = 0$
 $\Rightarrow R_{23} = -\frac{c30^\circ c75^\circ}{c30^\circ}$
- $|\hat{z}_1| = 1 \Rightarrow R_{33}^2 = 1 - c^2 75^\circ - R_{23}^2$
 $\Rightarrow R_{23} = +\sqrt{1 - c^2 75^\circ - \frac{c^2 30^\circ c^2 75^\circ}{c^2 30^\circ}} \quad (\text{acute angle between } \hat{z}_0 \text{ and } \hat{z}_1)$
- $\hat{y}_1 = \hat{z}_1 \times \hat{x}_1$

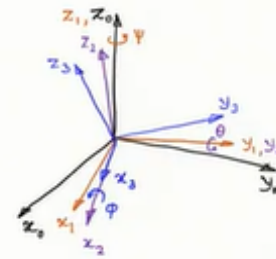


We consider the next problem below.

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Problem 2:

Given a 3x3 rotation matrix R , determine the yaw-pitch-roll angles required to achieve the orientation.



The solution is presented in the two slides below.

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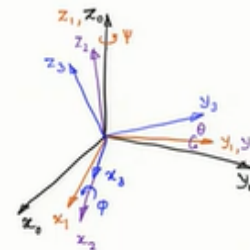
Rotation matrix with yaw-pitch-roll

$${}^0R_3 = R_{z_0, \psi} R_{y_1, \theta} R_{x_1, \phi}$$

$$R_{z_0, \psi} = \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad R_{y_1, \theta} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}$$

$$R_{x_1, \phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\phi & -s\phi \\ 0 & s\phi & c\phi \end{bmatrix}$$

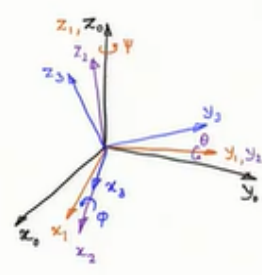
$$\Rightarrow {}^0R_3 = \begin{bmatrix} c\psi c\theta & -s\psi c\theta + c\psi s\theta s\phi & s\psi s\theta + c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\theta + s\psi s\theta s\phi & -c\psi s\theta + s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$



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Rotation matrix with yaw-pitch-roll

$${}^0R_3 = \begin{bmatrix} c\psi c\theta & -s\psi c\theta + c\psi s\theta s\phi & s\psi c\theta + c\psi s\theta c\phi \\ s\psi c\theta & c\psi c\theta + s\psi s\theta s\phi & -c\psi c\theta + s\psi s\theta c\phi \\ -s\theta & c\theta s\phi & c\theta c\phi \end{bmatrix}$$

$$= R = \begin{bmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{bmatrix}$$


$$\frac{R_{32}}{R_{33}} = \tan\phi \quad (R_{32} \neq 0, R_{33} \neq 0 \Rightarrow \theta \neq \frac{\pi}{2}, \frac{3\pi}{2})$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{R_{32}}{R_{33}}\right) = \text{atan2}(R_{32}, R_{33}) \quad (\text{correct quadrant})$$

$$\psi = \tan^{-1}\left(\frac{R_{21}}{R_{11}}\right) \quad (\theta \neq \frac{\pi}{2}, \frac{3\pi}{2})$$

$$\theta = \tan^{-1}\left(\frac{-R_{31}}{R_{32}/s\phi}\right)$$

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Summary

- Rotation matrix
- Combination of rotations
- Problems

Summary of the lecture is provided above.