Advanced Dynamics
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Module No # 01
Lecture No # 04
Relative Motion – II

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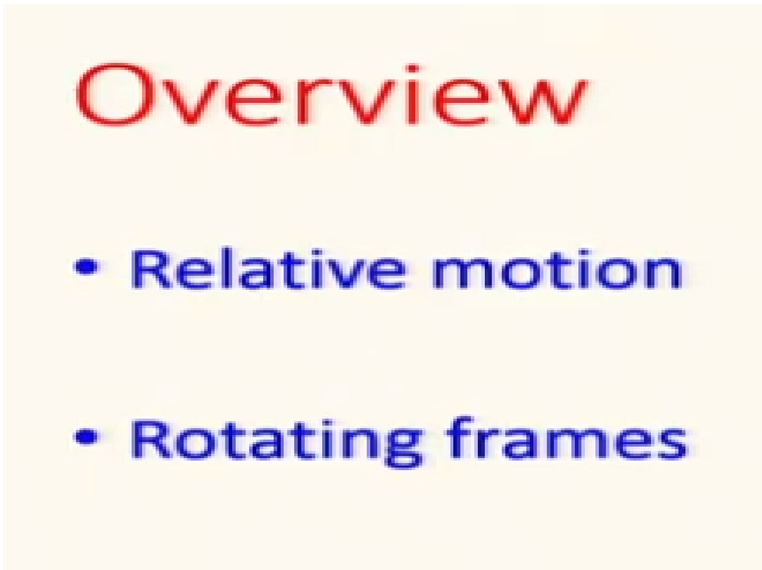

In this lecture we are going to continue our discussions on kinematics on the topic relative motion. And the overview of today's lecture is relative motion which we will be discussing for rotating frames. In the last lecture you saw relative motion in translating frames, here we are going to discuss relative motion in rotating frames.

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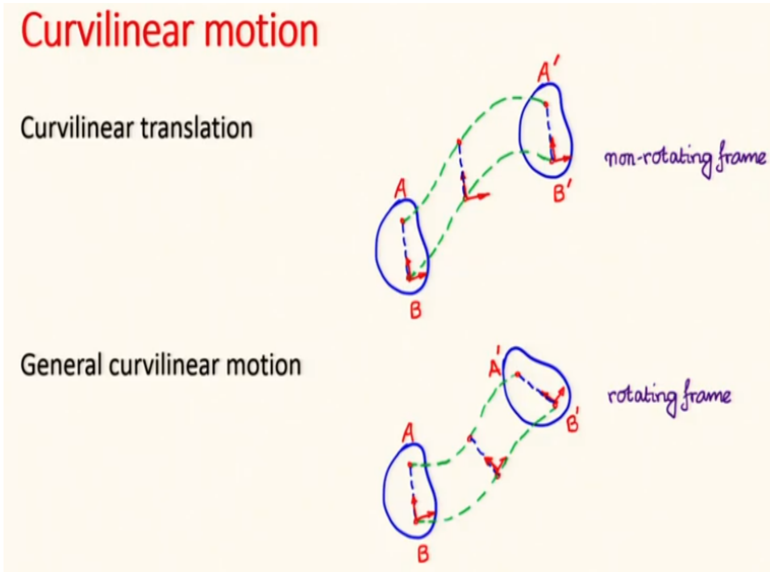

The above figure distinguishes curvilinear translation from general curvilinear motion. In curvilinear translation, the coordinate system does not change its orientation (non-rotating) as the frame moves on a general path. Whereas, for a general curvilinear motion, you can have rotation as well as translation of the frame.

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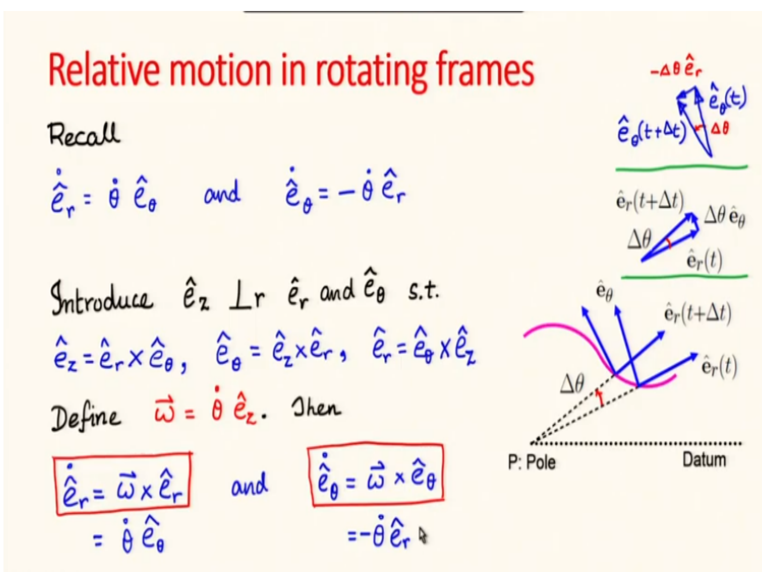

Let us recall the kinematics in plane polar coordinates as shown above. We defined the time derivatives of the frame vectors as

$$\dot{\hat{e}}_r = \dot{\theta}\,\hat{e}_\theta \quad \text{and} \quad \dot{\hat{e}}_\theta = -\dot{\theta}\,\hat{e}_r$$

This was derived in the previous lecture. Here, we are going to see an alternative way of representing this. We introduce a new frame unit vector

$$\hat{e}_z = \hat{e}_r \times \hat{e}_\theta$$

This vector comes out of the plane of the paper. Now define an omega vector angular velocity vector as

$$\vec{\omega} = \dot{\theta}\,\hat{e}_z$$

Then, it can be checked that

$$\boxed{\dot{\hat{e}}_r = \vec{\omega} \times \hat{e}_r} = \dot{\theta}\,\hat{e}_\theta \quad \text{and} \quad \boxed{\dot{\hat{e}}_\theta = \vec{\omega} \times \hat{e}_\theta} = -\dot{\theta}\,\hat{e}_r$$

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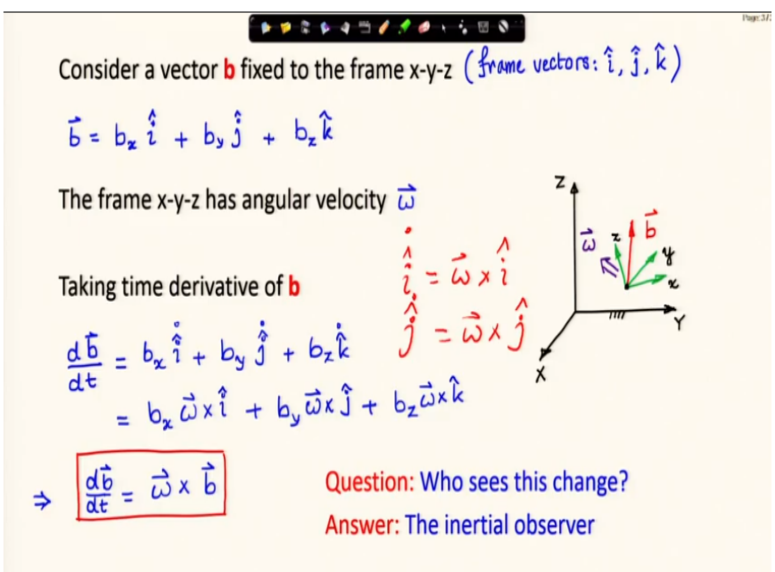

Now using this idea let us now understand the kinematics in the rotating frame. First consider a vector b shown as a red vector above, which is fixed to a frame xyz which is rotating. There is another frame XYZ which is fixed with distant stars.

The representation of the b vector is

$$\vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}$$

Let the frame xyz have angular velocity ω. If we take the time derivative of this vector b we have

$$\frac{d\vec{b}}{dt} = b_x \dot{\hat{i}} + b_y \dot{\hat{j}} + b_z \dot{\hat{k}}$$

$$= b_x \vec{\omega} \times \hat{i} + b_y \vec{\omega} \times \hat{j} + b_z \vec{\omega} \times \hat{k}$$

$$\Rightarrow \boxed{\frac{d\vec{b}}{dt} = \vec{\omega} \times \vec{b}}$$

Now the question is who sees this change who can perceive this change of vector b. Let me give you an analogy. Suppose you are sitting in a big auditorium and you are watching another person sitting in the same auditorium and on another seat. If you are asked, what is the rate of the change of the vector from you to that person, you will obviously say 0. This is because the person is not moving. That person is seated, and you are also seated in the same auditorium. But then remember our earth is rotating therefore this vector will change its direction. This change you cannot perceive being in that frame. But who can perceive who sees the change: the answer is the inertial observer can see this change. And why we are interested in what the inertial observer sees? It is because Newtonian mechanics requires the kinematics to be as observed by an inertial observer. Therefore we should be very careful about finding the rate of change of vectors as it should be as perceived or as seen or as measured by an inertial observer.

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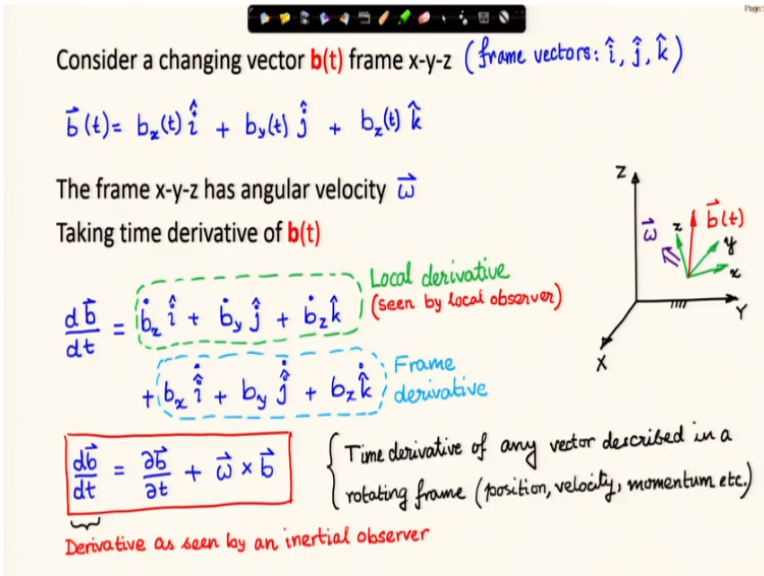

Next we let us consider of vector b which is a function of time. That means b is changing in the rotating green frame xyz. Now the components of b is xyz are also functions of time. Therefore if I differentiate with respect to time then I will have 2 kinds of terms: the local derivative due to the local change of vector b in the xyz frame, and the terms involving time derivative of the frame vectors. Hence, we have

$$\frac{d\vec{b}}{dt} = \underbrace{\dot{b}_x\hat{i} + \dot{b}_y\hat{j} + \dot{b}_z\hat{k}}_{\text{Local derivative (seen by local observer)}} + \underbrace{b_x\dot{\hat{i}} + b_y\dot{\hat{j}} + b_z\dot{\hat{k}}}_{\text{Frame derivative}}$$

In a compact form

$$\boxed{\underbrace{\frac{d\vec{b}}{dt}}_{\text{Derivative as seen by an inertial observer}} = \frac{\partial \vec{b}}{\partial t} + \vec{\omega}\times\vec{b}}$$

This is the convention we will follow now about the local time derivative, and derivative as seen by an inertial observer this time derivative of any vector now will follow this rule. For example, the vector could be position vector in a rotating frame velocity vector in a rotating frame linear or angular momentum in a rotating frame.

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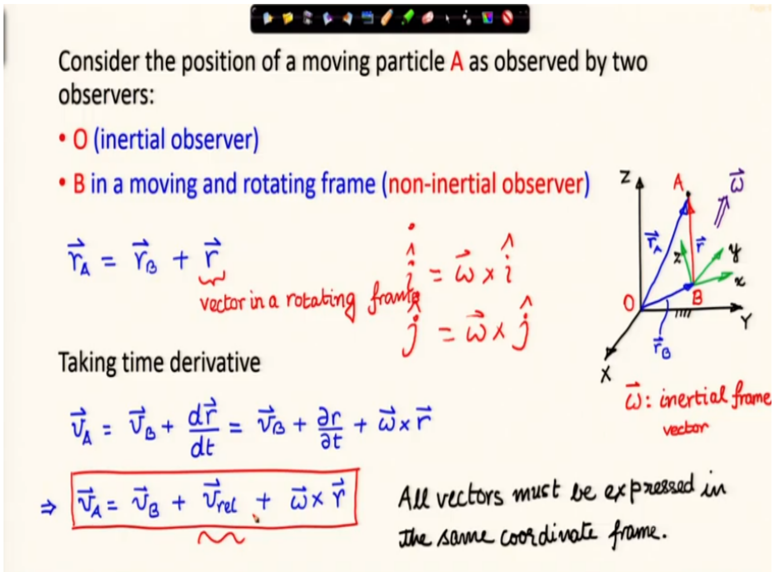

Now consider the position of a moving particle A as observed by 2 observers: one is B in a rotating frame, and O in the inertial frame.

We can write

$$\underbrace{\vec{r}_A = \vec{r}_B + \vec{r}}_{}$$

vector in a rotating frame

Differentiating with respect to time and following the prescription of derivative, we have

$$\vec{v}_A = \vec{v}_B + \frac{d\vec{r}}{dt} = \vec{v}_B + \frac{\partial r}{\partial t} + \vec{\omega} \times \vec{r}$$

$$\Rightarrow \boxed{\vec{v}_A = \vec{v}_B + \vec{v}_{rel} + \vec{\omega} \times \vec{r}}$$

This is a vector relation and should be true in any frame. However, all vectors must be represented in one frame.

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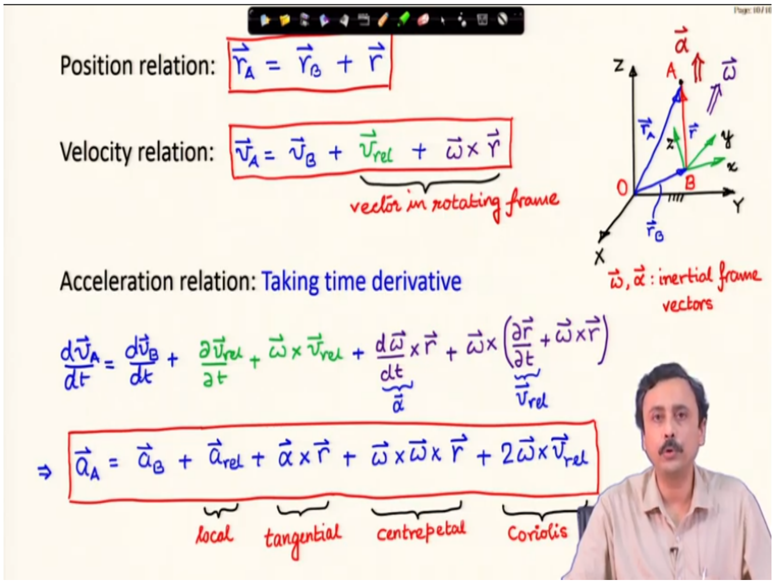

Now if I want to go for acceleration I take time derivative of the velocity vector and follow the same prescription to obtain

$$\frac{d\vec{v}_A}{dt} = \frac{d\vec{v}_B}{dt} + \frac{\partial \vec{v}_{rel}}{\partial t} + \vec{\omega} \times \vec{v}_{rel} + \underbrace{\frac{d\vec{\omega}}{dt}}_{\vec{\alpha}} \times \vec{r} + \vec{\omega} \times \Bigg(\underbrace{\frac{\partial \vec{r}}{\partial t}}_{\vec{v}_{rel}} + \vec{\omega} \times \vec{r}\Bigg)$$

$$\Rightarrow \boxed{\vec{a}_A = \vec{a}_B + \underbrace{\vec{a}_{rel}}_{\text{local}} + \underbrace{\vec{\alpha} \times \vec{r}}_{\text{tangential}} + \underbrace{\vec{\omega} \times \vec{\omega} \times \vec{r}}_{\text{centrepetal}} + \underbrace{2\vec{\omega} \times \vec{v}_{rel}}_{\text{Coriolis}}}$$

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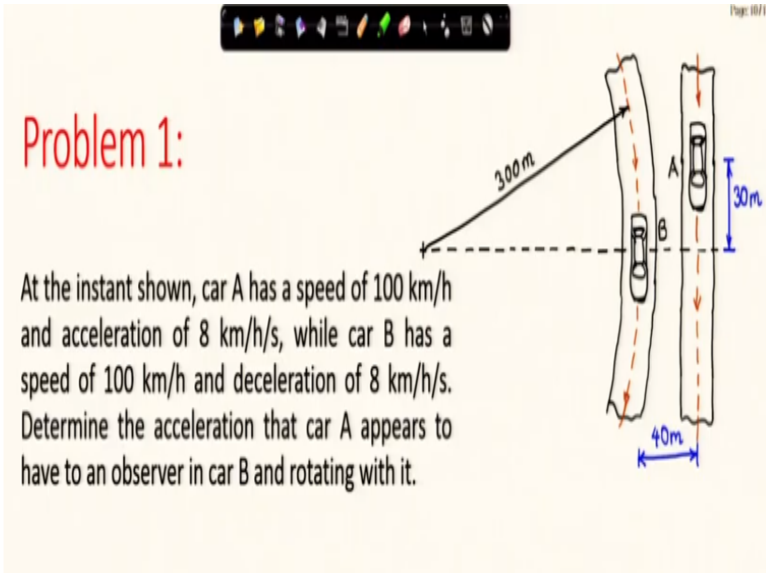

Let us look at at the above example. The solution is given as follows where we take the x-y frame attached to the car B so as to capture the observation of the observer in car B. All vectors are represented in this frame x-y.

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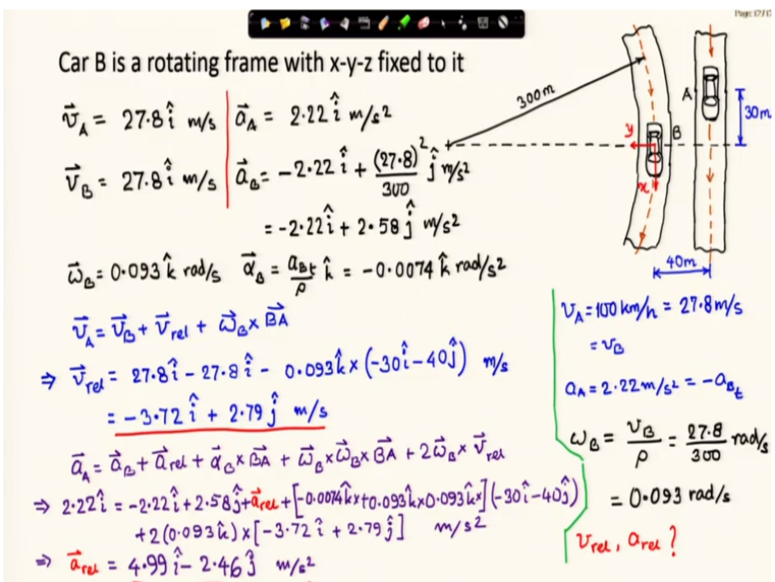

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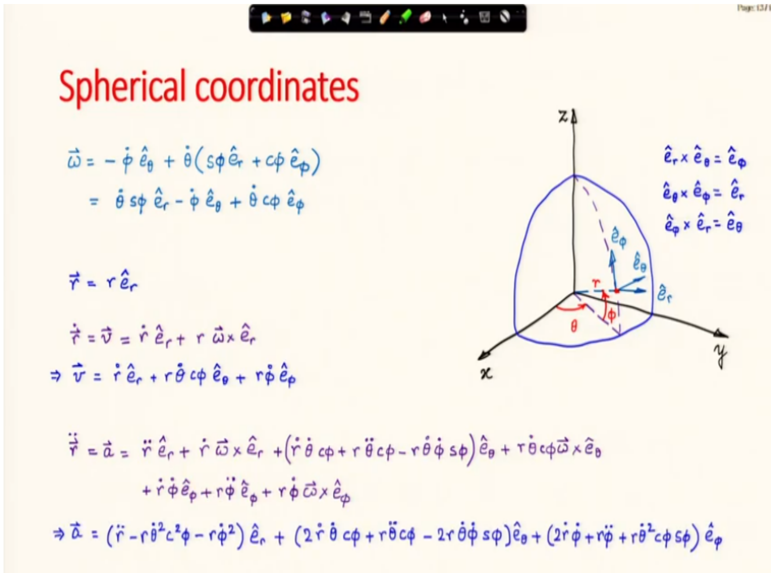

Now, based on what we have discussed, we discuss the representation of kinematics in a spherical coordinate system. Here I have shown one eight of its sphere. In the case of spherical coordinates we consider motion on an imaginary sphere

Let us join the center of the sphere to the particle by this blue dashed line and define that as the radial coordinate. The angular coordinate θ is measured from the x axis to the radial line on the equator in the equatorial plane. φ is the angle measured from the equatorial plane line to the radial line. So, we have 3 coordinates: r, θ and φ. Now we define our frame vectors as shown in the figure above such that

$$\hat{e}_r \times \hat{e}_\theta = \hat{e}_\phi$$
$$\hat{e}_\theta \times \hat{e}_\phi = \hat{e}_r$$
$$\hat{e}_\phi \times \hat{e}_r = \hat{e}_\theta$$

First we represent the angular velocity vector of the frame as the particle moves. This is given by

$$\vec{\omega} = -\dot{\phi}\,\hat{e}_\theta + \dot{\theta}\,(s\phi\,\hat{e}_r + c\phi\,\hat{e}_\phi)$$
$$= \dot{\theta}\,s\phi\,\hat{e}_r - \dot{\phi}\,\hat{e}_\theta + \dot{\theta}\,c\phi\,\hat{e}_\phi$$

The position vector is given by

$$\vec{r} = r\hat{e}_r$$

Time differentiating the position vector (following our prescription of derivative) gives the velocity vector

$$\dot{\vec{r}} = \vec{v} = \dot{r}\hat{e}_r + r\,\vec{\omega}\times\hat{e}_r$$

$$\Rightarrow \vec{v} = \dot{r}\hat{e}_r + r\dot{\theta}\,c\phi\,\hat{e}_\theta + r\dot{\phi}\,\hat{e}_\phi$$

Differentiating once again with respect to time gives the acceleration vector

$$\ddot{\vec{r}} = \vec{a} = \ddot{r}\hat{e}_r + \dot{r}\,\vec{\omega}\times\hat{e}_r + (\dot{r}\dot{\theta}\,c\phi + r\ddot{\theta}\,c\phi - r\dot{\theta}\dot{\phi}\,s\phi)\hat{e}_\theta + r\dot{\theta}\,c\phi\,\vec{\omega}\times\hat{e}_\theta$$
$$+\dot{r}\dot{\phi}\hat{e}_\phi + r\ddot{\phi}\hat{e}_\phi + r\dot{\phi}\,\vec{\omega}\times\hat{e}_\phi$$

$$\Rightarrow \vec{a} = (\ddot{r} - r\dot{\theta}^2c^2\phi - r\dot{\phi}^2)\hat{e}_r + (2\dot{r}\dot{\theta}\,c\phi + r\ddot{\theta}\,c\phi - 2r\dot{\theta}\dot{\phi}\,s\phi)\hat{e}_\theta + (2\dot{r}\dot{\phi} + r\ddot{\phi} + r\dot{\theta}^2c\phi\,s\phi)\hat{e}_\phi$$

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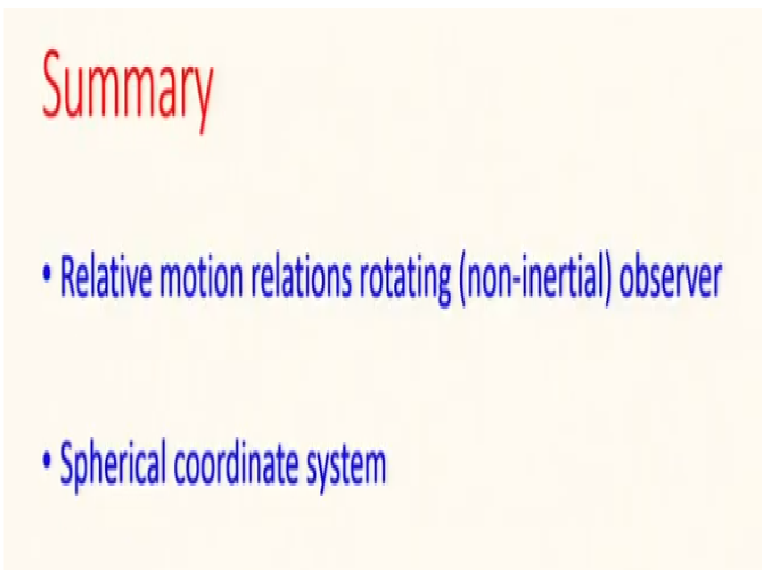

To summarize, we have looked at kinematics in rotating frames which represent non-inertial observers, and we have seen how to find out time derivative of vectors as seen by an inertial observer. This is the most important step in this lecture. We have also looked at the kinematics in spherical coordinate system.