

Advanced Dynamics
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Lecture - 38
Gyroscopic Motion - I

With this lecture I am going to start discussions on gyroscopic motions. Gyroscopic motions are observed in axisymmetric bodies rotating about its axis of symmetry about a fixed point in inertial space. We will first calculate the gyroscopic forces in spinning axisymmetric bodies and look at precessional motion of these bodies with and without external moment.

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Gyroscopic motion

Spinning axisymmetric body

Frame angular velocity

$$\vec{\Omega}_f = \dot{\theta} \hat{i} + \dot{\psi} (\sin\theta \hat{j} + \cos\theta \hat{k})$$

Spin velocity $\vec{p} = p \hat{k}$

Net angular velocity

$$\vec{\omega} = \vec{\Omega}_f + \vec{p} = \dot{\theta} \hat{i} + \dot{\psi} \sin\theta \hat{j} + (\dot{\psi} \cos\theta + p) \hat{k}$$

Nutation Precession Spin

The diagram shows a spinning top with a vertical z-axis and a horizontal x-y plane. The top's axis of symmetry is the z-axis. The angular velocities are represented by vectors: $\dot{\theta} \hat{i}$ (Nutation), $\dot{\psi} \sin\theta \hat{j}$ (Precession), and $(\dot{\psi} \cos\theta + p) \hat{k}$ (Spin). The total angular velocity vector $\vec{\omega}$ is the sum of these three components.

This is our prototype problem that we have discussed in the past lectures. We have discussed the setting up of the x-y-z frame which is not a body-fixed frame but ensures that the moment of inertia tensor is time invariant. The expression of the net angular velocity of the body represented in the x-y-z frame is presented in the above slide.

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Spinning axisymmetric body

Angular momentum

$$\vec{H}_O = I_O \dot{\theta} \hat{i} + I_O \dot{\psi} \sin \theta \hat{j} + I (\dot{\psi} \cos \theta + p) \hat{k}$$

Rotational dynamics

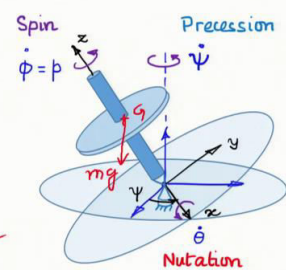
$$\dot{\vec{H}}_O = \vec{M}_O \Rightarrow \frac{\partial \vec{H}_O}{\partial t} + \vec{\Omega}_f \times \vec{H}_O = \vec{M}_O$$

$$\vec{\Omega}_f = \dot{\theta} \hat{i} + \dot{\psi} (\sin \theta \hat{j} + \cos \theta \hat{k})$$

$$p = \text{constant} \quad \vec{M}_O = M_x \hat{i}$$

$$\frac{\partial \vec{H}_O}{\partial t} = I_O \ddot{\theta} \hat{i} + I_O \ddot{\psi} \sin \theta \hat{j} + I_O \dot{\psi} \dot{\theta} \cos \theta \hat{j} + I (\ddot{\psi} \cos \theta - \dot{\psi} \dot{\theta} \sin \theta) \hat{k}$$

$$\Rightarrow I_O \ddot{\theta} \hat{i} + (I_O \ddot{\psi} \sin \theta + I_O \dot{\psi} \dot{\theta} \cos \theta) \hat{j} + I (\ddot{\psi} \cos \theta - \dot{\psi} \dot{\theta} \sin \theta) \hat{k}$$

$$+ [(I_O - I) \dot{\psi} \dot{\theta} \cos \theta - I p \dot{\theta}] \hat{j} + [I \dot{\psi} (\dot{\psi} \cos \theta + p) \sin \theta - I_O \dot{\psi}^2 \sin \theta \cos \theta] \hat{i} = \vec{M}_O$$


The diagram illustrates the three types of rotational motion for a spinning axisymmetric body. It shows a blue rod with a disk at the end, pivoted at point O. The body's angular velocity is decomposed into three components: Spin (rotation about the body's symmetry axis, labeled $\dot{\phi} = p$), Precession (rotation about the vertical Z-axis, labeled $\dot{\psi}$), and Nutation (rotation about the horizontal Y-axis, labeled $\dot{\theta}$). The center of mass is at a distance g from the pivot, and the weight mg acts downwards. The coordinate system (x, y, z) is shown with the z-axis along the symmetry axis.

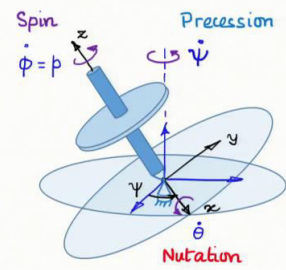
The calculation of the angular momentum about the fixed point O is shown in the above slide. Finally, using the rotational dynamics equation about O, we obtain the equation of motion.

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Spinning axisymmetric body

$$I_O \ddot{\theta} + I \dot{\psi} (\dot{\psi} \cos \theta + p) \sin \theta - I_O \dot{\psi}^2 \sin \theta \cos \theta = M_x$$

$$I_O \ddot{\psi} \sin \theta + 2 I_O \dot{\psi} \dot{\theta} \cos \theta - I \dot{\theta} (\dot{\psi} \cos \theta + p) = 0$$

$$I (\ddot{\psi} \cos \theta - \dot{\psi} \dot{\theta} \sin \theta) = 0$$


The diagram is identical to the one in the previous slide, showing the decomposition of angular velocity into Spin, Precession, and Nutation for a spinning axisymmetric body pivoted at point O.

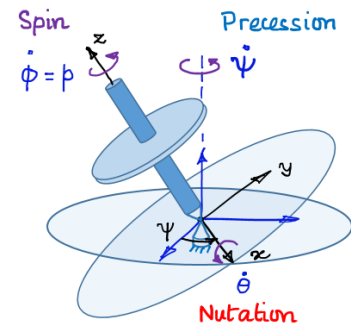
The components of the equation of motion is presented in the slide above.

(1) Steady precession $\dot{\theta} = 0$ $\dot{\psi} = \text{constant}$

$$I_0 \ddot{\theta} + I \dot{\psi} (\dot{\psi} \cos \theta + p) \sin \theta - I_0 \dot{\psi}^2 \sin \theta \cos \theta = M_x$$

$$~~I_0 \ddot{\psi} \sin \theta + 2 I_0 \dot{\psi} \dot{\theta} \cos \theta - I \dot{\theta} (\dot{\psi} \cos \theta + p) = 0~~$$

$$~~I (\ddot{\psi} \cos \theta - \dot{\psi} \dot{\theta} \sin \theta) = 0~~$$



For steady precession, we have simplifications as indicated in the slide above.

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(1) Steady precession $\dot{\theta} = 0$ $\dot{\psi} = \text{constant}$

$$I \dot{\psi} (\dot{\psi} \cos \theta + p) \sin \theta - I_0 \dot{\psi}^2 \sin \theta \cos \theta = M_x$$

Assuming $\dot{\psi} \ll p$, $p \gg 1$

$$I \dot{\psi} p \sin \theta = M_x$$

$$\Rightarrow \dot{\psi} = \frac{M_x}{I p \sin \theta}$$

Spinning top
 $M_x = mgr_g \sin \theta$

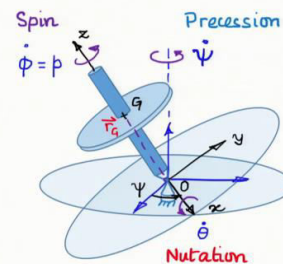
$$\Rightarrow \dot{\psi}_{\text{top}} = \frac{mgr_g}{I p}$$

Simplified derivation

$$\vec{H}_0 = I_0 \dot{\theta} \hat{i} + I_0 \dot{\psi} \sin \theta \hat{j} + I (\dot{\psi} \cos \theta + p) \hat{k} \approx I p \hat{k}$$

$$\vec{H}_0 = I p \vec{\Omega}_f \times \hat{k} = \vec{M}_0 \quad [\vec{\Omega}_f = \dot{\theta} \hat{i} + \dot{\psi} (\sin \theta \hat{j} + \cos \theta \hat{k})]$$

$$\Rightarrow I p \dot{\psi} \sin \theta \hat{i} = M_x \hat{i}$$



Assuming steady precession and spin speed to be much higher compared to the precession angular speed, we obtain a simple expression of the precession angular speed as shown in the slide above.

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(2) Steady torquefree precession $\theta = 0$ $\dot{\psi} = \text{constant}$

$$I \dot{\psi} (\dot{\psi} \cos \theta + p) \sin \theta - I_0 \dot{\psi}^2 \sin \theta \cos \theta = M_x^0$$

$$\Rightarrow \dot{\psi} \sin \theta [(I - I_0) \dot{\psi} \cos \theta + I p] = 0$$

$$\Rightarrow \dot{\psi} = 0 \quad \text{or} \quad \dot{\psi} = \frac{I p}{(I_0 - I) \cos \theta}$$

(a) $I_0 > I$: Direct precession

(b) $I_0 < I$: Retrograde precession

$$[I_0] = \begin{bmatrix} I_0 & 0 \\ 0 & I_0 \\ 0 & 0 & I \end{bmatrix}$$

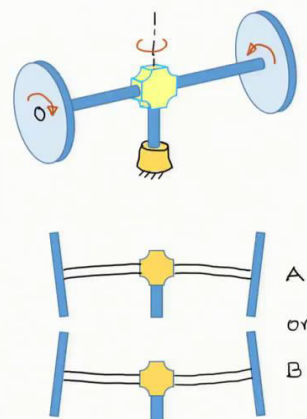
Next we look at the second simplification which is the steady torque-free precession as shown above. This leads us to two cases (a) direct/prograde precession (when $I_0 < I$), and (b) retrograde precession (when $I < I_0$). In the former case, the precessional angular velocity appears in the same direction as the spin angular velocity. These can also be understood kinematically as rolling of the body cone on the fixed space cone as shown in the figures.

We consider the following problem.

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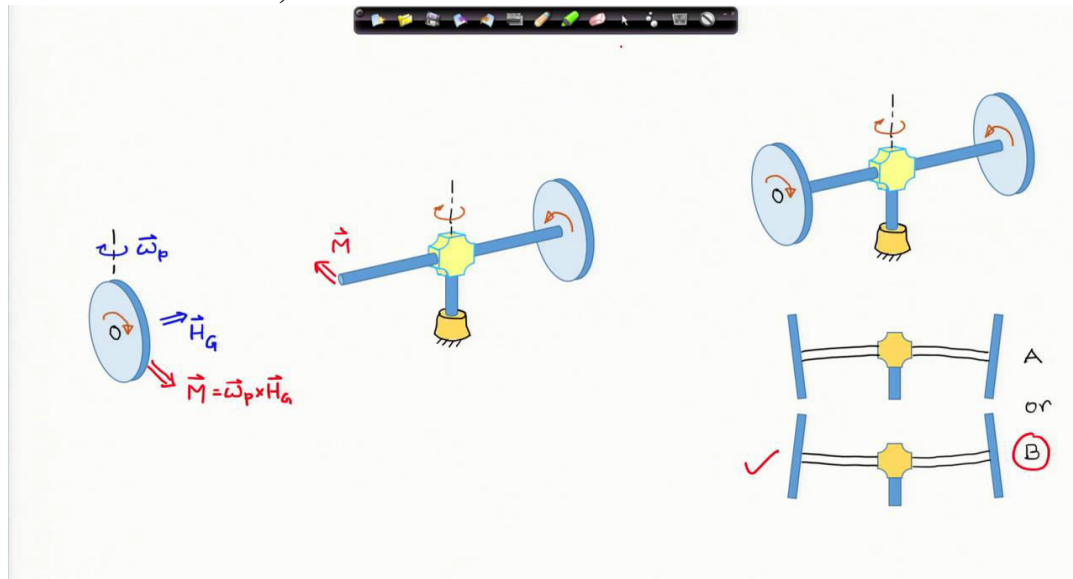
Problem 1:

Two identical discs are rapidly rotating freely on a shaft with angular velocities equal in magnitude and opposite in direction as shown. The shaft in turn is caused to rotate slowly about the vertical axis in the sense indicated. Determine whether the shaft bends as in A or as in B because of gyroscopic action.



Source: Dynamics, Meriam and Kraige

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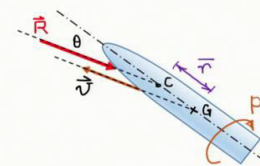
The solution is discussed in the above slide.

Consider the next problem as shown below.

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Problem 2:

A spinning projectile moving through the atmosphere with a velocity \mathbf{v} , which makes a small angle θ with its geometric axis, is subjected to a resultant aerodynamic force \mathbf{R} acting opposite to \mathbf{v} as shown. If \mathbf{R} passes through C slightly ahead of the mass center G on the symmetry axis, determine the minimum spin velocity p for which the projectile is spin-stabilized with $d\theta/dt = 0$. Take the moment of inertia about the symmetry axis as I_0 and about the transverse axes as I .



Source: Dynamics, Meriam and Kraige

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Governing equation

$$I \dot{\psi} (\dot{\psi} \cos \theta + p) \sin \theta - I_0 \dot{\psi}^2 \sin \theta \cos \theta = M_x = \bar{r} (R \sin \theta)$$

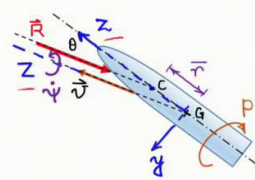

$$\Rightarrow (I - I_0) \cos \theta \dot{\psi}^2 + I p \dot{\psi} - \bar{r} R = 0 \quad (\sin \theta \neq 0)$$

For real solution of precession angular velocity

$$\Delta = I^2 p^2 + 4 \bar{r} R (I - I_0) \cos \theta \geq 0$$

$$\Rightarrow p \geq \frac{2}{I} \sqrt{\bar{r} R (I_0 - I) \cos \theta} \quad (I_0 > I)$$

Important: Choice of x-axis





First thing we fix up is the coordinate frame based on our previous discussions. This should be noted carefully. The solution steps are detailed in the slide above.

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Summary

- Gyroscopic forces in spinning axisymmetric bodies
- Precession with and without external moment
- Problems



The discussions are summarized as shown above.