

Advanced Dynamics
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Module No # 06
Lecture No # 35
Spatial Kinetics of Rigid Bodies – I

In this lecture we are going to start with spatial kinetics of rigid bodies.

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Overview

- Equations of motion of rigid bodies in 3D
- Moment of inertia tensor
- Work-energy relations
- Impulse-momentum relations

To give an overview, we are going to look at the equations of motion of rigid bodies in 3 dimensions. We are going to see what new things comes up when we study kinetics of rigid bodies in general 3D motion. We are going to look at the moment of inertial tensor. Then from these equations of motion we are going to derive the work energy relation and the impulse momentum relations.

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Equations of motion (using G)

Translational dynamics:

$$\dot{\vec{G}} = \vec{F} \quad \vec{G} = \int dm \vec{v}_G = m \vec{v}_G$$

$$\Rightarrow m \dot{\vec{v}}_G = \vec{F}$$

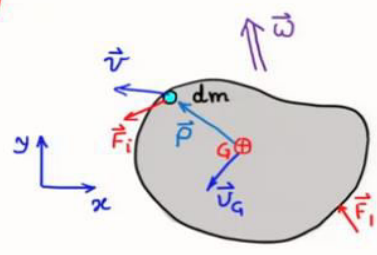

Rotational dynamics:

$$\dot{\vec{H}}_G = \vec{M}_G$$

$$\vec{H}_G = \int \vec{p} \times \dot{\vec{p}} dm = \int \vec{p} \times (\vec{\omega} \times \vec{p}) dm$$

$$\Rightarrow \vec{H}_G = \int [(\vec{p} \cdot \vec{p}) \vec{\omega} - (\vec{p} \cdot \vec{\omega}) \vec{p}] dm$$

($\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C}) \vec{B} - (\vec{A} \cdot \vec{B}) \vec{C}$)

The above slide recapitulates the translational and rotational dynamics of a rigid body. The rotational dynamics is written for the center of mass of the body.

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Angular momentum (about G)

$$\vec{H}_G = \int [(\vec{p} \cdot \vec{p}) \vec{\omega} - (\vec{p} \cdot \vec{\omega}) \vec{p}] dm$$

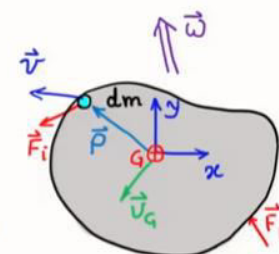

$$\rho = x\hat{i} + y\hat{j} + z\hat{k} \quad \vec{\omega} = \omega_x\hat{i} + \omega_y\hat{j} + \omega_z\hat{k}$$

$$\Rightarrow \vec{H}_G = \int \left[(x^2 + y^2 + z^2) \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} - \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} (x\omega_x + y\omega_y + z\omega_z) \right] dm$$

$$\Rightarrow \vec{H}_G = \int \begin{bmatrix} y^2 + z^2 & -xy & -xz \\ -yz & x^2 + z^2 & -yz \\ -zx & -zy & x^2 + y^2 \end{bmatrix} dm \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix}$$

$$\Rightarrow \vec{H}_G = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{Bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{Bmatrix} = [I_G] \vec{\omega}$$

Moment of inertia tensor (intrinsic)
• Time invariant

The calculation of the angular momentum vector leads us to the moment of inertia tensor calculated about the center of mass G, as shown in the slide above.

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Equations of motion (using G)

Translational dynamics:

$$\dot{\vec{G}} = \vec{F} \quad \vec{G} = \int dm \vec{v}_G = m \vec{v}_G$$

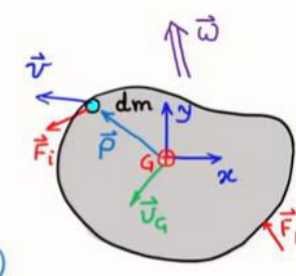
$$\Rightarrow \boxed{m \dot{\vec{v}}_G = \vec{F}}$$

Rotational dynamics: Body-fixed frame (rotating frame)

$$\dot{\vec{H}}_G = \vec{M}_G \quad \vec{H}_G = [I_G] \vec{\omega} \quad ([I_G] \text{ constant})$$

$$\frac{d\vec{H}_G}{dt} = \frac{\partial \vec{H}_G}{\partial t} + \vec{\omega} \times \vec{H}_G$$

$$\Rightarrow \boxed{[I_G] \dot{\vec{\omega}} + \vec{\omega} \times [I_G] \vec{\omega} = \vec{M}_G}$$

$$[I_G] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$


Finally, using a body-fixed frame, the equation of rotational dynamics can be written in terms of the center of mass G in the form shown in the slide above.

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Equations of motion (using G)

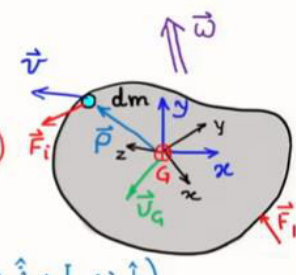
Simplification of rotational dynamics: Principal axes

$$[I_G] = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \quad (\text{Principal body-fixed frame})$$

$$[I_G] \dot{\vec{\omega}} + \vec{\omega} \times [I_G] \vec{\omega} = \vec{M}_G \quad ([I_G] \vec{\omega} = I_{xx} \omega_x \hat{i} + I_{yy} \omega_y \hat{j} + I_{zz} \omega_z \hat{k})$$

$$\begin{aligned} I_{xx} \dot{\omega}_x - (I_{yy} - I_{zz}) \omega_y \omega_z &= M_{Gx} \\ I_{yy} \dot{\omega}_y - (I_{zz} - I_{xx}) \omega_z \omega_x &= M_{Gy} \\ I_{zz} \dot{\omega}_z - (I_{xx} - I_{yy}) \omega_x \omega_y &= M_{Gz} \end{aligned}$$

Euler's equations



If we choose the x-y-z body-fixed frame as the principal frame of the body, the moment of inertia tensor becomes diagonal. In that case, we obtain the simplified equations of rotational dynamics which go by the name Euler's equations of motion.

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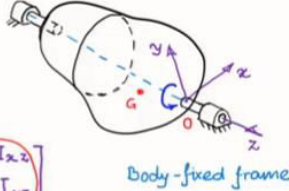
Fixed-axis rotation (parallel-plane motion)

$$m \dot{\vec{v}}_G = \vec{F}$$

$$[I_0] \dot{\vec{\omega}} + \vec{\omega} \times [I_0] \vec{\omega} = \vec{M}_0$$

$$\vec{\omega} = \begin{Bmatrix} 0 \\ 0 \\ \omega_z \end{Bmatrix}$$

$$[I_0] = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$



Body-fixed frame

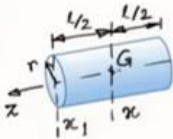
$$\Rightarrow \begin{cases} I_{yz} \omega_z^2 = M_{0x} \\ -I_{xz} \omega_z^2 = M_{0y} \\ I_{zz} \dot{\omega}_z = M_{0z} \end{cases}$$

(Reduces to planar kinetics if z is a principal axis and x-y plane contains G.)

Now we consider the fixed axis rotation or what is also known as parallel plane motion. It is called parallel plane motion because all points of the rigid body are moving in parallel planes. I have chosen the frame in such a way that the angular velocity is along the z axis. However, this frame is not guaranteed to be the principal frame of the rigid body. The equations of motion of the body is presented in the slide above. In case the frame happens to be the principal frame, and G lies on the fixed axis of rotation, the equations of rotational dynamics for such a body reduce to planar kinetics equation.

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Moment of inertia: examples



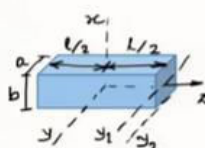
$$I_{xx} = \frac{1}{4} m r^2 + \frac{1}{12} m l^2$$

$$I_{x_1 x_1} = \frac{1}{4} m r^2 + \frac{1}{3} m l^2$$

$$I_{zz} = \frac{1}{2} m r^2$$



$$I_{zz} = \frac{2}{5} m r^2$$

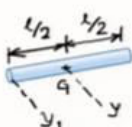


$$I_{xx} = \frac{1}{12} m (a^2 + l^2)$$

$$I_{zz} = \frac{1}{12} m (a^2 + b^2)$$

$$I_{y_1 y_1} = \frac{1}{12} m b^2 + \frac{1}{3} m l^2$$

$$I_{y_2 y_2} = \frac{1}{3} m (b^2 + l^2)$$



$$I_{yy} = \frac{1}{12} m l^2$$

$$I_{y_1 y_1} = \frac{1}{3} m l^2$$

$$I_{zz} \sim 0$$

The moments of inertia of some standard regular bodies are presented in the slide above.

Next, we discuss the work-energy relations.

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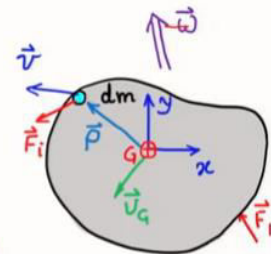
Work-energy relations

$$(m \dot{\vec{v}}_G = \vec{F}) \cdot \vec{v}_G$$

$$([I_G] \dot{\vec{\omega}} + \vec{\omega} \times [I_G] \vec{\omega} = \vec{M}_G) \cdot \vec{\omega}$$

$$\underbrace{\frac{d}{dt} \left(\frac{1}{2} m \vec{v}_G \cdot \vec{v}_G \right)}_{T_r} + \underbrace{\frac{d}{dt} \left(\frac{1}{2} \vec{\omega} \cdot [I_G] \vec{\omega} \right)}_{T_R} = \underbrace{\vec{F} \cdot \vec{v}_G + \vec{M}_G \cdot \vec{\omega}}_{P \text{ (power)}}$$

$$\Rightarrow \underbrace{\frac{d}{dt} \left(\frac{1}{2} \vec{v}_G \cdot \vec{G} + \frac{1}{2} \vec{\omega} \cdot \vec{H}_G \right)}_{T \text{ (total kinetic energy)}} = \vec{F} \cdot \vec{v}_G + \vec{M} \cdot \vec{\omega}$$



$$\vec{\omega} \times \vec{H}_G \cdot \vec{\omega} = 0$$

$$(\dot{\vec{\omega}} \cdot [I_G] \vec{\omega} = \vec{\omega} \cdot [I_G] \dot{\vec{\omega}})$$

The above slide presents the first steps in the derivation.

The kinetic energy expression is presented for the general motion and motion about a fixed point in the following slide.

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Kinetic energy expression

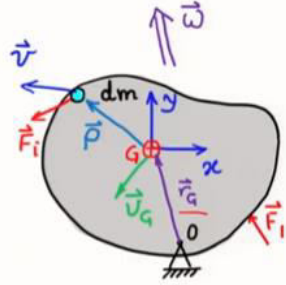

$$T = \frac{1}{2} \vec{v}_G \cdot \vec{G} + \frac{1}{2} \vec{\omega} \cdot \vec{H}_G$$

KE using a fixed point or point instantaneously at rest

$$\vec{v}_G = \vec{v}_O + \vec{\omega} \times \vec{r}_G$$

$$\vec{r}_G \times \vec{G} = \vec{\omega} \cdot \vec{H}_G$$

$$\Rightarrow T = \frac{1}{2} (\vec{\omega} \times \vec{r}_G) \cdot \vec{G} + \frac{1}{2} \vec{\omega} \cdot \vec{H}_G$$

$$\Rightarrow T = \frac{1}{2} \vec{\omega} \cdot (\vec{r}_G \times \vec{G} + \vec{H}_G) = \frac{1}{2} \vec{\omega} \cdot \vec{H}_O$$



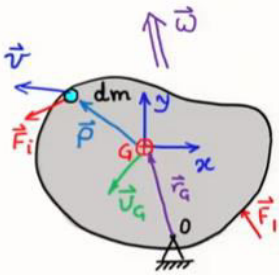

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Work-energy relations

$$\frac{dT}{dt} = \underbrace{\vec{F} \cdot \vec{v}_G}_{\frac{dT_T}{dt}} + \underbrace{\vec{M}_G \cdot \vec{\omega}}_{\frac{dT_R}{dt}}$$

$$\Rightarrow \Delta T = T_2 - T_1 = \int_{t_1}^{t_2} (\vec{F} \cdot \vec{v}_G + \vec{M}_G \cdot \vec{\omega}) dt$$

$$\Delta T_T = \int (\vec{F} \cdot \vec{v}_G) dt$$

$$\Delta T_R = \int (\vec{M}_G \cdot \vec{\omega}) dt$$



Finally, the work-energy relation is presented in the above slide. The decomposition of the kinetic energy in translational and rotational parts is used above.

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Impulse-momentum relation

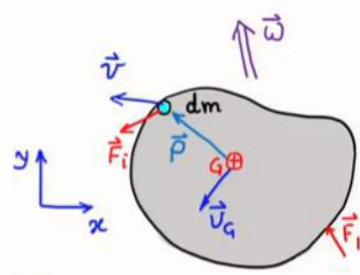
$$\dot{\vec{G}} = \vec{F}$$
$$\dot{\vec{H}}_G = \vec{M}_G$$

$$\Delta \vec{G} = \vec{G}_2 - \vec{G}_1 = \int_{t_1}^{t_2} \vec{F} dt$$


Linear impulse - momentum

$$\Delta \vec{H}_G = \vec{H}_2 - \vec{H}_1 = \int_{t_1}^{t_2} \vec{M}_G dt$$

Angular impulse - momentum



The diagram shows a rigid body with a center of mass G. A coordinate system (x, y) is shown. A small mass element dm is at position vector \vec{p} from G. A force \vec{F}_i acts on dm. The velocity of dm is \vec{v} . The velocity of G is \vec{v}_G . The angular velocity of the body is $\vec{\omega}$. A force \vec{F}_I acts on the body at a point.



The impulse momentum relations involving the linear and angular impulse are presented in the above slide.

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Summary

- Equations of motion of rigid bodies in 3D
- Moment of inertia tensor
- Work-energy relations
- Impulse-momentum relations

The discussions are summarized above.