

Advanced Dynamics
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Module No # 05
Lecture No # 24
Kinetics of a System of Particles – II

We will continue our discussions on kinetics of system of particles.

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Overview

- Generalized Newton's second law of motion
- Impulse-momentum relation
- Angular momentum

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Impulse-momentum form

$$\vec{r}_i = \vec{r}_G + \vec{\rho}_i \Rightarrow \dot{\vec{r}}_i = \dot{\vec{r}}_G + \dot{\vec{\rho}}_i$$

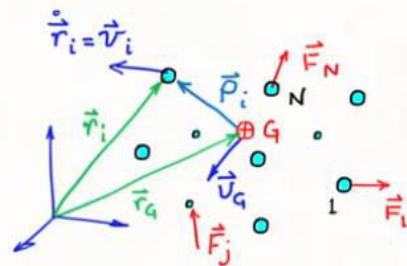
linear momentum $\vec{G}_i = m_i \dot{\vec{r}}_i$

$$\Rightarrow \vec{G} = \sum \vec{G}_i = \sum (m_i \dot{\vec{r}}_G + m_i \dot{\vec{\rho}}_i)$$

$$\Rightarrow \boxed{\vec{G} = m \dot{\vec{r}}_G} \quad (\sum m_i \dot{\vec{\rho}}_i = 0)$$

Newton's 2nd law: $\dot{\vec{G}} = m \ddot{\vec{r}}_G = \vec{F}$

$$\boxed{\Delta \vec{G} = \vec{G}_2 - \vec{G}_1 = \int_{t_1}^{t_2} \vec{F} dt} \quad (\text{Linear impulse})$$



The impulse-momentum form on the generalized Newton's second law for a system of particles is presented in the slide above. It should be noted that linear momentum of a system of particles is given by $\vec{G} = m \, d\vec{r}_G/dt$, where m is the total mass of the system of particles and $d\vec{r}_G/dt$ is the velocity of the center of mass of the system.

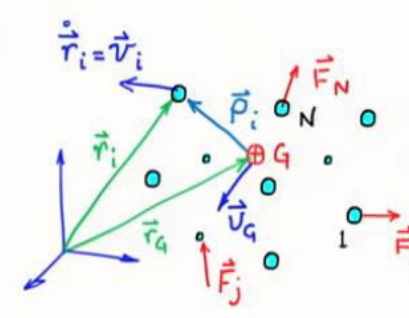
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Impulse-momentum form

$$\dot{\vec{G}} = m \ddot{\vec{r}}_G = \vec{F}$$

$$\Delta \vec{G} = \vec{G}_2 - \vec{G}_1 = \int_{t_1}^{t_2} \vec{F} \, dt$$

- If net external force is zero, linear momentum of C.M. is conserved
- Internal forces do not contribute

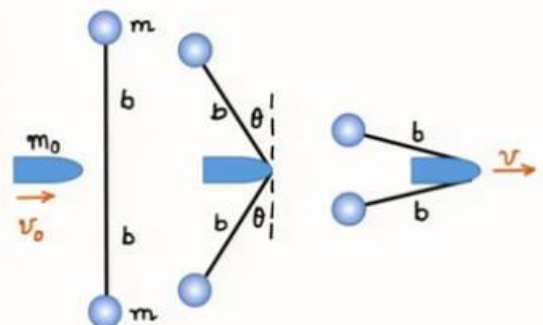


As shown above, if there is no external force on the system of particles, then the linear momentum of the center of the mass is conserved. Here we notice once again that internal forces do not contribute.

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Problem 1:

Two small spheres (treated as particles), each of mass m , are connected by a cord of length $2b$ (measured up to the sphere centers) and initially at rest on a smooth horizontal surface. A body of mass m_0 travelling in a straight line with velocity v_0 hits the cord perpendicularly in the middle causing deflection of the two parts as shown. Determine the velocity v of the mass m_0 as the two spheres near contact with θ approaching 90 deg. Also, find $d\theta/dt$ for this condition.



Source: Dynamics, Meriam and Kraige

We consider the problem as shown in the slide above.

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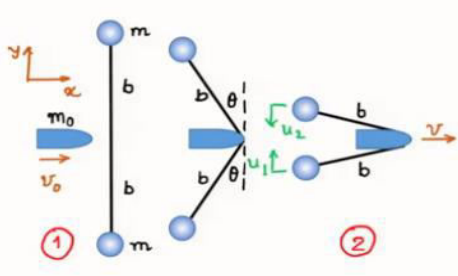

(i) Net external force is zero \Rightarrow conservation of linear momentum of c.m.
(ii) Absence of dissipative forces/work done by internal forces
 \Rightarrow conservation of energy

(i) $\vec{G}_2 = \vec{G}_1$

$$\Rightarrow m_0 v \hat{i} + m(v \hat{i} + u_1 \hat{j}) + m(v \hat{i} - u_2 \hat{j})$$

$$= m_0 v_0 \hat{i}$$

$$\Rightarrow (m_0 + 2m)v = m_0 v_0, \quad u_1 = u_2 = u = b\dot{\theta}$$

$$\Rightarrow \underline{v = \frac{m_0}{m_0 + 2m} v_0}$$



Since there is no external force acting on the system, we first use the linear momentum conservation to determine the velocities of the two balls in the folded configuration.

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(i) Net external force is zero \Rightarrow conservation of linear momentum of c.m.
(ii) Absence of dissipative forces/work done by internal forces
 \Rightarrow conservation of energy

(ii) $E_2 = E_1 \Rightarrow \frac{1}{2} m_0 v^2 + 2 \left[\frac{1}{2} m (v^2 + u^2) \right]$

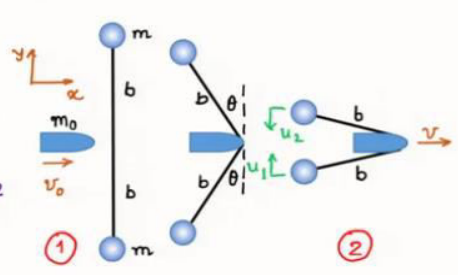

$$= \frac{1}{2} m_0 v_0^2$$

$$\Rightarrow \frac{1}{2} (m_0 + 2m) \frac{m_0^2}{(m_0 + 2m)^2} v_0^2 + m b^2 \dot{\theta}^2 = \frac{1}{2} m_0 v_0^2$$

$$\Rightarrow m b^2 \dot{\theta}^2 = \frac{m m_0}{m_0 + 2m} v_0^2$$

$$\Rightarrow b \dot{\theta} = \sqrt{\frac{m_0}{m_0 + 2m}} v_0$$

$$\Rightarrow \underline{\dot{\theta} = \sqrt{\frac{m_0}{m_0 + 2m}} \left(\frac{v_0}{b} \right)}$$

$$\underline{v = \frac{m_0}{m_0 + 2m} v_0}$$



In the second step, we use the conservation of energy since there are no dissipative forces, as shown in the slide above, to completely solve the problem.

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Angular impulse-momentum

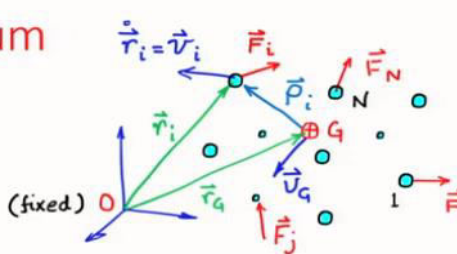
$$\vec{r}_i \times (m_i \ddot{\vec{r}}_i = \vec{F}_i + \sum_{j=1}^N \vec{f}_{ij})$$

$$\vec{r}_N \times (m_N \ddot{\vec{r}}_N = \vec{F}_N + \sum_{j=1}^N \vec{f}_{Nj})$$

$$\frac{d}{dt} \left(\sum \vec{r}_i \times m_i \dot{\vec{r}}_i \right) = \sum \vec{r}_i \times \vec{F}_i$$

Moment of linear momentum about O
Moment of all forces about O

(Angular momentum about O)



$$\begin{cases} \vec{r}_i \times m_i \frac{d\dot{\vec{r}}_i}{dt} = \frac{d}{dt} (\vec{r}_i \times m_i \dot{\vec{r}}_i) \\ \vec{r}_i \times \vec{f}_{ij} + \vec{r}_j \times \vec{f}_{ji} = (\vec{r}_i - \vec{r}_j) \times \vec{f}_{ij} = 0 \end{cases}$$

Now we will define angular momentum and look at angular impulse momentum relation. We start by writing the Newton's 2nd law for the individual particles, taking a cross product of the ith equation with the position vector of the ith particles as shown in the slide above, and summing over all the particles. This finally leads to

$$\frac{d}{dt} \left(\sum \vec{r}_i \times m_i \dot{\vec{r}}_i \right) = \sum \vec{r}_i \times \vec{F}_i$$

Moment of linear momentum about O
Moment of all forces about O

(Angular momentum about O)

$$\Rightarrow \boxed{\frac{d\vec{H}_o}{dt} = \vec{M}_o}$$

It is observed that the internal forces do not contribute to this equation.

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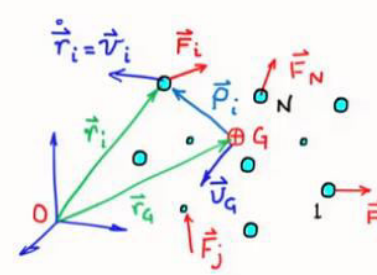

Angular impulse-momentum

$$\frac{d}{dt} \left(\underbrace{\sum \vec{r}_i \times m_i \dot{\vec{r}}_i}_{\text{Angular momentum about O } (\vec{H}_O)} \right) = \underbrace{\sum \vec{r}_i \times \vec{F}_i}_{\text{Moment of all forces about O } (\vec{M}_O)}$$

$$\Rightarrow \boxed{\frac{d\vec{H}_O}{dt} = \vec{M}_O}$$

$$\Rightarrow \boxed{\Delta \vec{H}_O = \vec{H}_{O_2} - \vec{H}_{O_1} = \int_{t_1}^{t_2} \vec{M}_O dt}$$

Angular impulse

Integrating the angular momentum equation over time between two time instants t_1 and t_2 , we obtain the angular impulse – angular momentum relation as presented in the slide above.

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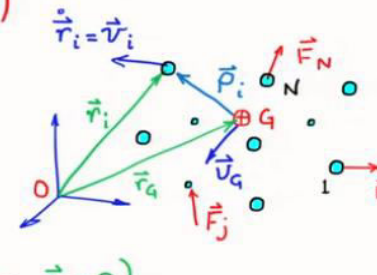
Angular momentum (about O)

$$\vec{H}_O = \sum \vec{r}_i \times m_i \dot{\vec{r}}_i$$

$$\vec{r}_i = \vec{r}_G + \vec{\rho}_i \Rightarrow \dot{\vec{r}}_i = \dot{\vec{r}}_G + \dot{\vec{\rho}}_i$$

$$\Rightarrow \vec{H}_O = \sum (\vec{r}_G + \vec{\rho}_i) \times (m_i \dot{\vec{r}}_G + m_i \dot{\vec{\rho}}_i)$$

$$\Rightarrow \vec{H}_O = \sum (\vec{r}_G \times m_i \dot{\vec{r}}_G + \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i) \quad \left(\sum m_i \dot{\vec{\rho}}_i = 0 \right)$$

$$\Rightarrow \vec{H}_O = \underbrace{\vec{r}_G \times m \dot{\vec{r}}_G}_{\text{Angular momentum due to motion of center of mass}} + \underbrace{\sum \vec{\rho}_i \times m_i \dot{\vec{\rho}}_i}_{\text{Angular momentum due to relative motion about center of mass } \vec{H}_G^{\text{rel}}}$$


The angular momentum about a fixed point O is calculated in the slide above.

It is observed that the total angular momentum of the system about O is the sum of the angular momentum because of the motion of the center of mass with the whole mass concentrated at the

center of mass at G and the angular momentum because of the relative motion of the individual particle about the center of mass.

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Angular momentum (about G)

$$\vec{H}_G = \sum \vec{p}_i \times m_i \dot{\vec{r}}_i$$

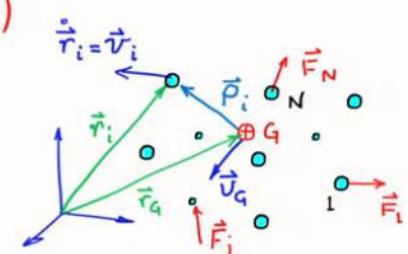
$$\vec{r}_i = \vec{r}_G + \vec{\rho}_i \Rightarrow \dot{\vec{r}}_i = \dot{\vec{r}}_G + \dot{\vec{\rho}}_i$$

$$\Rightarrow \vec{H}_G = \sum \vec{\rho}_i \times (m_i \dot{\vec{r}}_G + m_i \dot{\vec{\rho}}_i)$$

$$\Rightarrow \vec{H}_G = \underbrace{\sum (\vec{\rho}_i \times m_i \dot{\vec{\rho}}_i)}_{\vec{H}_G^{rel}} \quad \left(\sum m_i \dot{\vec{\rho}}_i = 0 \right)$$

Rigid frame

$$\vec{H}_G = \sum (\vec{\rho}_i \times m_i \omega \times \vec{\rho}_i)$$



For a rigid frame

$$\dot{\vec{\rho}}_i = \frac{\partial \vec{\rho}_i}{\partial t} + \vec{\omega} \times \vec{\rho}_i$$

$$\Rightarrow \dot{\vec{\rho}}_i = \vec{\omega} \times \vec{\rho}_i$$

In the above slide, it is shown that the angular momentum about the center of mass G is equal to the relative angular momentum about G. It is to be noted that, in the former, the absolute velocities of the particles are used, while in the latter, the relative velocities of the particles with respect to G are used. The angular momentum expression for a system of particles held by a rigid frame is also presented in the slide above.

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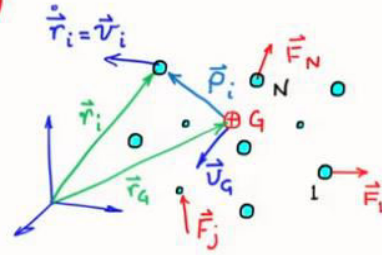
Angular momentum (about G)

$$\vec{H}_G = \sum \vec{p}_i \times m_i \dot{\vec{r}}_i$$

$$\begin{aligned} \frac{d\vec{H}_G}{dt} &= \sum \dot{\vec{p}}_i \times m_i \dot{\vec{r}}_i + \sum \vec{p}_i \times m_i \ddot{\vec{r}}_i \\ &= \sum \dot{\vec{p}}_i \times (m_i \dot{\vec{r}}_G + m_i \dot{\vec{p}}_i) + \sum \vec{p}_i \times \vec{F}_i \end{aligned}$$

$$(\dot{\vec{r}}_i = \dot{\vec{r}}_G + \dot{\vec{p}}_i)$$

$$\Rightarrow \boxed{\dot{\vec{H}}_G = \sum \vec{p}_i \times \vec{F}_i = \vec{M}_G} \quad (\sum m_i \dot{\vec{p}}_i = 0)$$



We continue with this angular momentum about the center of mass. If we take the time derivative of the angular momentum about the center of mass, we obtain

$$\boxed{\dot{\vec{H}}_G = \sum \vec{p}_i \times \vec{F}_i = \vec{M}_G}$$

This equation tells us how the angular momentum is going to change about the center of mass for a system of particles.

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Summary

- Generalized Newton's second law of motion
- Impulse-momentum relation
- Angular momentum

The discussions are summarized above.