

Advanced Dynamics
Prof. Anirvan Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology - Kharagpur

Module No # 05
Lecture No # 23
Kinetics of a System of Particles – I

In this lecture we are going to discuss the kinetics of a system of particles.

(Refer Slide Time: 00:57)

Overview

- Generalized Newton's second law of motion
- Work-energy relation

We will generalize Newton's second law of motion for a single particle to a system of particles and look at the work energy relations.

(Refer Slide Time: 01:08)

Generalized Newton's second law of motion

For individual particles

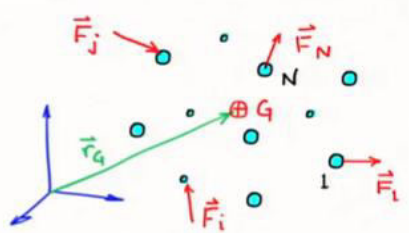
$$m_1 \ddot{\vec{r}}_1 = \vec{F}_1 + \sum_{j=1}^N \vec{f}_{1j}$$

⋮

$$m_N \ddot{\vec{r}}_N = \vec{F}_N + \sum_{j=1}^N \vec{f}_{Nj}$$

$$\sum_{i=1}^N m_i \ddot{\vec{r}}_i = \sum_{i=1}^N \vec{F}_i + \sum_{i=1}^N \sum_{j=1}^N \vec{f}_{ij}$$

$$\Rightarrow \boxed{m \ddot{\vec{r}}_G = \vec{F}} \quad \left(\sum m_i \vec{r}_i = m \vec{r}_G \right)$$



The diagram shows a 3D coordinate system with a green vector \vec{r}_G pointing to the center of mass G of a system of particles. Several particles are shown with external force vectors $\vec{F}_1, \vec{F}_i, \vec{F}_j, \vec{F}_N$ and internal force vectors \vec{f}_{ij} between them.

$\vec{f}_{ij} = -\vec{f}_{ji}$: internal force between i and j ($\vec{f}_{ii} = 0$)
 \vec{F}_i : external force
 G : center of mass

Consider a system of n particles as shown in the figure above. The position vectors of the individual particles are denoted by \mathbf{r}_i , the interaction force on the i^{th} particle due to the j^{th} particles is denoted by \mathbf{f}_{ij} , and the external force on the i^{th} particle is written as \mathbf{F}_i . By Newton's 3rd law, $\mathbf{f}_{ij} = -\mathbf{f}_{ji}$. The equations of motion of the particles are presented in the slide above and summed over all particles. This leads to the extension of Newton's 2nd law for a system of particles as

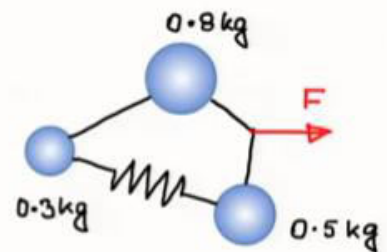
$$m \ddot{\mathbf{r}}_G = \vec{\mathbf{F}}$$

where \mathbf{r}_G is the position vector of the center of mass of the system of particles, and \mathbf{F} is the summation of all external forces acting on the system.

(Refer Slide Time: 07:40)

Problem 1:

The three small spheres (treated as point masses) are connected by cords and a spring, and placed on a smooth horizontal surface. If a force $F=6.4$ N is applied to one of the cords as shown, find the acceleration a_G of the center of mass of the spheres for the instant depicted.



Source: Dynamics, Meriam and Kraige

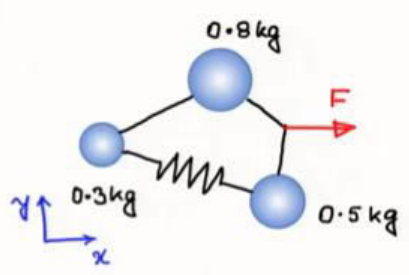
We consider the above problem.

(Refer Slide Time: 08:40)

Equation of motion of center of mass

$$m \vec{a}_G = \vec{F} = 6.4 \hat{i} \text{ N}$$

$$\Rightarrow 1.6 \vec{a}_G = 6.4 \hat{i} \text{ N}$$

$$\Rightarrow \underline{\vec{a}_G = 4 \hat{i} \text{ m/s}^2}$$


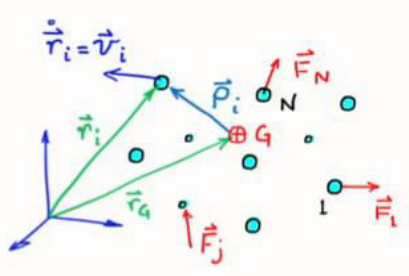

The solution is presented in the slide above.

(Refer Slide Time: 10:27)

Work-energy form

$$\begin{pmatrix} m_1 \ddot{\vec{r}}_1 = \vec{F}_1 + \sum_{j=1}^N \vec{f}_{1j} \\ \vdots \\ m_N \ddot{\vec{r}}_N = \vec{F}_N + \sum_{j=1}^N \vec{f}_{Nj} \end{pmatrix} \cdot \dot{\vec{r}}_i$$

$$\frac{d}{dt} \left(\underbrace{\sum \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i}_{T \text{ (kinetic energy)}} \right) = \sum \vec{F}_i \cdot \dot{\vec{r}}_i + \sum \sum \vec{f}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j)$$

$$\Rightarrow \boxed{\frac{dT}{dt} = \sum \vec{F}_i \cdot \dot{\vec{r}}_i + \sum \sum \vec{f}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j)}$$



Next we will look at the work energy relation for a system of particles. First, writing out the Newton's 2nd law for the individual particles, we take dot product of the i^{th} particle equation with the velocity vector $d\vec{r}_i/dt$, and sum over all particles as shown in the slide above. This leads to the rate of change of kinetic energy of the system as

$$\boxed{\frac{dT}{dt} = \sum \vec{F}_i \cdot \dot{\vec{r}}_i + \sum \sum \vec{f}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j)}$$

(Refer Slide Time: 14:15)

Work-energy form

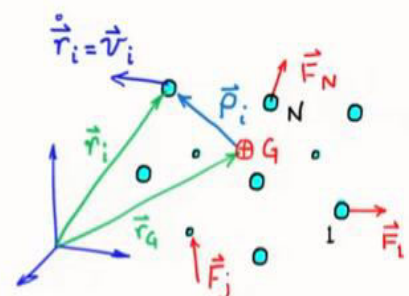
$$\frac{dT}{dt} = \sum \vec{F}_i \cdot \dot{\vec{r}}_i + \sum \sum \vec{f}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j)$$

(i) No interaction $\Rightarrow \vec{f}_{ij} = 0$

$$\Rightarrow \int_a^b dT = \sum \int_a^b \vec{F}_i \cdot d\vec{r}_i = \sum W_{i a-b}$$

$\Rightarrow \Delta T = T_b - T_a = W_{a-b}$

$W_{a-b} = 0 \Rightarrow T_b = T_a$



We consider two cases. The case (i) of no interaction between particles (i.e., $f_{ij}=0$) is shown in the slide above. This leads to the work-energy relation

$\Delta T = T_b - T_a = W_{a-b}$

If the work done by all external forces is zero, then kinetic energy of the system of particles is conserved.

(Refer Slide Time: 16:42)

Work-energy form

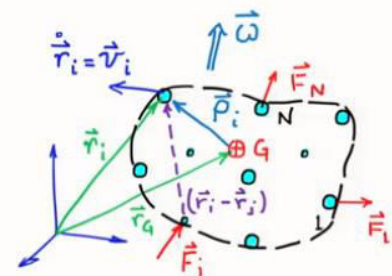
$$\frac{dT}{dt} = \sum \vec{F}_i \cdot \dot{\vec{r}}_i + \sum \sum \vec{f}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j)$$

(ii) Rigid frame $\Rightarrow \dot{\vec{r}}_i - \dot{\vec{r}}_j = \vec{\omega} \times (\vec{r}_i - \vec{r}_j)$
 (Inextensible connection) $\Rightarrow \sum \sum \vec{f}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j) = 0$

$$\Rightarrow \int_a^b dT = \sum \int_a^b \vec{F}_i \cdot d\vec{r}_i = \sum W_{i a-b}$$

$\Rightarrow \Delta T = T_b - T_a = W_{a-b}$

$W_{a-b} = 0 \Rightarrow T_b = T_a$



Next, we consider case (ii) of a rigid frame, as shown in the slide above. In this case, the work done term due to internal forces becomes zero. Hence, we again get back the work-energy relation

$$\Delta T = T_b - T_a = W_{a-b}$$

Now we have considered these 2 situations where the second term on the right hand side drops off. This happens when, either there is no interaction between the particles, or the particles are connected by rigid frame.

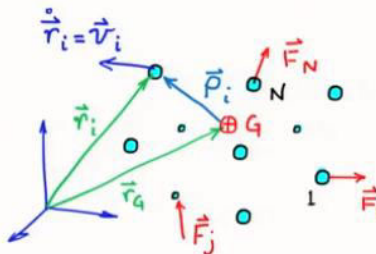
(Refer Slide Time: 22:13)

Work-energy form (special)


$$\frac{dT}{dt} = \sum \vec{F}_i \cdot \dot{\vec{r}}_i + \sum \sum \vec{f}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j)$$

$$\int_a^b dE = \sum \int_a^b \vec{F}_j^N \cdot d\vec{r}_j$$

$$\Rightarrow \Delta E = E_b - E_a = W_{a-b}^N$$

$$W_{a-b}^N = 0 \Rightarrow E_b = E_a$$


- Special case of non-interacting or rigidly connected particles
- Condition of conservation of kinetic or total mechanical energy



In the case when potential forces are present, we can define potential functions corresponding to each force, and the work done by such forces can be expressed as the negative of the change in the potential energy of the system as

$$\int_a^b dT = \int_a^b \left(\underbrace{\sum \vec{F}_i^c \cdot d\vec{r}_i}_{-dV_i} + \sum \vec{F}_j^N \cdot d\vec{r}_j \right)$$

This leads to

$$\int_a^b dT = \int_a^b (-dV + \sum \vec{F}_j^N \cdot d\vec{r}_j) \Rightarrow \int_a^b d(\underbrace{T+V}_E) = \sum \int_a^b \vec{F}_j^N \cdot d\vec{r}_j$$

(Mechanical energy)

and finally, we obtain

$$\Delta E = E_b - E_a = W_{a \rightarrow b}^N$$

(Refer Slide Time: 26:23)

Decomposition of kinetic energy

$$\frac{d}{dt} \left(\underbrace{\sum \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i}_T \right) = \sum \vec{F}_i \cdot \dot{\vec{r}}_i + \sum \sum \vec{F}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j)$$

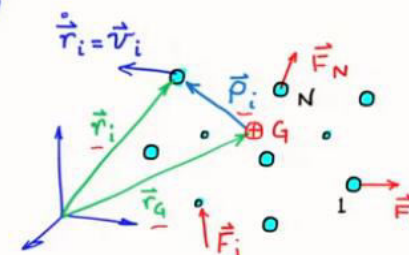
T (kinetic energy)

$$\vec{r}_i = \vec{r}_G + \vec{\rho}_i$$

$$\Rightarrow \dot{\vec{r}}_i = \dot{\vec{r}}_G + \dot{\vec{\rho}}_i$$

$$\Rightarrow T = \sum \frac{1}{2} m_i (\dot{\vec{r}}_G + \dot{\vec{\rho}}_i) \cdot (\dot{\vec{r}}_G + \dot{\vec{\rho}}_i)$$

$$\Rightarrow T = \sum \frac{1}{2} m_i \dot{\vec{r}}_G \cdot \dot{\vec{r}}_G + \sum \frac{1}{2} m_i \dot{\vec{\rho}}_i \cdot \dot{\vec{\rho}}_i$$



$\sum m_i \vec{r}_i = \sum m_i \vec{r}_G$
 $\Rightarrow \sum m_i (\vec{r}_i - \vec{r}_G) = 0$
 $\Rightarrow (\sum m_i \dot{\vec{\rho}}_i = \vec{0})$

Now let us look at the decomposition of the kinetic energy as shown in the slide above.

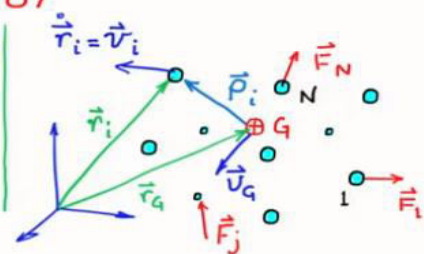
(Refer Slide Time: 29:33)

Decomposition of kinetic energy

$$\frac{d}{dt} \left(\underbrace{\sum \frac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i}_{T \text{ (kinetic energy)}} \right) = \sum \vec{F}_i \cdot \dot{\vec{r}}_i + \sum \sum \vec{f}_{ij} \cdot (\dot{\vec{r}}_i - \dot{\vec{r}}_j)$$

$$T = \sum \frac{1}{2} m_i \dot{\vec{r}}_G \cdot \dot{\vec{r}}_G + \sum \frac{1}{2} m_i \dot{\vec{\rho}}_i \cdot \dot{\vec{\rho}}_i$$

$$\Rightarrow T = \underbrace{\frac{1}{2} m v_G^2}_{\text{K.E. of center of mass}} + \underbrace{\sum \frac{1}{2} m_i \dot{\vec{\rho}}_i \cdot \dot{\vec{\rho}}_i}_{\text{K.E. of relative motion about center of mass}}$$



For a rigid frame

$$\dot{\vec{\rho}}_i = \frac{\partial \vec{\rho}_i}{\partial t} + \vec{\omega} \times \vec{\rho}_i$$

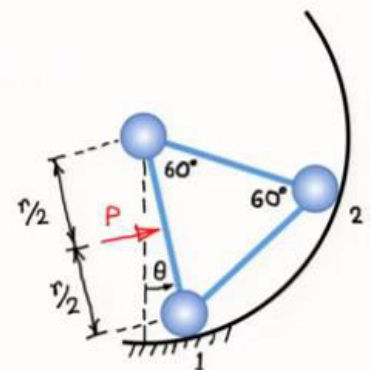
$$\Rightarrow \dot{\vec{\rho}}_i = \vec{\omega} \times \vec{\rho}_i$$

As shown in the above slide, the total kinetic energy of the system of particles is sum of 2 parts: (i) the kinetic energy due to the motion of the center of mass with all the mass concentrated at the center of mass, and (ii) the kinetic energy of all particles moving relative to the center of mass.

(Refer Slide Time: 32:58)

Problem 2:

Three identical small spheres (treated as particles), each of mass m , are fixed by three identical light rigid rods as shown. The assembly moves on a smooth circular track in the vertical plane. If the unit starts from rest at $\theta=0$ with a force of constant magnitude P always acting perpendicularly at the mid-point of a rod as shown, determine (a) the minimum force P_{\min} which will bring the unit to rest at $\theta=60$ deg, and (b) the common velocity v of spheres 1 and 2 when $\theta=60$ deg if $P=2P_{\min}$.



Source: Dynamics, Meriam and Kraige

We consider the above example.

(Refer Slide Time: 33:57)

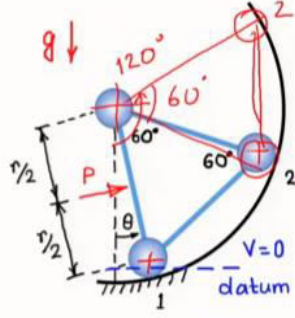
System with rigid frame

(a) $\Delta E = E_b - E_a = W_{a \rightarrow b}^N$

$$\Rightarrow T_b^0 + V_b - (T_a^0 + V_a) = \int_0^{\pi/3} P \frac{r}{2} d\theta$$

$$\Rightarrow mgr(1 - \cos \frac{\pi}{3}) + mgr(1 - \cos \frac{2\pi}{3}) - mgr(1 - \cos 0) - mgr(1 - \cos \frac{\pi}{3})$$

$$= \frac{\pi}{6} Pr$$

$$\Rightarrow \underline{P_{\min} = \frac{9mg}{\pi}}$$


The solution is presented in the slide above using the work-energy relation to determine P_{\min} required to take the system to the configuration with $\theta=60^\circ$.

(Refer Slide Time: 39:11)

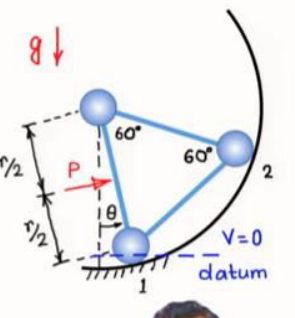
System with rigid frame

(b) $T_b + V_b - (T_a^0 + V_a) = \int_0^{\pi/3} 2P_{\min} \frac{r}{2} d\theta$

where $P_{\min} = \frac{9mg}{\pi}$

$$\Rightarrow 2\left(\frac{1}{2}mv^2\right) + mgr(1 - \cos \frac{\pi}{3}) + mgr(1 - \cos \frac{2\pi}{3}) - mgr(1 - \cos 0) - mgr(1 - \cos \frac{\pi}{3}) = 3mgr$$

$$\Rightarrow mv^2 = \frac{3}{2}mgr$$

$$\Rightarrow \underline{v = \sqrt{\frac{3}{2}gr}}$$


In a second part of this problem we have to find out the common velocity of the 2 particles 1 and 2 when $2P_{\min}$ is applied in place of P_{\min} . The solution is again approached using the work-energy relation as shown in the slide above.

(Refer Slide Time: 41:03)

Summary

- Generalized Newton's second law of motion
- Work-energy relation

The above slide summarizes the discussion.