Advanced Dynamics
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Module No # 01
Lecture No # 02
Coordinate Systems – II

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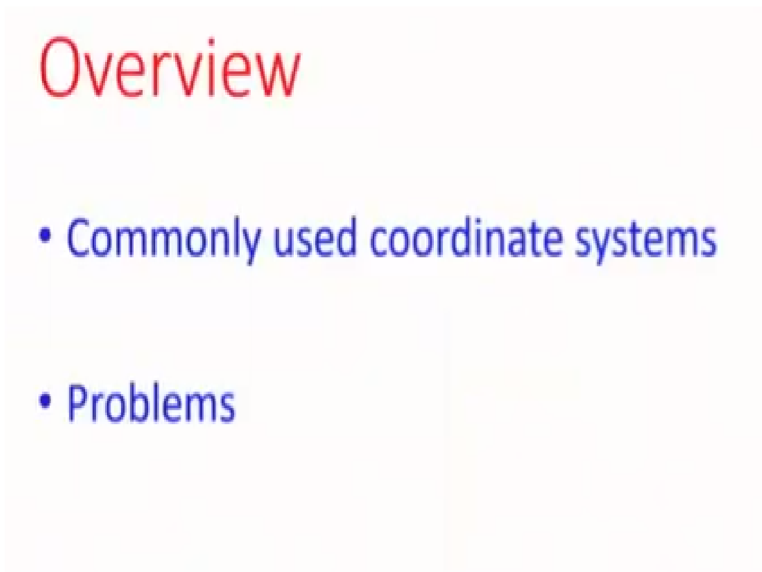

Here we are going to continue our discussions on the commonly used coordinate systems and look at some problems.

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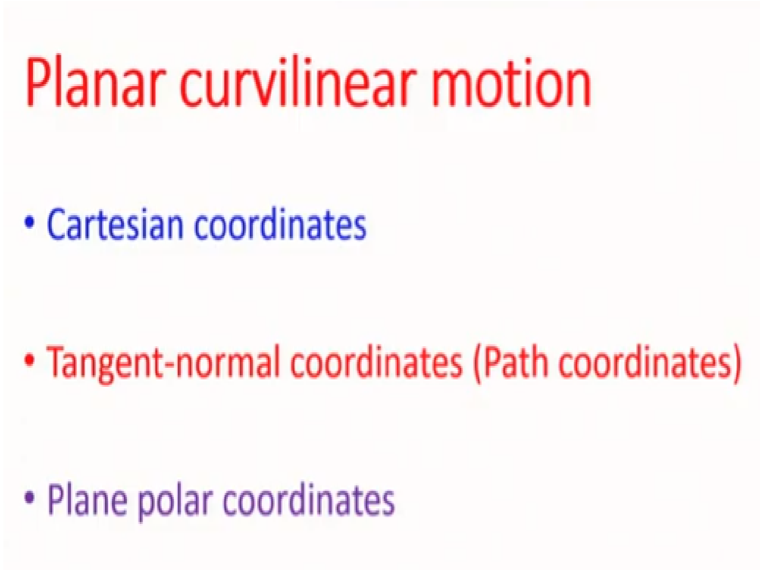

In the last lecture I had started with the Cartesian coordinates and in this lecture I am going to discuss about the tangent normal coordinates which are also known as the path coordinates, and the plane polar coordinates.

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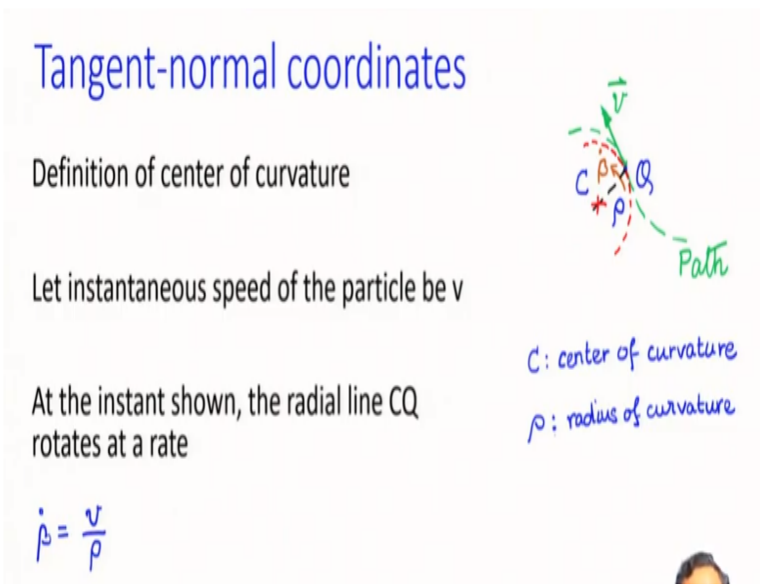

The tangent normal coordinates comprises of, as the name itself suggests, a tangent vector and a normal vector. Suppose there is a path in the plane on which a particle Q is travelling. I can define the center of curvature of this path at any point. You know that 3, infinitesimally separated points on this path uniquely defines a circle, and the center of the circle defines the center of curvature of the path at that point around which I have taken the 3 infinitesimally separated points. The center of curvature at the current location of the particle indicated by a red plus symbol. The point C is the center of curvature and rho is the radius of the curvature. Let the instantaneous speed of the particle, which is the magnitude of the velocity vector, be v. The velocity vector is always tangent to the path I have shown and the magnitude is v. Then you can imagine that this line CQ, where Q is the particle.

The radial line CQ will be rotating as the particle and I can represent the rate of rotation as

$$\dot{\beta} = \frac{v}{\rho}$$

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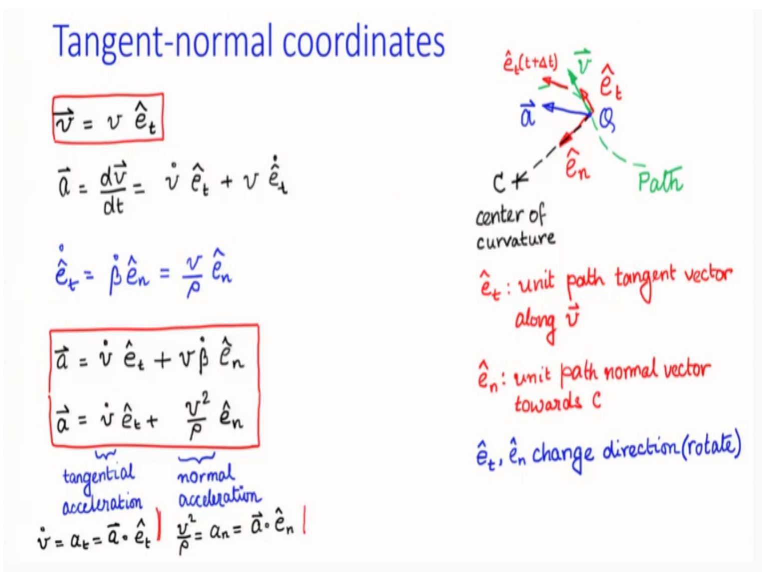

With this definition we come to the figure now where I start defining my coordinate system. The unit tangent and normal vectors are defined as

$\hat{e}_t$: unit path tangent vector along $\vec{v}$

$\hat{e}_n$: unit path normal vector towards C

Now, the velocity vector in this coordinate system is defined by

$$\boxed{\vec{v} = v\,\hat{e}_t}$$

For acceleration we differentiate the velocity vector. Now as mentioned the unit vectors are changing as the particle moves. Therefore, when we differentiate the velocity vector, not only do we generate the time rate of the change of the speed, but also generate the time rate of the change of the unit tangent vector to obtain

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{v}\,\hat{e}_t + v\,\dot{\hat{e}}_t$$

The time rate of change of the unit tangent vector is obtained as

$$\begin{aligned}\dot{\hat{e}}_t &= \lim_{\Delta t \to 0} \frac{\hat{e}_t(t+\Delta t) - \hat{e}_t(t)}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{\Delta \beta}{\Delta t}\,\hat{e}_n \\ &= \dot{\beta}\,\hat{e}_n = \frac{v}{\rho}\,\hat{e}_n\end{aligned}$$

The expressions of acceleration can now be written as

$$\boxed{\begin{aligned}\vec{a} &= \dot{v}\,\hat{e}_t + v\dot{\beta}\,\hat{e}_n \\ \vec{a} &= \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\,\hat{e}_n\end{aligned}}$$

The second expression of acceleration is very useful in finding out the radius of curvature ρ of a moving particle.

The acceleration terms have some names. The first term is the tangential acceleration which is obtained by taking dot product of the acceleration vector with the unit tangential vector. And if you take dot product with the unit normal vector, you get the normal acceleration. The normal acceleration is related to the radius of curvature of the path.

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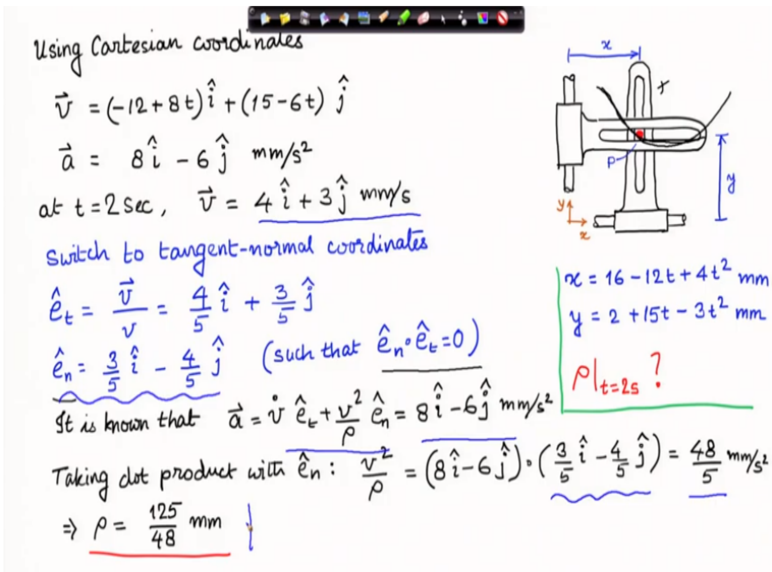

In the above example the slotted links move according to this expressions of x and y, where x and y are in millimeter, and t is in second. The horizontal slot moves vertically and the vertical slot moves horizontally. At t = 2 second, we have to determine the radius of curvature ρ of the path of the constrained pin. We will use the Cartesian coordinates (x, y).

The velocity of the pin is obtained by differentiating the position vector of the pin as

$$\vec{v} = (-12+8t)\hat{i} + (15-6t)\hat{j}$$

The acceleration is obtained as

$$\vec{a} = 8\hat{i} - 6\hat{j} \quad mm/s^2$$

At time t = 2 second the velocity vector is

$$\vec{v} = 4\hat{i} + 3\hat{j} \quad mm/s$$

The direction of the unit tangent vector is same as the velocity direction (unit vector along v)

$$\hat{e}_t = \frac{\vec{v}}{v} = \frac{4}{5}\hat{i} + \frac{3}{5}\hat{j}$$

We complete the tangent-normal coordinate system by defining the unit normal vector. But, we do not know in which direction the center of the curvature of the path lies. So, we just assume a certain direction as

$$\hat{e}_n = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

Now the expression of acceleration as determined from the tangent normal system is equated to the representation of absolute acceleration in the Cartesian frame

$$\vec{a} = \dot{v}\,\hat{e}_t + \frac{v^2}{\rho}\hat{e}_n = 8\hat{i} - 6\hat{j} \text{ mm/s}^2$$

Now, take dot product of both sides of this acceleration with the unit normal vector to obtain

$$\frac{v^2}{\rho} = (8\hat{i} - 6\hat{j}) \cdot \left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right) = \frac{48}{5} \text{ mm/s}^2$$

$$\Rightarrow \underline{\rho = \frac{125}{48} \text{ mm}}$$

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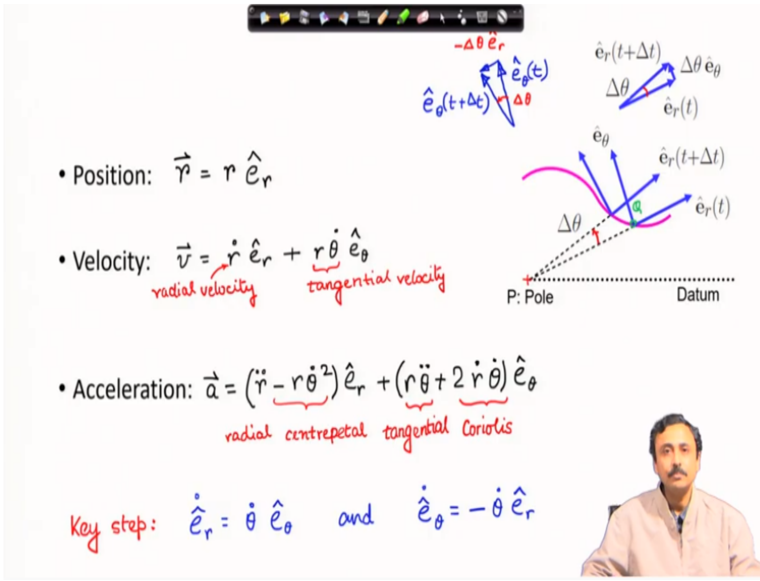

Next we move on to the plane polar coordinates. Imagine a path as shown on which a particle Q is travelling. We first arbitrarily fix a point called the pole P and also choose a line known as the datum line from where we will measure all angles. From the datum line we measure the angle to the line joining the pole and the particle, which is θ. We define the position vector as

$$\vec{r} = r\,\hat{e}_r$$

Where the direction of the unit vector is from the pole to the particle in the direction of increasing r. We also define a unit vector cap in the direction of increasing theta as shown in the figure. The velocity vector is found by differentiating with respect to time the position vector. We obtain

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\,\hat{e}_r + r\,\dot{\hat{e}}_r$$

Following the rules of differentiation of frame vectors we obtain

$$\dot{\hat{e}}_r = \lim_{\Delta t \to 0} \frac{\hat{e}_r(t+\Delta t) - \hat{e}_r(t)}{\Delta t}$$

$$= \dot{\theta}\,\hat{e}_\theta$$

Finally, we obtain the velocity vector as

$$\vec{v} = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta$$

Now we obtain the acceleration vector by differentiating the velocity vector

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\,\hat{e}_r + \dot{r}\,\dot{\hat{e}}_r + \dot{r}\dot{\theta}\,\hat{e}_\theta + r\ddot{\theta}\,\hat{e}_\theta + r\dot{\theta}\,\dot{\hat{e}}_\theta$$

Again following the rules of differentiation of frame vectors we obtain

$$\frac{d\hat{e}_r}{dt} = \dot{\theta}\,\hat{e}_\theta$$

And finally

$$\vec{a} = (\underbrace{\ddot{r}}_{\text{radial}} \underbrace{- r\dot{\theta}^2}_{\text{centrepetal}})\,\hat{e}_r + (\underbrace{r\ddot{\theta}}_{\text{tangential}} + \underbrace{2\dot{r}\dot{\theta}}_{\text{Coriolis}})\,\hat{e}_\theta$$

where the names of the different acceleration terms are indicated.

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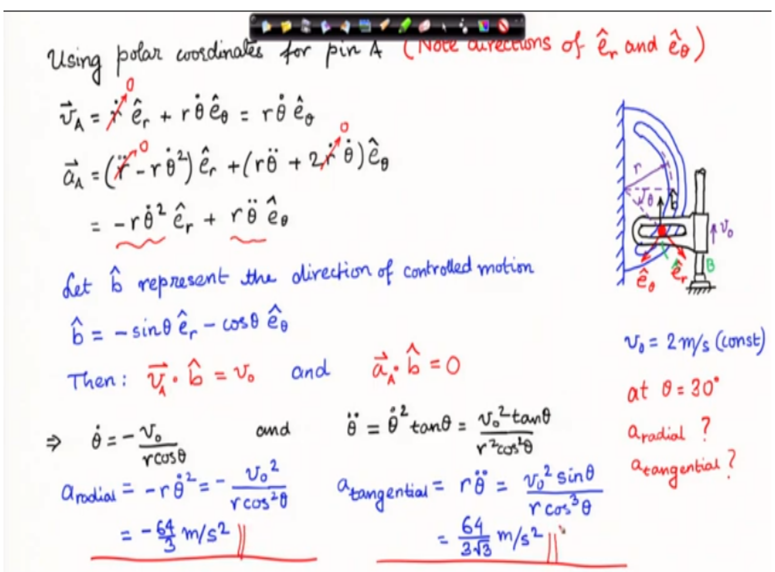
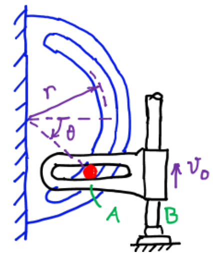


We consider a problem.

The pin A moves in a semi-circular slot, driven by a horizontal slot moving upward with a constant velocity of v_0=2 m/s. Calculate the normal and tangential components of the acceleration of the pin as it passes the position with θ=30 deg.



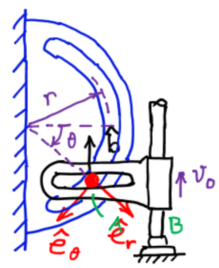
Solution:

Using polar coordinates for pin A (Note directions of $\hat{e}_r$ and $\hat{e}_\theta$)

$$\vec{v}_A = \cancel{\dot{r}}^{\,0}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta = r\dot{\theta}\,\hat{e}_\theta$$

$$\vec{a}_A = (\cancel{\ddot{r}}^{\,0} - r\dot{\theta}^2)\,\hat{e}_r + (r\ddot{\theta} + 2\cancel{\dot{r}}^{\,0}\dot{\theta})\,\hat{e}_\theta$$

$$= -r\dot{\theta}^2\,\hat{e}_r + r\ddot{\theta}\,\hat{e}_\theta$$



Let $\hat{b}$ represent the direction of controlled motion

$$\hat{b} = -\sin\theta\,\hat{e}_r - \cos\theta\,\hat{e}_\theta$$

$v_0 = 2\,m/s$ (const)

at $\theta = 30°$

a_{radial} ?

$a_{tangential}$?

Then: $\vec{v}_A \cdot \hat{b} = v_0$ and $\vec{a}_A \cdot \hat{b} = 0$

$\Rightarrow\ -r\dot{\theta}\cos\theta = v_0$ and $r\dot{\theta}^2\sin\theta - r\ddot{\theta}\cos\theta = 0$

$\Rightarrow\ \dot{\theta} = -\dfrac{v_0}{r\cos\theta}$ and $\ddot{\theta} = \dot{\theta}^2\tan\theta = \dfrac{v_0^2\tan\theta}{r^2\cos^2\theta}$

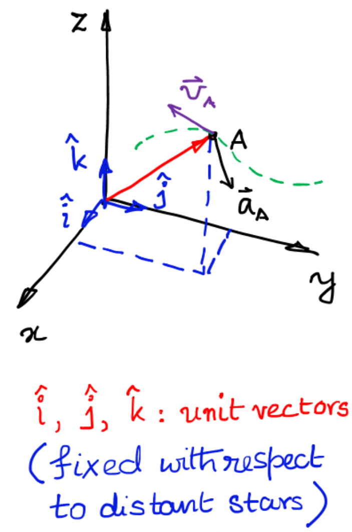
Plugging in the values, we obtain

$$a_{radial} = -r\dot{\theta}^2 = -\frac{v_0^2}{r\cos^2\theta} = -\frac{64}{3}\,m/s^2$$

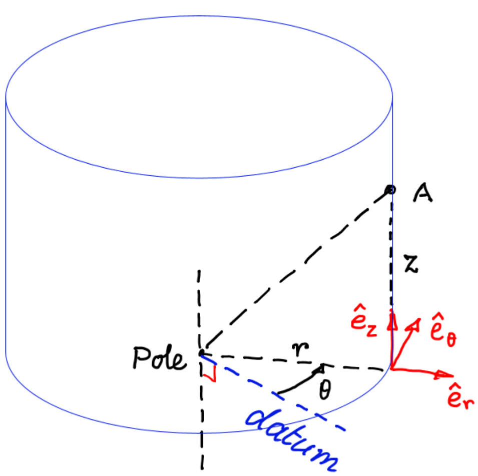
$$a_{tangential} = r\ddot{\theta} = \frac{v_0^2\sin\theta}{r\cos^3\theta} = \frac{64}{3\sqrt{3}}\,m/s^2$$

Next we move to coordinate systems in 3 dimensions.

$$\vec{p}_A = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$$
$$\vec{v}_A = \dot{x}\,\hat{i} + \dot{y}\,\hat{j} + \dot{z}\,\hat{k}$$
$$\vec{a}_A = \ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \ddot{z}\,\hat{k}$$



This is the Cartesian coordinate system. Again, the unit vectors are fixed with respect to distant stars. The position, velocity and acceleration vectors are shown. Note that the derivative of the frame vectors is zero because they are fixed with respect to distance stars.



Let us move to the cylindrical polar coordinates. Imagine a particle A moving in certain way right now located on a hypothetical cylinder with an axis as shown here by this dashed black line.

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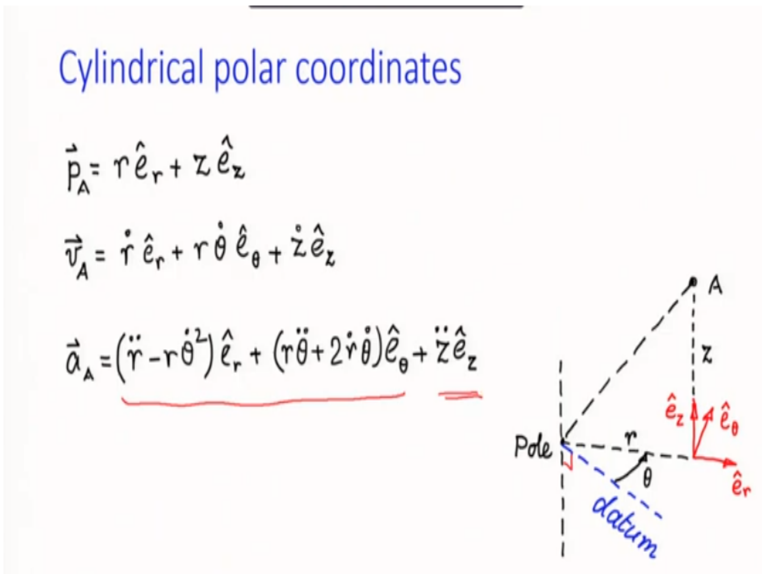
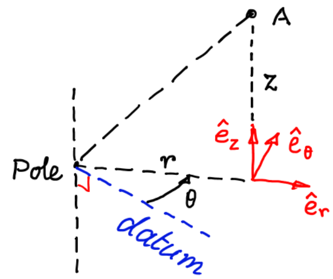


We choose arbitrarily a pole on this axis of the cylinder and I join from the pole to A by a dashed line then I define a datum line which is perpendicular to the axis of the cylinder in an arbitrary direction. Then we drop down from A, a line which is parallel to the axis of the cylinder till it meets the plane of the datum line. Our coordinates here are (r, θ, z) with the corresponding unit vectors as shown.

$$\vec{p}_A = r\hat{e}_r + z\hat{e}_z$$

$$\vec{v}_A = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + \dot{z}\,\hat{e}_z$$

$$\vec{a}_A = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_\theta + \ddot{z}\,\hat{e}_z$$



We can define the position, velocity and acceleration vectors as shown, where we use the rules of time derivative of the unit frame vectors as discussed for plane polar coordinates. Note that the unit frame vector along the z-coordinate direction is non rotating. Hence, its time derivative vanishes.

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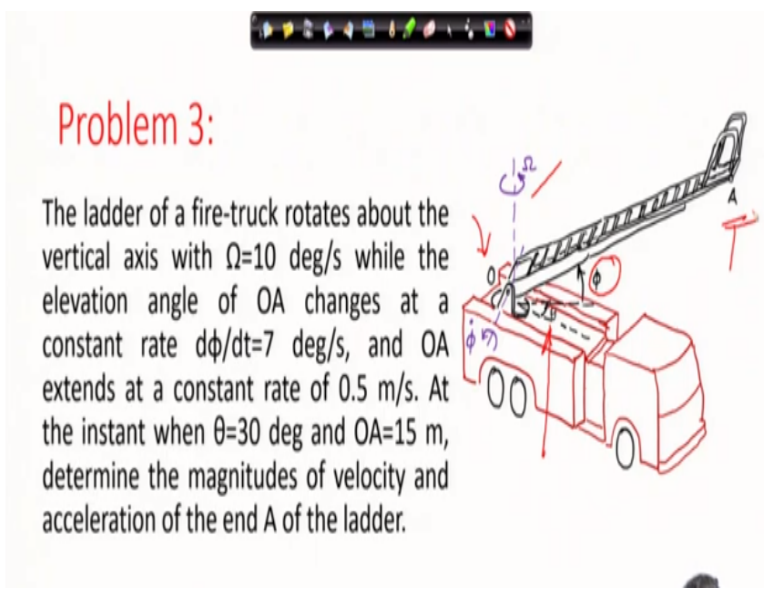

Let us look at the above problem and solve in the cylindrical polar coordinates.

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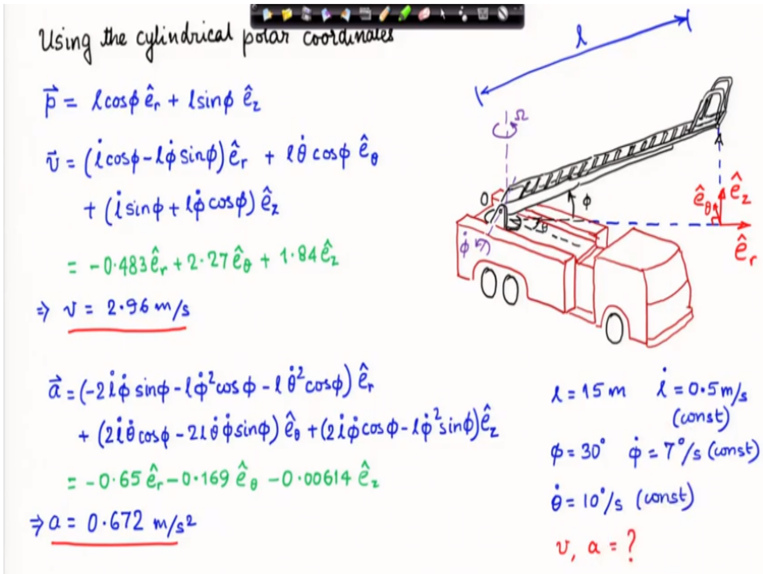

I will use the cylindrical polar coordinates here. The position vector is given as

$$\vec{p} = l\cos\phi\,\hat{e}_r + l\sin\phi\,\hat{e}_z$$

The velocity vector is obtained as

$$\vec{v} = (\dot{l}\cos\phi - l\dot{\phi}\sin\phi)\,\hat{e}_r + l\dot{\theta}\cos\phi\,\hat{e}_\theta + (\dot{l}\sin\phi + l\dot{\phi}\cos\phi)\,\hat{e}_z$$

$$= -0.483\,\hat{e}_r + 2.27\,\hat{e}_\theta + 1.84\,\hat{e}_z$$

$$\Rightarrow \underline{v = 2.96\ m/s}$$

Similarly, the acceleration vector is obtained as

$$\vec{a} = (-2\dot{l}\dot{\phi}\sin\phi - l\dot{\phi}^2\cos\phi - l\dot{\theta}^2\cos\phi)\,\hat{e}_r + (2\dot{l}\dot{\theta}\cos\phi - 2l\dot{\theta}\dot{\phi}\sin\phi)\,\hat{e}_\theta + (2\dot{l}\dot{\phi}\cos\phi - l\dot{\phi}^2\sin\phi)\,\hat{e}_z$$

$$= -0.65\,\hat{e}_r - 0.169\,\hat{e}_\theta - 0.00614\,\hat{e}_z$$

$$\Rightarrow \underline{a = 0.672\ m/s^2}$$

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So to summarize we have discussed the Cartesian tangent normal and the polar coordinate systems both in 2D and 3D and we have discussed some problems.