

Advanced Dynamics
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Module No # 04
Lecture No # 19
Central Force Motion – III

In this lecture we are going to continue our discussions on central force field motion and we are going to look at some problems. The theory is first recapitulated in the following slides.

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Central force motion

$$\frac{d^2 u}{d\theta^2} + u = \frac{GM}{h^2}$$

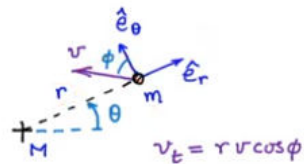
General solution $u = \frac{1}{r} = C \cos(\theta + \delta) + \frac{GM}{h^2}$

C, δ : constants of integration

At $\theta = 0, r = r_{\min} \Rightarrow \delta = 0$

$$\Rightarrow \frac{1}{r} = C \cos \theta + \frac{GM}{h^2}$$

$$h = r^2 \dot{\theta} = r v_{\perp} \quad (v_{\perp} = r \dot{\theta} \text{ tangential velocity})$$



In the above slide the equation of motion of a particle in a central force field is presented along with the solution of the motion.

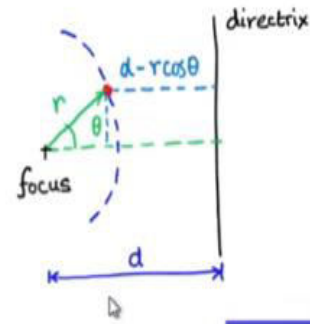
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Conic sections

General representation of conic sections (Geometry)

$$\frac{r}{d - r \cos \theta} = e \text{ (constant) eccentricity}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{d} \cos \theta + \frac{1}{ed}$$



The mathematical representation of conic sections is presented in the above slide. A similarity of this representation with the structure of solution of a massive particle in a central force field is clearly observed.

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Circular path

$$\frac{1}{r} = C \cos \theta + \frac{GM}{h^2} \quad (\text{Kinetics})$$

$$\text{Comparing} \quad d = \frac{1}{C} \quad e = \frac{Ch^2}{GM} \quad (ed = \frac{h^2}{GM})$$

(i) $C = 0 \Rightarrow e = 0$: $r = r_c$ (constant) Circle

$$r_c = \frac{h^2}{GM} = \frac{r_c^4 \dot{\theta}^2}{GM} = \frac{r_c^2 v_c^2}{GM} \quad (v_t = r_c \dot{\theta} = v_c)$$

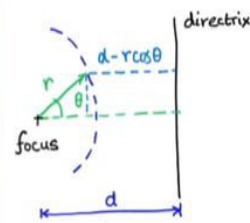
$$\Rightarrow \boxed{v_c = \sqrt{\frac{GM}{r_c}}} \quad (\text{Alternative: } \frac{mv_c^2}{r_c} = \frac{GMm}{r_c^2})$$

(Orbital speed)

$$\text{Orbital period } T: \dot{\theta} = \frac{2\pi}{T} \Rightarrow \frac{T^2}{r_c^3} = \frac{4\pi^2}{GM} = \text{constant} \quad (\text{Kepler's 3rd law})$$

Geometry:

$$\frac{1}{r} = \frac{1}{d} \cos \theta + \frac{1}{ed}$$

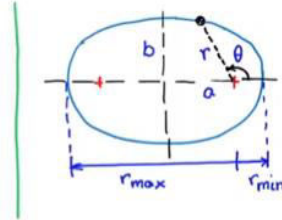


A comparison between the conic section and the solution is presented above. The characteristics of a circular trajectory is also shown.

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Elliptic path

$$\left. \begin{aligned} \frac{1}{r} &= C \cos \theta + \frac{GM}{h^2} \\ \frac{1}{r} &= \frac{1}{d} \left(\cos \theta + \frac{1}{e} \right) \end{aligned} \right\} d = \frac{1}{C} \quad e = \frac{Ch^2}{GM}$$



(ii) $0 < e < 1$ ($\frac{1}{r} \neq 0$ bounded motion) Ellipse

$$a = \frac{ed}{1-e^2} \quad b = a\sqrt{1-e^2}$$

$$\frac{1}{r_{\min}} + \frac{1}{r_{\max}} = \frac{2GM}{h^2}$$

$$r_{\min} = a(1-e) \quad r_{\max} = a(1+e)$$

Orbital period

$$T = \frac{A}{\dot{A}} = \frac{\pi ab}{\frac{1}{2} r^2 \dot{\theta}} = \frac{\pi ab}{h/2}$$

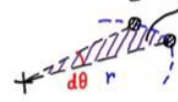
$$= \frac{2\pi a^2 \sqrt{1-e^2}}{h} = \frac{2\pi a^2}{h} \sqrt{\frac{ed}{a}}$$

$$\Rightarrow T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

$$\Rightarrow \boxed{\frac{T^2}{a^3} = \frac{4\pi^2}{GM}}$$

Kepler's 3rd law

$$dA = \frac{1}{2} r(r d\theta)$$

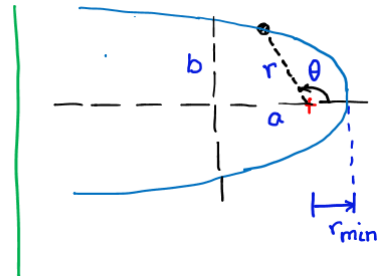


Characteristics of the elliptic path are presented above.

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Parabolic path

$$\left. \begin{aligned} \frac{1}{r} &= C \cos \theta + \frac{GM}{h^2} \\ \frac{1}{r} &= \frac{1}{d} \left(\cos \theta + \frac{1}{e} \right) \end{aligned} \right\} d = \frac{1}{C} \quad e = \frac{Ch^2}{GM}$$



(iii) $e = 1$ ($\frac{1}{r} = 0$ at $\theta = \pi$: unbounded motion) Parabola

$$\frac{GM}{h^2} = C \Rightarrow r = \frac{h^2}{GM(1+\cos \theta)}$$

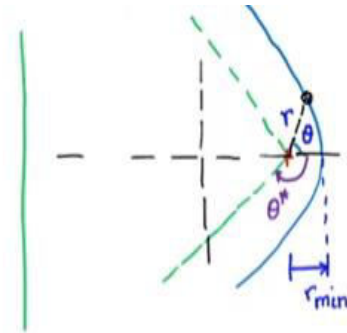
$$r_{\min} = \frac{h^2}{2GM}$$

Characteristics of the parabolic path are presented above.

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Hyperbolic path

$$\left. \begin{aligned} \frac{1}{r} &= C \cos \theta + \frac{GM}{h^2} \\ \frac{1}{r} &= \frac{1}{d} \left(\cos \theta + \frac{1}{e} \right) \end{aligned} \right\} d = \frac{1}{C} \quad e = \frac{Ch^2}{GM}$$

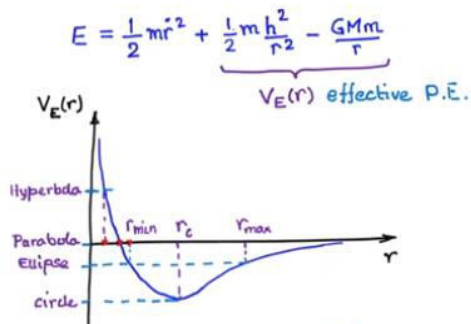


(iv) $e > 1$ ($\frac{1}{r} = 0$ at $\theta^* = \pm \cos^{-1}(-\frac{1}{e})$: unbounded motion) Hyperbola

Characteristics of the hyperbolic path are presented in the slide above.

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Central force motion: Energetics



For circle $\frac{dV_E}{dr} = 0 \Rightarrow -\frac{r_c^2 v_t^2}{r_c^3} + \frac{GM}{r_c^2} = 0$
 $\Rightarrow v_t = \sqrt{\frac{GM}{r_c}} \quad (h = r_c v_t)$

Trajectory	e	E
Circle	$e=0$	$-\frac{G^2 M^2 m}{2 h^2}$
Ellipse	$0 < e < 1$	$-\frac{G^2 M^2 m}{2 h^2} < E < 0$
Parabola	$e=1$	$E=0$
Hyperbola	$e>1$	$E>0$



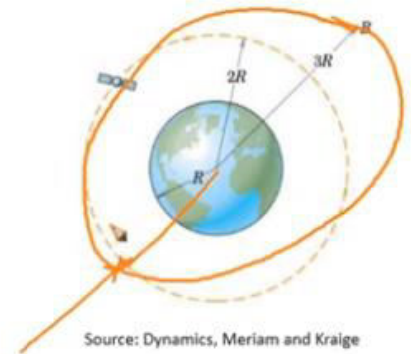
The energetics of motion of a massive particle in a central force field is presented in the slide above. We plotted and discussed the effective potential energy of the particle. We found that, for bounded trajectories, the effective energy is negative, while for unbounded trajectories it is 0 or greater. We also calculated escape velocity etc.

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Wec

Problem 1:

A satellite is in a circular earth orbit of radius $2R$, where R is the radius of the earth. What is the minimum velocity boost Δv necessary to reach point B , which is a distance $3R$ from the center of the earth? At what point in the original circular orbit should the velocity increment be added?



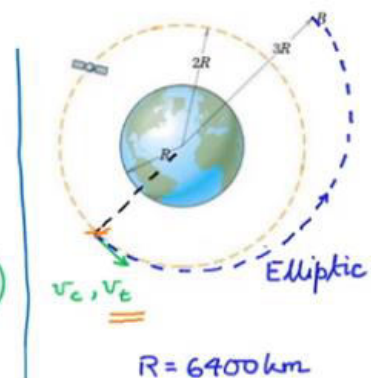
We consider the above problem. An important consideration is the determination of minimum velocity boost. This will happen when the satellite will reach B as the furthest point on the new trajectory. Thus, the new trajectory should be an ellipse with B as the apogee. And therefore the diametrically opposite point should be the perigee.

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Circular trajectory: $v_c = \sqrt{\frac{GM_E}{2R}} = \sqrt{\frac{g_E R}{2}} = \sqrt{\frac{g_E R}{2}}$
 $\Rightarrow v_c = 5603 \text{ m/s}$

Elliptic trajectory: $\frac{1}{r} = C \cos \theta + \frac{GM_E}{h^2}$
 $\Rightarrow \frac{1}{r_{\max}} + \frac{1}{r_{\min}} = \frac{2GM_E}{h^2} = \frac{2g_E R^2}{(2Rv_t)^2} = \frac{g_E}{2v_t^2} \quad (h = r^2 \dot{\theta})$
 $\Rightarrow v_t = \sqrt{\frac{3}{5} g_E R} = 6138 \text{ m/s} \quad (r_{\max} = 3R, r_{\min} = 2R)$

$\Delta v = v_t - v_c = 535 \text{ m/s}$

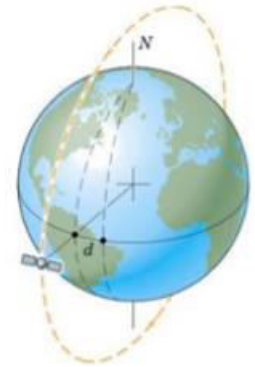


With these considerations, the solution is presented in the slide above.

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Problem 2:

A satellite is in a circular polar orbit of altitude 300 km. Determine the separation d at the equator between the ground tracks (shown dashed) associated with two successive overhead passes of the satellite.



Source: Dynamics, Meriam and Kraige

The next problem is presented in the slide above.

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Diagram illustrating the satellite orbit and ground tracks, showing the separation d at the equator between two successive overhead passes.

Handwritten calculations for the satellite orbit:

$$r_c = 6400 + 300 = 6700 \text{ km}$$
$$T = 2\pi \sqrt{\frac{r_c^3}{GM_E}} = 2\pi \sqrt{\frac{r_c^3}{g_E R_E^2}} = 5436 \text{ sec} = 90.6 \text{ min}$$
$$\omega_E = \frac{2\pi}{24 \times 3600} = 7.27 \times 10^{-5} \text{ rad/s}$$
$$d = R_E \omega_E T = 2529 \text{ km}$$

Diagram labels:

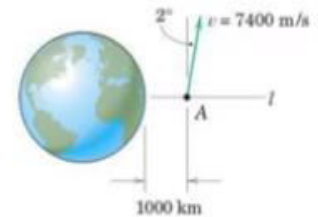
- $R_E = 6400 \text{ km}$
- $h_s = 300 \text{ km}$
- $d = ?$

The solution is presented in the slide above.

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Problem 3:

A spacecraft in an elliptical orbit has the position and velocity indicated in the figure at a certain instant. Determine the semimajor axis length a of the orbit and find the acute angle α between the semi-major axis and the line l . Does the spacecraft eventually strike the earth?



Source: Dynamics, Meriam and Kraige

The next problem is shown in the slide above. We have to calculate the trajectory of the spacecraft and also check whether it strikes earth or not. It will strike the earth if the minimum separation for the center of the earth falls below the radius of the earth.

$$h = r^2 \dot{\theta} = r v_t$$

$$\Rightarrow h = (74 \times 10^5) 7400 \cos 2^\circ = 5.473 \times 10^{10} \text{ m}^2/\text{s}$$

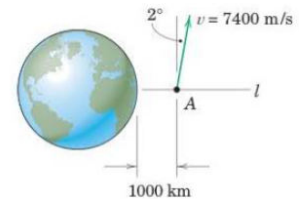
$$\dot{r} = v \sin 2^\circ = 258.26 \text{ m/s}$$

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r} = \frac{1}{2} m v^2 - \frac{g_E R_E^2}{r} m$$
$$= -2.69 \times 10^7 \text{ m} \quad (\text{constant})$$

At perigee and apogee

$$E = \frac{1}{2} m \frac{h^2}{r^2} - \frac{g_E R_E^2}{r} m = -2.69 \times 10^7 \text{ m}$$

$$\Rightarrow 2.69 \times 10^7 r^2 - 4.018 \times 10^{14} r + 1.4978 \times 10^{21} = 0$$



$$R_E = 6400 \text{ km}$$

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$$2.69 \times 10^7 r^2 - 4.018 \times 10^7 r + 1.4978 \times 10^7 = 0$$

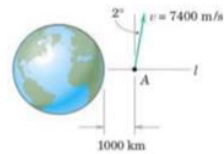
$$\Rightarrow r = \frac{1}{5.38} \left[4.018 \times 10^7 \pm \sqrt{4.018^2 - 4(2.69)(2.995) \times 10^7} \right]$$

$$= (0.7468 \pm 0.0311) \times 10^7 \text{ m}$$

$$\Rightarrow r_{\max} = 7779 \text{ km} \quad r_{\min} = 7157 \text{ km}$$

Since $r_{\min} > R_E$, the satellite will not hit the Earth, but move on an elliptic trajectory.

$$\text{Eccentricity } e = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} = 0.0416$$

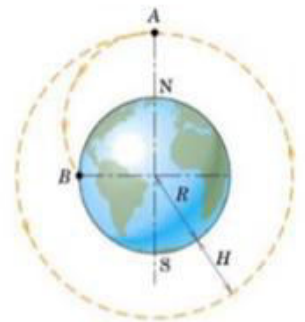


The solution is presented in the 2 slides above.

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Problem 4:

A satellite is placed in a circular polar orbit a distance H above the earth. As the satellite goes over the north pole at A , its retro-rocket is activated to produce a burst of negative thrust which reduces its velocity to a value which will ensure an equatorial landing. Derive the expression for the required reduction Δv_A of velocity at A . Note that A is the apogee of the elliptical path.



Source: Dynamics, Meriam and Kraige

The next problem is presented above. It is to be understood that the path will be elliptic with A as the apogee.

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Circular trajectory: $v_A^C = \sqrt{\frac{GM_E}{R+H}}$

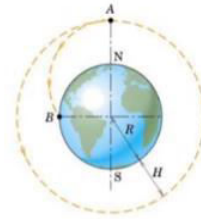
Elliptic trajectory: $\frac{1}{r} = C \cos \theta + \frac{GM_E}{h^2}$

At $\theta = -\frac{\pi}{2}$ $r = R \Rightarrow \frac{1}{R} = \frac{GM_E}{h^2}$

$\Rightarrow h = \sqrt{GM_E R} = (R+H) v_A^E$

$\Rightarrow v_A^E = \frac{\sqrt{GM_E R}}{R+H}$

$\Delta v_A = v_A^E - v_A^C = \left(\sqrt{\frac{R}{R+H}} - 1 \right) \sqrt{\frac{GM_E}{R+H}}$



The solution is presented in the slide above.

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Overview

- Newton's law of gravitation
- Particle motion under a central force
- Problems

The summary is presented in the slide above.