

Advanced Dynamics
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Module No # 04
Lecture No # 18
Central Force Motion – II

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Overview

- Newton's law of gravitation
- Particle orbits under a central force
- Energetics

We will continue our discussions on central force motion.

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Central force motion

$$\frac{d^2 u}{d\theta^2} + u = \frac{GM}{h^2}$$

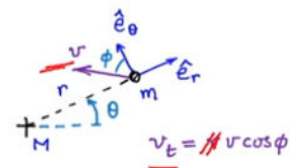
General Solution $u = \frac{1}{r} = C \cos(\theta + \delta) + \frac{GM}{h^2}$

C, δ : constants of integration

At $\theta = 0, r = r_{\min} \Rightarrow \delta = 0$

$$\Rightarrow \frac{1}{r} = C \cos \theta + \frac{GM}{h^2}$$

$$h = r^2 \dot{\theta} = r v_t \quad (v_t = r \dot{\theta} \text{ tangential velocity})$$



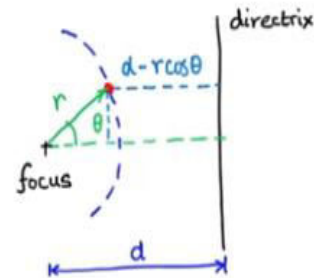
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Conic sections

General representation of conic sections (Geometry)

$$\frac{r}{d-r\cos\theta} = e \text{ (constant) eccentricity}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{d}\cos\theta + \frac{1}{ed}$$

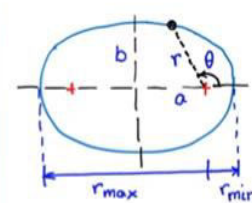


The 2 slides above recapitulate the discussions on the equation of motion of a particle in a central force field, and also presents its solution. This solution is compared with the general representation of conic sections.

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Elliptic path

$$\left. \begin{aligned} \frac{1}{r} &= C \cos\theta + \frac{GM}{h^2} \\ \frac{1}{r} &= \frac{1}{d} \left(\cos\theta + \frac{1}{e} \right) \end{aligned} \right\} d = \frac{1}{C} \quad e = \frac{Ch^2}{GM}$$



(ii) $0 < e < 1$ ($\frac{1}{r} \neq 0$ bounded motion) Ellipse

$$d = \frac{a(1-e^2)}{e}$$

Perigee/Perihelion ($\theta = 0$)

$$r_{\min} = \frac{1}{C + \frac{GM}{h^2}} = \frac{ed}{1+e}$$

Apogee/Aphelion ($\theta = \pi$)

$$r_{\max} = \frac{1}{-C + \frac{GM}{h^2}} = \frac{ed}{1-e}$$

$$r_{\min} + r_{\max} = 2a = \frac{2ed}{1-e^2}$$

$$\Rightarrow a = \frac{ed}{1-e^2}$$

$$b = a\sqrt{1-e^2}$$

$$\frac{1}{r_{\min}} + \frac{1}{r_{\max}} = \frac{2GM}{h^2}$$

$$\begin{aligned} r_{\min} &= a(1-e) \\ r_{\max} &= a(1+e) \end{aligned}$$

$$\frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}} = e$$

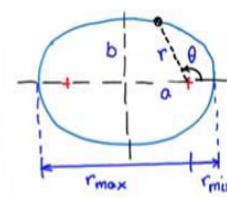
The above slide presents the characteristics of an elliptic path.

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Elliptic path

$$\left. \begin{aligned} \frac{1}{r} &= C \cos \theta + \frac{GM}{h^2} \\ \frac{1}{r} &= \frac{1}{d} \left(\cos \theta + \frac{1}{e} \right) \end{aligned} \right\} d = \frac{1}{C} \quad e = \frac{Ch^2}{GM}$$

$$ed = \frac{h^2}{GM}$$



(ii) $0 < e < 1$ ($\frac{1}{r} \neq 0$ bounded motion) Ellipse

$$a = \frac{ed}{1-e^2}$$

$$b = a\sqrt{1-e^2}$$

$$\frac{1}{r_{min}} + \frac{1}{r_{max}} = \frac{2GM}{h^2}$$

$$r_{min} = a(1-e) \quad r_{max} = a(1+e)$$

Orbital period

$$T = \frac{A}{\dot{A}} = \frac{\pi ab}{\frac{1}{2} r^2 \dot{\theta}} = \frac{\pi ab}{h/2}$$

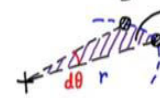
$$= \frac{2\pi a^2 \sqrt{1-e^2}}{h} = \frac{2\pi a^2}{h} \sqrt{\frac{ed}{a}}$$

$$\Rightarrow T = \frac{2\pi a^{3/2}}{\sqrt{GM}}$$

$$\Rightarrow \frac{T^2}{a^3} = \frac{4\pi^2}{GM}$$

Kepler's 3rd law

$$dA = \frac{1}{2} r(r d\theta)$$

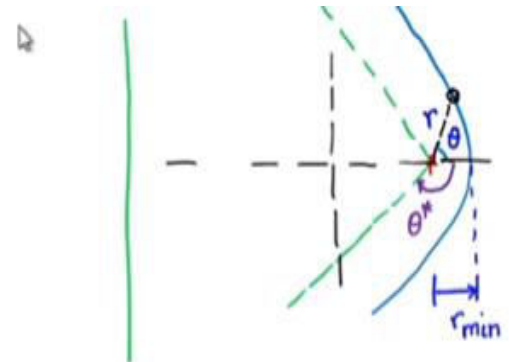


The expression of orbital time period in terms of the semi-major axis of the ellipse clearly shows the validity of Kepler's 3rd law.

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Hyperbolic path

$$\left. \begin{aligned} \frac{1}{r} &= C \cos \theta + \frac{GM}{h^2} \\ \frac{1}{r} &= \frac{1}{d} \left(\cos \theta + \frac{1}{e} \right) \end{aligned} \right\} d = \frac{1}{C} \quad e = \frac{Ch^2}{GM}$$



(iv) $e > 1$ ($\frac{1}{r} = 0$ at $\theta^* = \pm \cos^{-1}(-\frac{1}{e})$: unbounded motion) Hyperbola

The characteristics of a hyperbolic trajectory is shown above.

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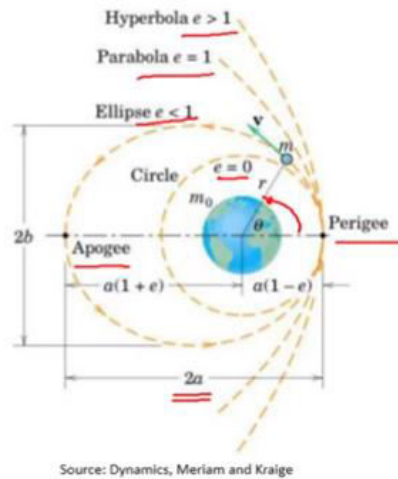
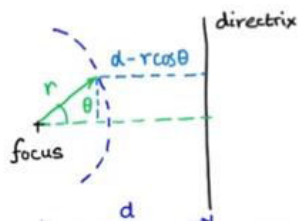
Central force motion

Kinetics:

$$\frac{1}{r} = C \cos \theta + \frac{GM}{h^2}$$

Geometry: conic section

$$\frac{1}{r} = \frac{1}{d} \cos \theta + \frac{1}{ea}$$



The above slide puts together all kinds of possibilities of motion of a particle under a central force field.

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Central force motion: Energetics

Equation of motion

$$m \ddot{\mathbf{a}}_m = \mathbf{F} = -\frac{GMm}{r^2} \hat{\mathbf{e}}_r$$

$$\ddot{\mathbf{a}}_m = (\ddot{r} - r\dot{\theta}^2) \hat{\mathbf{e}}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) \hat{\mathbf{e}}_\theta$$

Dot product with velocity $\dot{\mathbf{v}} = \dot{r} \hat{\mathbf{e}}_r + r\dot{\theta} \hat{\mathbf{e}}_\theta$

$$m \dot{r} (\ddot{r} - r\dot{\theta}^2) + m r \dot{\theta} (r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -\frac{GMm}{r^2} \dot{r}$$

$$\Rightarrow m (\dot{r}\ddot{r} + r\dot{\theta}^2 + r^2\dot{\theta}\ddot{\theta}) = -\frac{GMm}{r^2} \dot{r}$$

$$\Rightarrow \frac{d}{dt} \left(\underbrace{\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2}_{T = \frac{1}{2} m \dot{\mathbf{v}} \cdot \dot{\mathbf{v}}} \right) = \frac{d}{dt} \left(\underbrace{\frac{GMm}{r}}_{-V_g} \right) \Rightarrow \boxed{\frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{GMm}{r} = E \text{ (constant)}}$$

Now we come to the energetics. The energy relation is derived starting from Newton's 2nd law, as shown in the slide above.

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Central force motion: Energetics

$$E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{GMm}{r} \quad \left. \begin{array}{l} \\ r^2 \dot{\theta} = h \text{ (constant)} \end{array} \right\} E = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m \frac{h^2}{r^2} - \frac{GMm}{r}$$

$$\frac{1}{r} = C \cos \theta + \frac{GM}{h^2} \quad \left. \begin{array}{l} ed = \frac{h^2}{GM} \\ \text{At } \theta = 0, r = r_{\min} = \frac{ed}{1+e} \end{array} \right\} r_{\min} = \frac{h^2}{GM(1+e)}$$

$$\frac{1}{r} = \frac{1}{d} \left(\cos \theta + \frac{1}{e} \right) \quad \text{at } r = r_{\min} \quad \vec{r} = 0$$

Substituting in E at $r = r_{\min}$ ($\dot{r} = 0$)

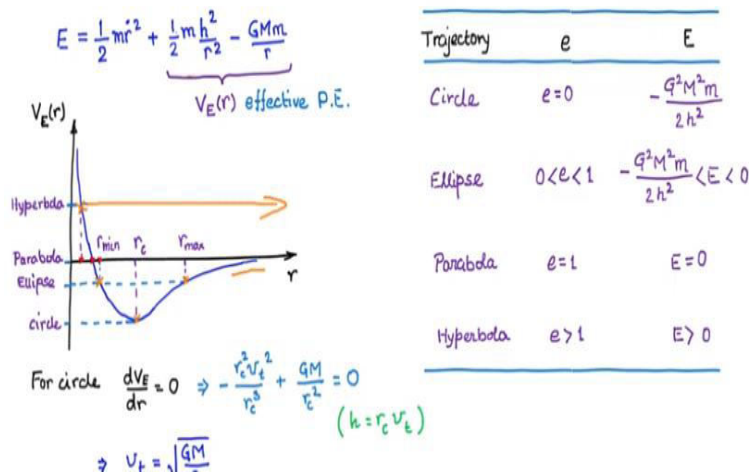
$$E = \frac{1}{2} m \frac{h^2}{r_{\min}^2} - \frac{GMm}{r_{\min}} = \frac{GM^2 m}{2h^2} (e^2 - 1) \Rightarrow E = \frac{GM^2 m}{2h^2} (e^2 - 1)$$

$$\Rightarrow e = \sqrt{1 + \frac{2Eh^2}{GM^2 m}}$$

The relation of eccentricity in terms of the energy of the particle is presented in the slide above.

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Central force motion: Energetics



The table above summarizes the eccentricity of the path and energy of the particle on the different kinds of paths possible. It also shows the effective potential energy landscape, and the possibilities of various paths depending on the level of effective potential energy.

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Overview

- Newton's law of gravitation
- Particle orbits under a central force
- Energetics

The summary is provided above.