Advanced Dynamics
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Module No # 03
Lecture No # 11
Work-Energy Relation – I

In this lecture we are going to discuss the work energy relation.

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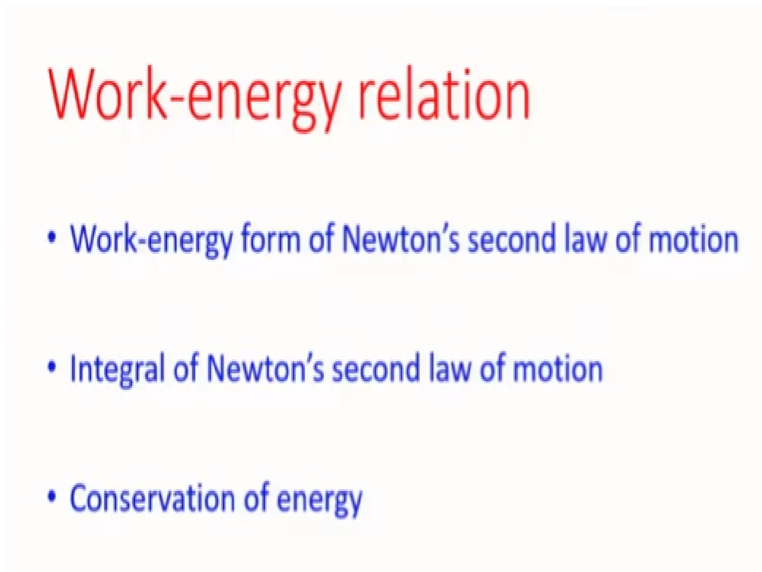

To understand the work energy relation we have to look at the Newton's second law of motion. The work energy relation is an integral of Newton second law. Therefore it will not generate any new information that is not already present in Newton's second law.

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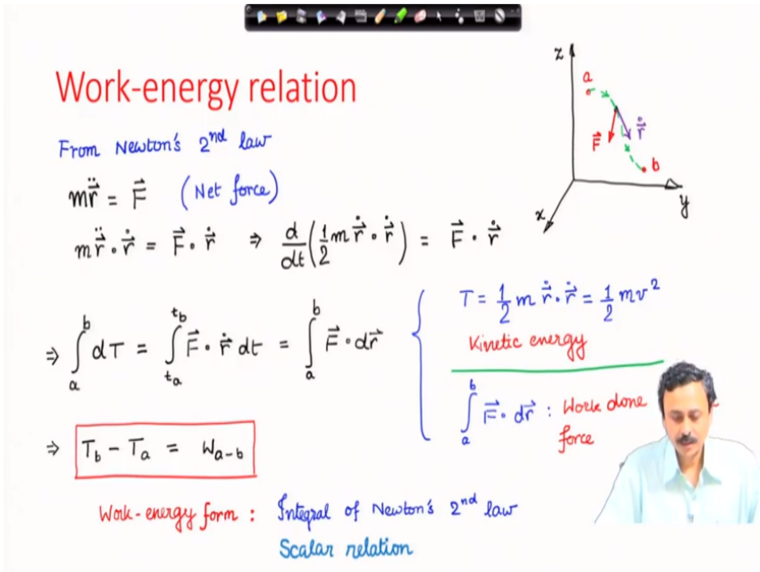

Let us start with the Newton's second law for a constant mass system. The equation of motion of a particle moving on a path from point a to point b can be written as shown in the slide above. Taking dot product of the equation of motion with the velocity vector we have

$$m\ddot{\vec{r}}\cdot\dot{\vec{r}} = \vec{F}\cdot\dot{\vec{r}}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}}\right) = \vec{F}\cdot\dot{\vec{r}}$$

We integrate this equation over the path of the particle from point a to b to obtain

$$\Rightarrow \int_a^b dT = \int_{t_a}^{t_b}\vec{F}\cdot\dot{\vec{r}}\,dt = \int_a^b \vec{F}\cdot d\vec{r}$$

$$\Rightarrow \boxed{T_b - T_a = W_{a-b}}$$

Thus, the work done by the net force acting on the particle as it moves from position a to position b is equal to the change in the kinetic energy of the particle. This is the known as the work energy form of the Newton's second law. This is an integral of the Newton second law because it involves one time derivative less than that in Newton second law. Newton's second law involves acceleration which is the double time derivative of the position vector. Whereas, in the work-energy relation, it involves only velocity. Further, the work energy form gives us a scalar relationship.

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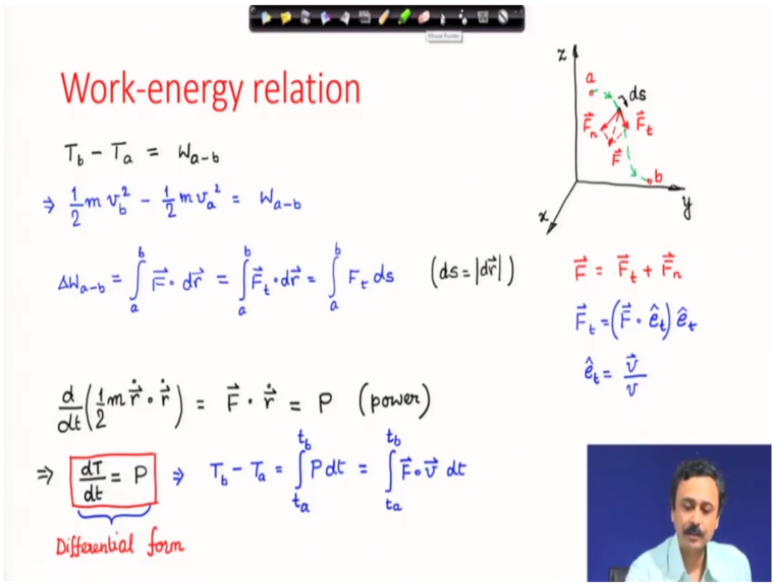

Now the expression of work done can be restructured by decomposed the force vector in the tangent and normal components with respect to the path. The decomposition is given by

$$\vec{F} = \vec{F}_t + \vec{F}_n$$

$$\vec{F}_t = \left(\vec{F}\cdot\hat{e}_t\right)\hat{e}_t$$

$$\hat{e}_t = \frac{\vec{v}}{v}$$

Using this decomposition one can then write

$$\Delta W_{a-b} = \int_a^b \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}_t \cdot d\vec{r} = \int_a^b F_t\, ds \qquad \left(ds = |d\vec{r}|\right)$$

We can also have a derivative form of the work energy relation as follows

$$\frac{d}{dt}\left(\frac{1}{2} m \dot{\vec{r}} \cdot \dot{\vec{r}}\right) = \vec{F} \cdot \dot{\vec{r}} = P \quad \text{(power)}$$

$$\Rightarrow \underbrace{\boxed{\frac{dT}{dt} = P}}_{\text{Differential form}} \Rightarrow T_b - T_a = \int_{t_a}^{t_b} P\, dt = \int_{t_a}^{t_b} \vec{F} \cdot \vec{v}\, dt$$

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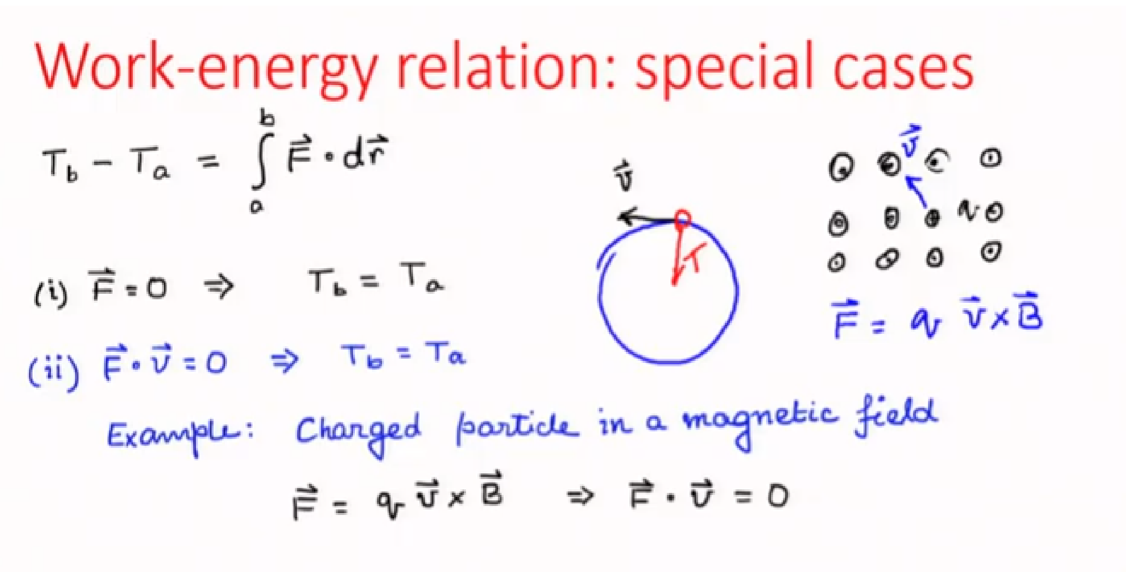

Let us look at some special cases of the work energy relation as shown above. The first and simplest special case is force equal to 0. Then, kinetic energy at position b must be equal to kinetic energy at position a. In other words, kinetic energy is conserved. This will also happen when the force vector is perpendicular to the velocity vector at all instants between the points a and b, as shown in the second case above. Examples of this case are a stone tied to one end of a string and rotated, and the motion of a charged particle in a magnetic field.

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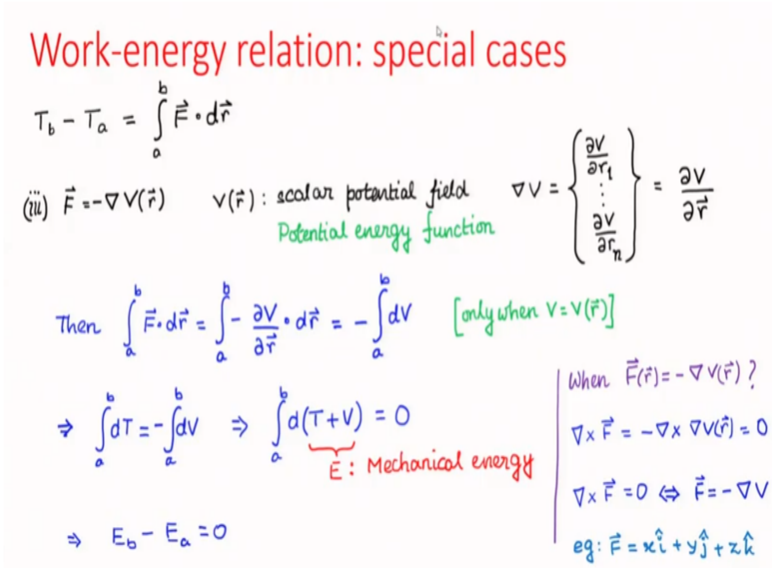

The third special case of the work energy relation is when the force F is negative gradient of a potential scalar potential V, as shown above. In this case, as derived above, the total mechanical energy written as E=T+V of the system is conserved.

Now the question arises, given a force field, when can the force field be represented as a negative gradient of a potential, i.e.,

When $\vec{F}(\vec{r}) = -\nabla V(\vec{r})$?

The answer is obtained by taking curl of both sides of this equation to obtain

$$\nabla\times\vec{F} = -\nabla\times\nabla V(\vec{r}) = 0$$

$$\nabla\times\vec{F} = 0 \Leftrightarrow \vec{F} = -\nabla V$$

Therefore if the curl of the force vector field is 0 then it can be represented as a negative gradient of a scalar field.

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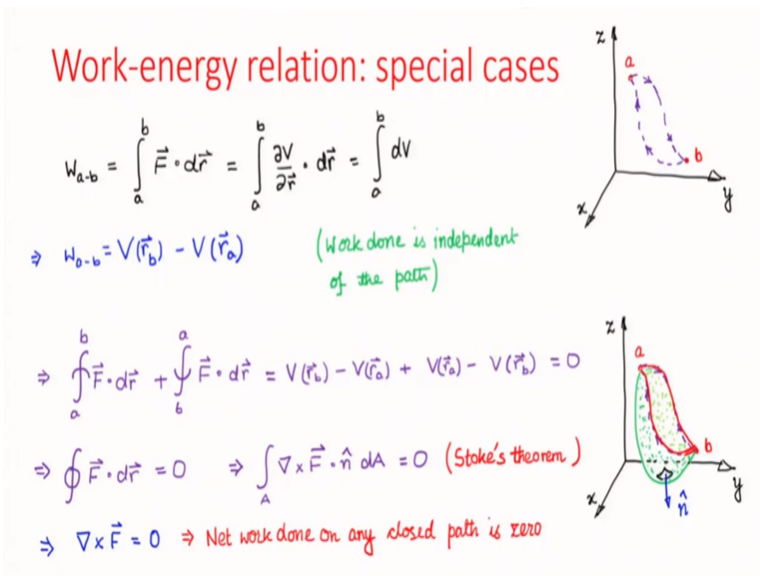

As shown above, the work done by the force in moving the particle from point a to b is nothing but the difference in the potential energy at the two point. In other words, the work done is independent of the path taken by the particle between the two points. Therefore, it follows that the net work done in moving from point a back to point a by any arbitrary path should be zero. This leads to

$$\oint \vec{F}\cdot d\vec{r} = 0 \quad \Rightarrow \int_A \nabla\times\vec{F}\cdot\hat{n}\, dA = 0 \quad \text{(Stoke's theorem)}$$

$$\Rightarrow \nabla\times\vec{F} = 0 \Rightarrow \text{Net work done on any closed path is zero}$$

Thus, if the curl of the force F vanishes, the force is representable as a negative gradient of a scalar potential function.

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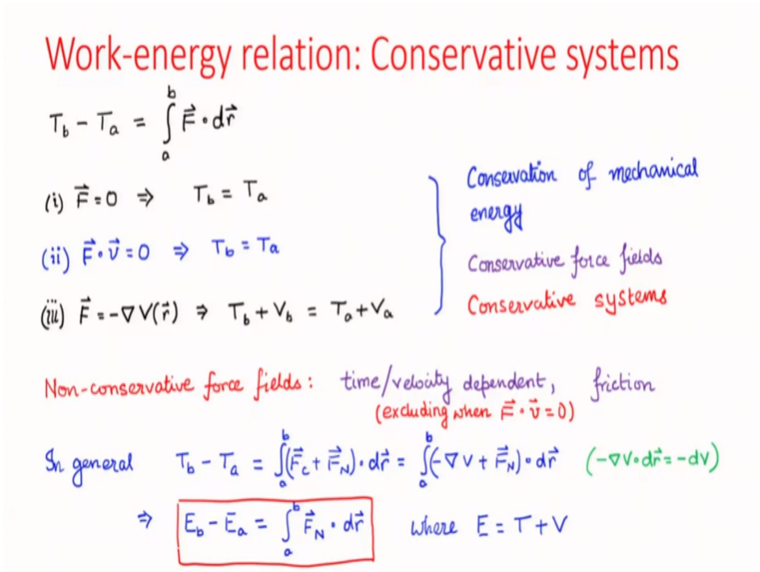

The above slide enumerates three cases of conservative systems, i.e., systems for which the mechanical energy is conserved. Force fields which are either always perpendicular to the velocity direction of the particle, or are representable a negative gradient of a scalar potential are conservative force fields. Also enumerated above are some non-conservative force fields. Whenever the force is function of time or velocity (excluding the case of the force on a charged particle in a magnetic field), it is non-conservative. Friction force is a common example of a non-conservative force.

The total force field, in general, can be divided into 2 parts: conservative force fields and non-conservative force fields. In that case, we can express the work-energy relation as

$$\boxed{E_b - E_a = \int_a^b \vec{F}_N \cdot d\vec{r}} \qquad \text{where } E = T + V$$

This is the most general form of work energy relation which says that the change in the total mechanical energy is equal to the work done by all non-conservative forces over the path.

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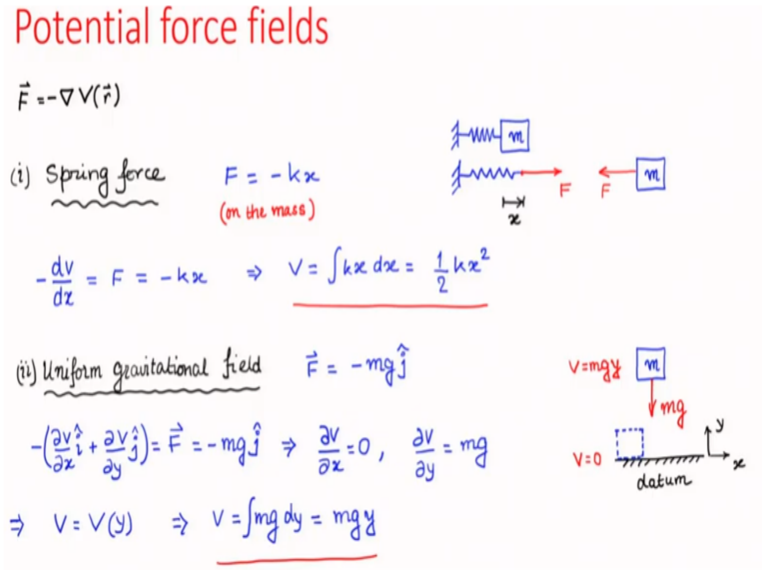

Let us look at some examples of potential force fields. The spring force F = -k x acts on the mass. The potential energy stored in a spring is presented in the slide above. The potential energy in a uniform gravitational field is also presented.

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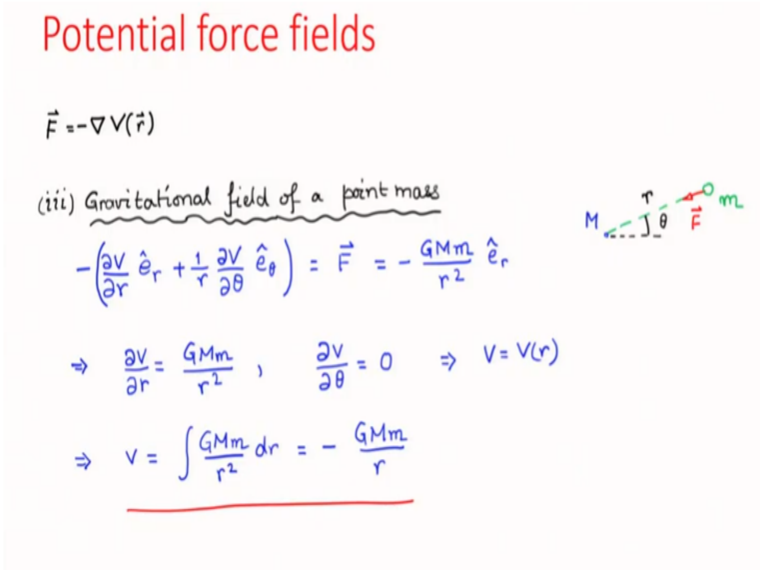

Let us look at gravitational field of a point mass M as shown above. Using the Newton's law of gravitation, one can determine the potential energy of a test mass at any radius r from the gravitating mass M, as shown in the above slide.

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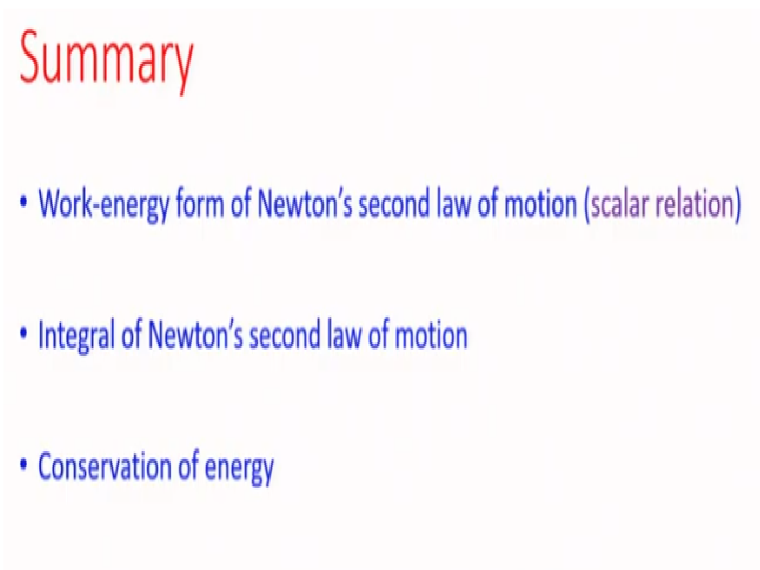

To summarize we have looked at the work energy form of Newton's second law. As mentioned, this relation is an integral of the Newton second law and does not generate any new information which is not already present in Newton second law. We have looked at conservation of energy, and discussed conservative force fields, and non-conservative force fields.