

Advanced Dynamics
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Module No # 01
Lecture No # 01
Coordinate Systems – I

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Dynamics

- Dynamics is a study of motion of particles/extended bodies consisting of two parts:

Kinematics: study of motion without any reference to forces.
Mass/inertia properties do not appear in kinematics problems.

Kinetics: study of effect of forces on motion. Involves mass/inertia properties of particles/bodies

To begin with, let us see what dynamics is all about. Dynamics is a study of motion of particles or extended bodies under the action of forces. This study consists of 2 parts. The first part is called kinematics which is the study of motion without any reference to forces. Because there are no forces involved it is purely a geometric study.

So inertia properties, mass of particles or inertia of the bodies, do not appear in the study of kinematics. On the other hand, the second part of the study of dynamics is kinetics. Kinetics is the study of effect of forces on the motion and hence it involves the inertia properties of bodies involved. So the 2 parts of dynamics consist of kinematics and kinetics. Now, out of these 2, kinematics is tricky, while kinetics is a straight forward step, more or less. So we are going to start with the study of kinematics which is the study of motion.

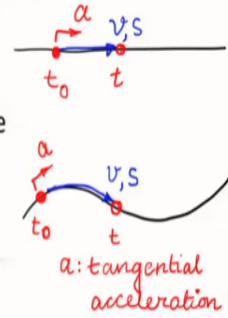
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Kinematics: Motion in 1 dimension

- **Given:** acceleration along the path
- **To determine:** speed and displacement/distance

Cases:

- Constant acceleration case
- Time dependent acceleration case
- Location dependent acceleration case



In this lecture, we are going to study motion in 1 dimension. We are going to look at commonly used coordinate systems and we are going to look at some problems. Whenever we say motion in one dimension, the typical picture in our mind is a particle moving on a straight path. But that need not always be the requirement; it could be constrained motion (1 degree of freedom) on a curved path but the acceleration specification must always be tangential to the path.

So, for constrained motion, if the acceleration is specified tangential to the path in that case it can be treated kinematically as 1 dimensional motion. And the displacement or distance traveled should be along the path. Velocity is always tangent to the path; the problem is specified with the acceleration along the path.

We are required to find out the speed which is the first integral of the acceleration and we are required to find out what is the displacement of the particle or the distance travelled by the particle along the path. We are going to look at some cases which you have seen right from schools. First case is the constant acceleration case, followed by the time dependent acceleration case and the location dependent acceleration case.

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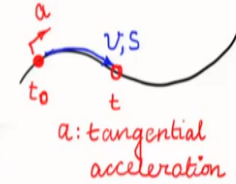
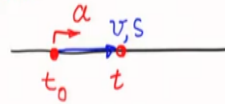
Kinematics: Motion in 1 dimension

- Constant acceleration case

$$\frac{dv}{dt} = a \text{ (constant)}$$

$$v(t) = \underbrace{v(t_0)}_u + a(t - t_0)$$

$$\underline{s(t) - s(t_0)} = \underline{u(t - t_0)} + \frac{1}{2} a(t - t_0)^2$$



When you have constant acceleration, you know that acceleration is defined as $a = dv/dt$. You can integrate this equation once easily because the acceleration is constant and what you will get is v as a function of time t as

$$v(t) = \underbrace{v(t_0)}_u + a(t - t_0)$$

One can integrate this once again, to determine the distance travelled as

$$s(t) - s(t_0) = u(t - t_0) + \frac{1}{2} a(t - t_0)^2$$

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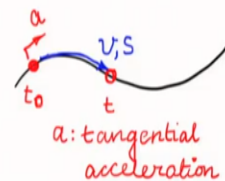
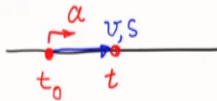
Kinematics: Motion in 1 dimension

- Constant acceleration case

$$\frac{dv}{dt} = a \quad \left| \quad \frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v \frac{dv}{ds}$$

$$\Rightarrow v \frac{dv}{ds} = a$$

$$\underline{v^2(s) - v^2(s_0) = 2a(s - s_0)}$$



$$v(s) = \frac{ds}{dt} = \pm \sqrt{v_0^2 + 2a(s - s_0)}$$

$$\Rightarrow \underline{t - t_0 = \pm \int_{s_0}^s \frac{ds}{\sqrt{v_0^2 + 2a(s - s_0)}}}$$

We proceed further and write dv, dt as dv, ds and ds, dt, dv, ds times ds, dt .

$$\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v \frac{dv}{ds}$$

Therefore the acceleration can be written as

$$v \frac{dv}{ds} = a$$

Because acceleration is a constant I can integrate this equation

$$v^2(s) - v^2(s_0) = 2a(s - s_0)$$

Now we can obtain

$$v(s) = \frac{ds}{dt} = \pm \sqrt{v_0^2 + 2a(s - s_0)}$$

$$\Rightarrow t - t_0 = \pm \int_{s_0}^s \frac{ds}{\sqrt{v_0^2 + 2a(s - s_0)}}$$

This result should lead us to the distance travelled result that we just now discussed. Here it looks in a different form in terms of the time elapsed. The sign depends on the velocity at that point. If the velocity is positive/negative then you have to take the positive/negative sign

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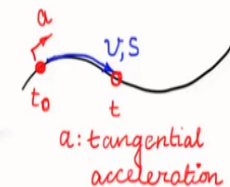
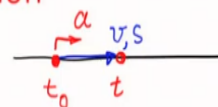
Kinematics: Motion in 1 dimension

- Time dependent acceleration case

$$\frac{dv}{dt} = a(t)$$

$$v(t) = v(t_0) + \int_{t_0}^t a(t) dt$$

$$s(t) = \int_{t_0}^t v(t) dt$$



Next we look at the time dependent case. Here, $dv/dt=a(t)$. Integrating, we can write

$$\frac{dv}{dt} = a(t)$$

$$v(t) = v(t_0) + \int_{t_0}^t a(t) dt$$

$$s(t) = \int_{t_0}^t v(t) dt$$

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Kinematics: Motion in 1 dimension

- Displacement dependent acceleration case

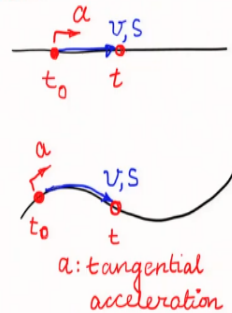
$$\frac{dv}{dt} = a(s)$$

$$\Rightarrow v \frac{dv}{ds} = a(s)$$

$$v^2 - v_0^2 = 2 \int_{s_0}^s a(s) ds$$

$$v(s) = \frac{ds}{dt} \Rightarrow \int_{t_0}^t dt = \int_{s_0}^s \frac{ds}{v(s)}$$

$$\Rightarrow t - t_0 = \int_{s_0}^s \frac{ds}{v(s)}$$



Next is the displacement along the path dependent acceleration case. In this case $dv/dt=a(s)$. We recast this equation as

$$v \frac{dv}{ds} = a(s)$$

Integrating, we have

$$v^2 - v_0^2 = 2 \int_{s_0}^s a(s) ds$$

Once we have velocity, we can write

$$v(s) = \frac{ds}{dt} \Rightarrow \int_{t_0}^t dt = \int_{s_0}^s \frac{ds}{v(s)}$$

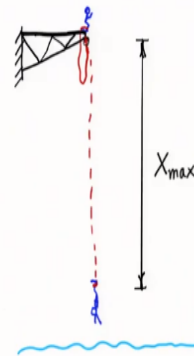
$$\Rightarrow t - t_0 = \int_{s_0}^s \frac{ds}{v(s)}$$

And if you can perform this integral then you will get the flow of time. Again there was a sign to choose you have to be careful about.

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Problem 1:

A bungee jumper uses a chord of free length $L=15$ m, and reaches a maximum jump depth of 22 m from the point of jump. If the acceleration of the jumper with the chord in tension is assumed to be of the form $a=g-by$, where $g=10$ m/s² is the acceleration due to gravity and y is the stretch of the chord in meter, determine the value of b for the jumper. Also, determine the time to reach the maximum jump depth.



This example is about a bungee jumper who uses a cord whose free length is 15 meter and reaches a maximum jump depth of 22 meter. This depth is measured from the point of jump as I have shown $X_{\max}=22$ meter from the point of jump. There are 2 acceleration regimes

$$\text{I: } a_1 = 10 \text{ m/s}^2 \quad x \leq 15 \text{ m (free fall)}$$

$$\text{II: } a_2 = 10 - by \quad (\text{taut string})$$

$$= 10 - b(x-15) \quad x > 15 \text{ m}$$

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Two acceleration regimes:

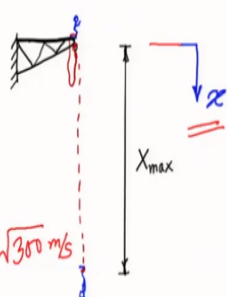
I: $a_1 = 10 \text{ m/s}^2$ $x \leq 15 \text{ m}$ (free fall)

II: $a_2 = 10 - by$ (taut string)
 $= 10 - b(x-15)$ $x > 15 \text{ m}$

I: $u_1 = 0$, $v_1 = \sqrt{2g(15)} = \sqrt{30g} = \sqrt{300} \text{ m/s}$

II: $u_2 = v_1$, $\frac{dv}{dt} = v \frac{dv}{dx} = 10 - b(x-15)$

$\Rightarrow \int_{v_1}^0 d\left(\frac{v^2}{2}\right) = \int_{15}^{x_{\max}} [10 - b(x-15)] dx$



Now I start integrating the acceleration. In regime I, the jumper starts with zero initial velocity. Integrating, we have for regime I

$$\text{I: } u_1 = 0, \quad v_1 = \sqrt{2g(15)} = \sqrt{30g}$$

Now in the regime II the cord starts getting stretched. The initial velocity for this phase is the final velocity of the first phase velocity. Integrating, we have for regime II

$$\text{II: } u_2 = v_1, \quad \frac{dv}{dt} = v \frac{dv}{dx} = 10 - b(x-15)$$

$$\Rightarrow \int_{v_1}^0 d\left(\frac{v^2}{2}\right) = \int_{15}^{x_{\max}} [10 - b(x-15)] dx$$

Performing this integration finally we arrive at the value of b as

$$\Rightarrow \int_{\sqrt{300}}^0 d\left(\frac{v^2}{2}\right) = \int_{15}^{22} [10 - b(x-15)] dx$$

$$\Rightarrow \left. \frac{v^2}{2} \right|_{\sqrt{300}}^0 = \left[10x - \frac{b}{2}(x-15)^2 \right]_{15}^{22}$$

$$\Rightarrow -\frac{300}{2} = 10(7) - \frac{b}{2}(7)^2$$

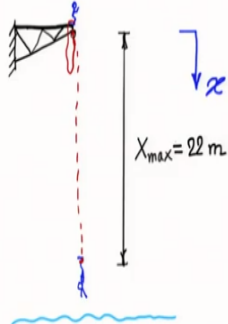
$$\Rightarrow \boxed{b = \frac{440}{49} \text{ 1/s}^2}$$

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$$\Rightarrow \int_{\sqrt{300}}^0 d\left(\frac{v^2}{2}\right) = \int_{15}^{22} [10 - b(x-15)] dx$$

$$\Rightarrow \frac{v^2}{2} \Big|_{\sqrt{300}}^0 = \left[10x - \frac{b}{2}(x-15)^2 \right]_{15}^{22}$$

$$\Rightarrow -\frac{300}{2} = 10(7) - \frac{b}{2}(7)^2$$

$$\Rightarrow \boxed{b = \frac{440}{49} \text{ 1/s}^2}$$


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Total time $T = \text{time of free fall} + \text{time of restrained fall}$

$$t_1 = \sqrt{\frac{2(15)}{10}} = \sqrt{3} \text{ sec}$$

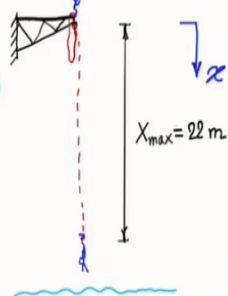
Restrained phase

$$\int_{v_1}^v d\left(\frac{v^2}{2}\right) = \int_{15}^x [10 - b(x-15)] dx \quad (x \geq 15 \text{ m})$$

$$\Rightarrow \frac{v^2}{2} \Big|_{\sqrt{300}}^v = \left[10x - \frac{b}{2}(x-15)^2 \right]_{15}^x$$

$$\Rightarrow \frac{v^2}{2} - 150 = 10(x-15) - \frac{b}{2}(x-15)^2$$

$$\Rightarrow v = \left[300 + 20x - bz^2 \right]^{1/2} \quad (z = x - 15 \quad v = \dot{x} = \dot{z})$$

$$\Rightarrow dt = \frac{dz}{[300 + 20z - bz^2]^{1/2}}$$


The next part of the problem is to determine of the total time it takes for the jumper to start from the top and reach the maximum jump depth X_{\max} . Now the total time is equal to the time of free fall plus the time of the restrain fall. The time of free fall is very easy to find $t_1 = \sqrt{3}$ sec. The time of travel in the restrained phase is a little tricky. Here, we first obtain the velocity as

$$\int_{v_1}^v d\left(\frac{v^2}{2}\right) = \int_{15}^x [10 - b(x-15)] dx \quad (x \geq 15 \text{ m})$$

$$\Rightarrow \frac{v^2}{2} \Big|_{\sqrt{300}}^v = \left[10x - \frac{b}{2}(x-15)^2 \right]_{15}^x$$

$$\Rightarrow \frac{v^2}{2} - 150 = 10(x-15) - \frac{b}{2}(x-15)^2$$

Now, we use the definition $z = x - 15$

$$v = \left[300 + 20z - bz^2 \right]^{1/2} \quad (z = x - 15 \quad v = \dot{x} = \dot{z})$$

$$\Rightarrow dt = \frac{dz}{[300 + 20z - bz^2]^{1/2}}$$

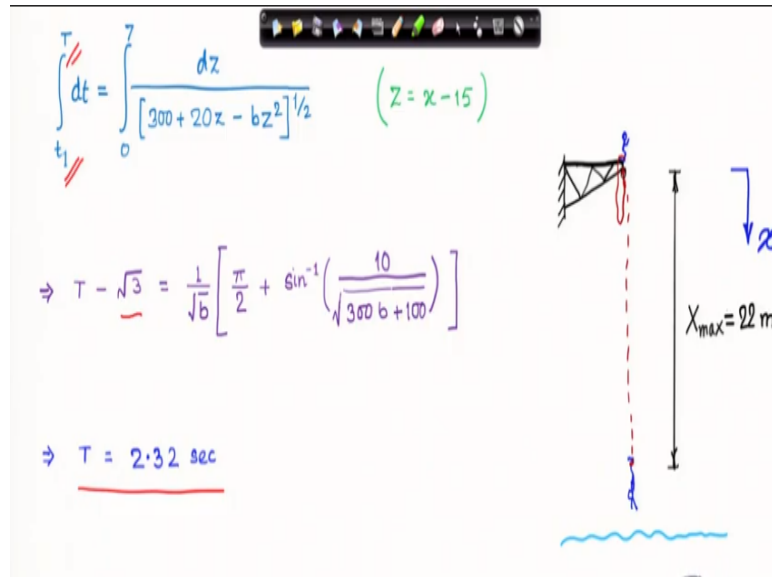
Integrating, we have

$$\int_{t_1}^T dt = \int_0^z \frac{dz}{[300 + 20z - bz^2]^{1/2}} \quad (z = x - 15)$$

$$\Rightarrow T - \sqrt{3} = \frac{1}{\sqrt{b}} \left[\frac{\pi}{2} + \sin^{-1} \left(\frac{10}{\sqrt{300b + 100}} \right) \right]$$

$$\Rightarrow \underline{T = 2.32 \text{ sec}}$$

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The image shows a handwritten derivation for the time of flight T of a projectile. The derivation starts with the integral equation:

$$\int_{t_1}^T dt = \int_0^7 \frac{dz}{[300 + 20z - bz^2]^{1/2}} \quad (Z = x - 15)$$

Then, it solves for T using the formula:

$$\Rightarrow T - \sqrt{3} = \frac{1}{\sqrt{b}} \left[\frac{\pi}{2} + \sin^{-1} \left(\frac{10}{\sqrt{300b + 100}} \right) \right]$$

Finally, it gives the numerical result:

$$\Rightarrow T = 2.32 \text{ sec}$$

To the right of the equations is a diagram of a projectile launched from a height. A dashed vertical line represents the path. A horizontal line at the top is labeled $X_{\max} = 22 \text{ m}$. A blue arrow labeled x points downwards, indicating the direction of motion.

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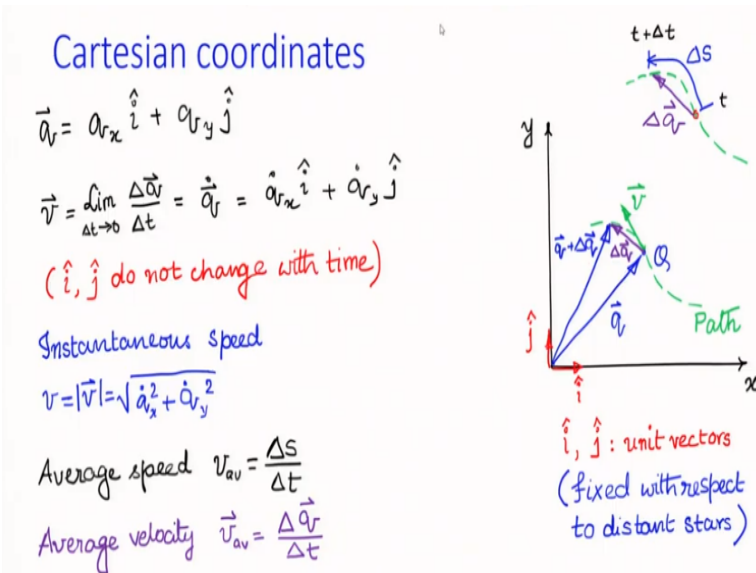
Planar curvilinear motion

- Cartesian coordinates
- Tangent-normal coordinates (Path coordinates)
- Plane polar coordinates

Representation of **vectors** (acceleration, velocity, position, displacement) + (force)

Next we are going to discuss some commonly used coordinate systems in planar curvilinear motion. We need coordinate systems to represent various vectorial quantities such as acceleration velocity position displacement etc. We start with the Cartesian coordinates, then move to tangent normal coordinates or what are also known as path coordinates, and then discuss the plane polar coordinates. In this lecture, we are going to start with the Cartesian coordinates.

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I will present a few important points here. Here, for the Cartesian coordinate system, I will keep the coordinate system fixed with respect distant stars. This is an important point because later, we are going to look at what happens when this is not fixed, for e.g., when it is rotating.

So right now I am fixing this coordinate system with respect to distant stars and that is because we are discussing kinematics, and I want to keep the kinematics correct. The Cartesian coordinate frame is defined by the unit vectors i-cap and j-cap which are the unit basis vectors along x and y directions.

Now imagine a particle q moving on a path as shown here in green. Vectors will be represented by putting a half arrow on top. Unit vectors will be represented by this caret sign or a hat sign. Therefore the representation of this q vector is

$$\vec{r} = r_x \hat{i} + r_y \hat{j}$$

Now velocity is defined as

$$\vec{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{r}}{\Delta t} = \dot{\vec{r}} = \dot{r}_x \hat{i} + \dot{r}_y \hat{j}$$

(\hat{i}, \hat{j} do not change with time)

Again I want to point out that the unit vectors \hat{i} -cap and \hat{j} -cap are fixed with respect to the distance stars, and hence, they do not change. The reason I fixed with distant stars is that they will not change direction they do not change magnitude of course since they are all unit vectors but they will not even change direction.

So when I fix a frame with respect to distant stars, I assume that their derivatives are 0 or negligible. We will consider that they are 0, and therefore they do not change direction. Now based on this, I can define the instantaneous speed which is the magnitude of the velocity vector, average speed and average velocity as

Instantaneous speed

$$v = |\vec{v}| = \sqrt{\dot{r}_x^2 + \dot{r}_y^2}$$

Average speed $v_{av} = \frac{\Delta s}{\Delta t}$

Average velocity $\vec{v}_{av} = \frac{\Delta \vec{r}}{\Delta t}$

Since speed is the magnitude of the velocity vector, it is always positive. Average speed is the ratio of finite distance travelled and the corresponding finite time elapsed. Average velocity is the finite displacement vector divided by the corresponding finite time elapsed.

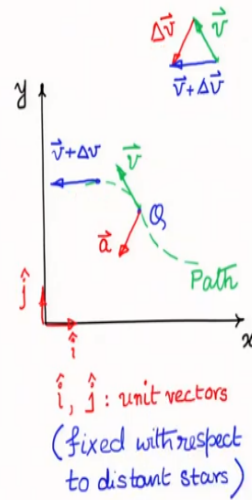
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Cartesian coordinates

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \ddot{v}_x \hat{i} + \ddot{v}_y \hat{j}$$

Note:

1. Velocity is always tangential to the path
2. No such restriction on acceleration



Acceleration is defined as the time derivative of the velocity vector. Using the velocity vector at two infinitesimally separated time instants, we can write

$$\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \ddot{v}_x \hat{i} + \ddot{v}_y \hat{j}$$

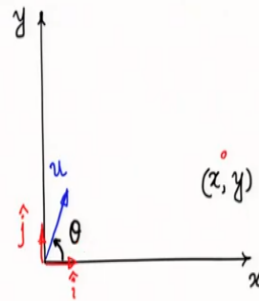
Now you should note that very carefully that velocity is always tangential to the path but there is no such restrictions on acceleration; it can be arbitrarily directed.

Now let us look at a problem.

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Problem 2:

Determine the launch angle θ of a projectile with initial velocity u for reaching a point (x, y) , and hence determine the reachable region in the x - y plane.



The reachable region of a projectile this is a very important calculation for artillery guns for example.

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Particle coordinates

$$x = u \cos \theta t \rightarrow t = \frac{x}{u \cos \theta}$$

$$y = u \sin \theta t - \frac{1}{2} g t^2$$

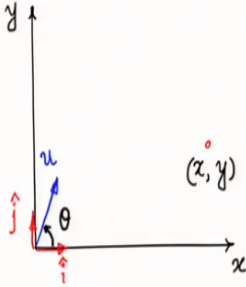

Path equation: $y = \tan \theta x - \frac{1}{2} g \frac{x^2}{u^2 \cos^2 \theta}$

$$\Rightarrow y u^2 \cos^2 \theta - x u^2 \sin 2\theta = -y u^2 - g x^2$$

Substituting $\cos 2\theta = \frac{1-z^2}{1+z^2}$, $\sin 2\theta = \frac{2z}{1+z^2}$
 where $z = \tan \theta$, we get

$$g x^2 z^2 - 2 u^2 x z + 2 y u^2 + g x^2 = 0$$

$A \cos 2\theta + B \sin 2\theta = C$

The coordinates of the particle can be written as

$$x = u \cos \theta t$$

$$y = u \sin \theta t - \frac{1}{2} g t^2$$

Eliminating the time t between the two, we obtain the path equation

$$y u^2 \cos^2 \theta - x u^2 \sin 2\theta = -y u^2 - g x^2$$

Given the coordinates of the target point (x, y) , we have to determine the launch angle θ .

Using standard substitutions

$$\cos 2\theta = \frac{1-z^2}{1+z^2}, \quad \sin 2\theta = \frac{2z}{1+z^2}$$

we obtain

$$g x^2 z^2 - 2 u^2 x z + 2 y u^2 + g x^2 = 0$$

where

$$z = \tan \theta$$

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$$g x^2 z^2 - 2u^2 x z + 2y u^2 + g x^2 = 0$$

$$z_{\pm} = \frac{1}{g x^2} \left[u^2 x \pm \sqrt{u^4 x^2 - g x^2 (2y u^2 + g x^2)} \right]$$

$\Delta > 0$: 2 real solutions
 $\theta_{\pm} = \tan^{-1}(z_{\pm})$

$\Delta < 0$: complex conjugate solutions
 Point (x, y) is beyond the reach

$\Delta = 0$: 2 repeated real solutions
 Point (x, y) is on the boundary of the reach

This quadratic equation has a solution of the form

$$z_{\pm} = \frac{1}{g x^2} \left[u^2 x \pm \sqrt{u^4 x^2 - g x^2 (2y u^2 + g x^2)} \right]$$

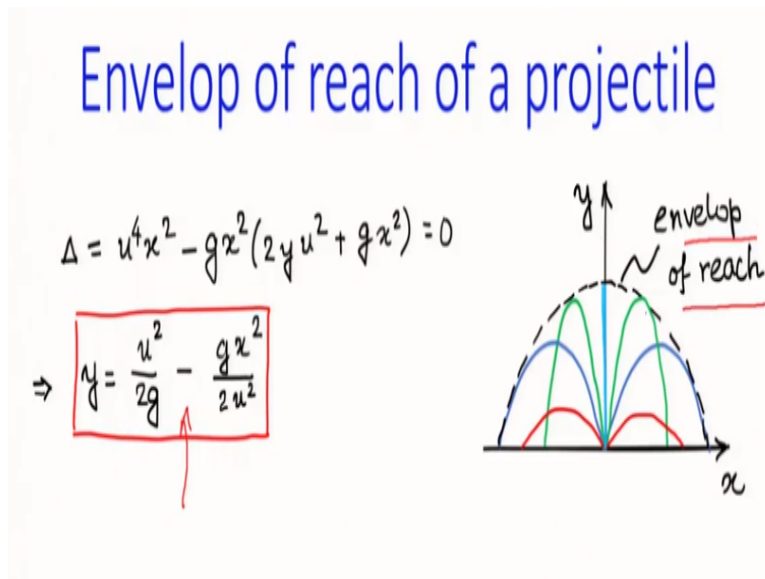
Now, depending on the value of delta, we can have 2 real solutions, 1 real solution (repeated), or no real solution. A complex solution means that the point (x, y) is unreachable. Only 1 real solution implies the boundary of the reachable space. Therefore, putting the discriminant to zero, we obtain the boundary of the reachable space as

$$\Delta = u^4 x^2 - g x^2 (2y u^2 + g x^2) = 0$$

$$\Rightarrow \boxed{y = \frac{u^2}{2g} - \frac{g x^2}{2 u^2}}$$

which is a parabola. This concave downwards parabola defines the envelope of reach of the projectile.

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Summary

- Motion in 1-dimension
- Cartesian coordinates

To summarize we have discussed motion in 1 dimension, and we are looked at the Cartesian coordinate system.