Advanced Dynamics
Prof. Anirvan Dasgupta
Department of Mechanical Engineering
Indian Institute of Technology - Kharagpur

Module No # 01
Lecture No # 01
Coordinate Systems – I

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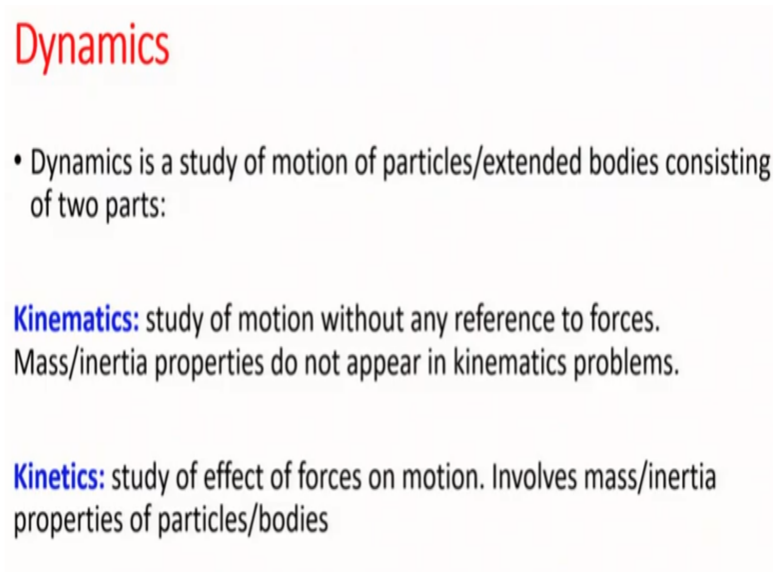

To begin with, let us see what dynamics is all about. Dynamics is a study of motion of particles or extended bodies under the action of forces. This study consists of 2 parts. The first part is called kinematics which is the study of motion without any reference to forces. Because there are no forces involved it is purely a geometric study.

So inertia properties, mass of particles or inertia of the bodies, do not appear in the study of kinematics. On the other hand, the second part of the study of dynamics is kinetics. Kinetics is the study of effect of forces on the motion and hence it involves the inertia properties of bodies involved. So the 2 parts of dynamics consist of kinematics and kinetics. Now, out of these 2, kinematics is tricky, while kinetics is a straight forward step, more or less. So we are going to start with the study of kinematics which is the study of motion.

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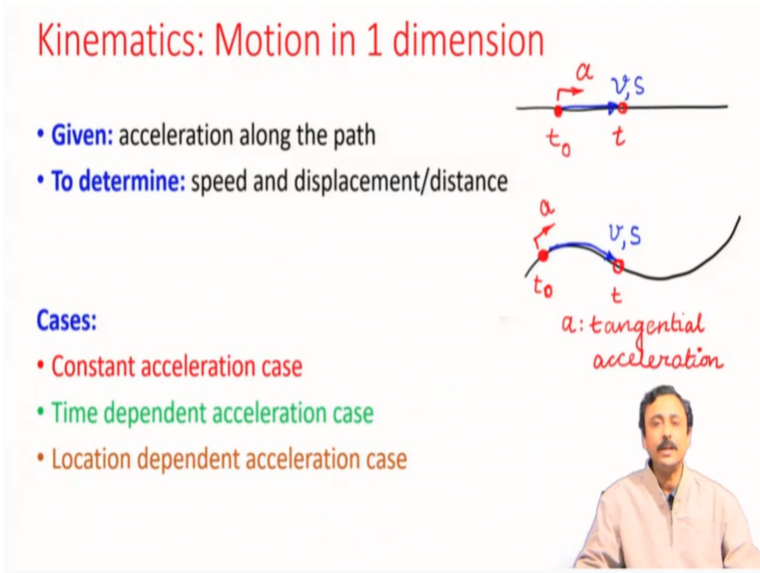

In this lecture, we are going to study motion in 1 dimension. We are going to look at commonly used coordinate systems and we are going to look at some problems. Whenever we say motion in one dimension, the typical picture in our mind is a particle moving on a straight path. But that need not always be the requirement; it could be constrained motion (1 degree of freedom) on a curved path but the acceleration specification must always be tangential to the path.

So, for constrained motion, if the acceleration is specified tangential to the path in that case it can be treated kinematically as 1 dimensional motion. And the displacement or distance traveled should be along the path. Velocity is always tangent to the path; the problem is specified with the acceleration along the path.

We are required to find out the speed which is the first integral of the acceleration and we are required to find out what is the displacement of the particle or the distance travelled by the particle along the path. We are going to look at some cases which you have seen right from schools. First case is the constant acceleration case, followed by the time dependent acceleration case and the location dependent acceleration case.

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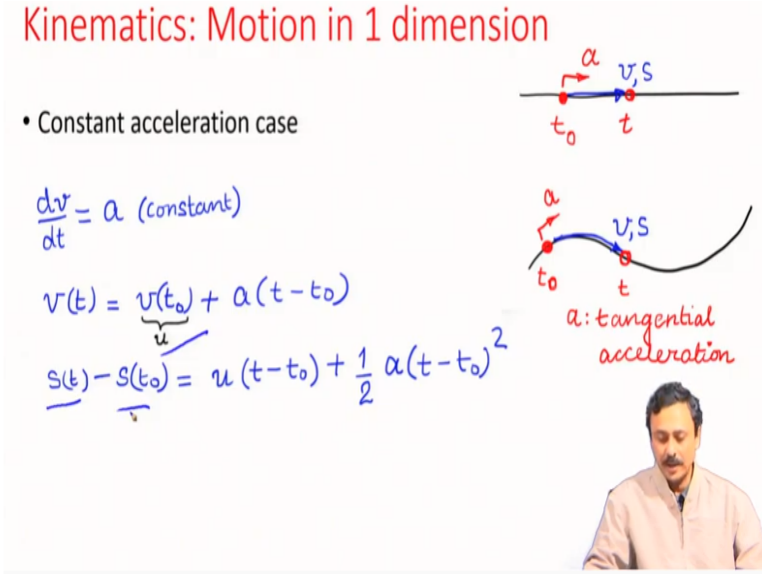

When you have constant acceleration, you know that acceleration is defined as a=dv/dt. You can integrate this equation once easily because the acceleration is constant and what you will get is v as a function of time t as

$$v(t) = \underbrace{v(t_0)}_{u} + a(t-t_0)$$

One can integrate this once again, to determine the distance travelled as

$$s(t) - s(t_0) = u(t-t_0) + \frac{1}{2}a(t-t_0)^2$$

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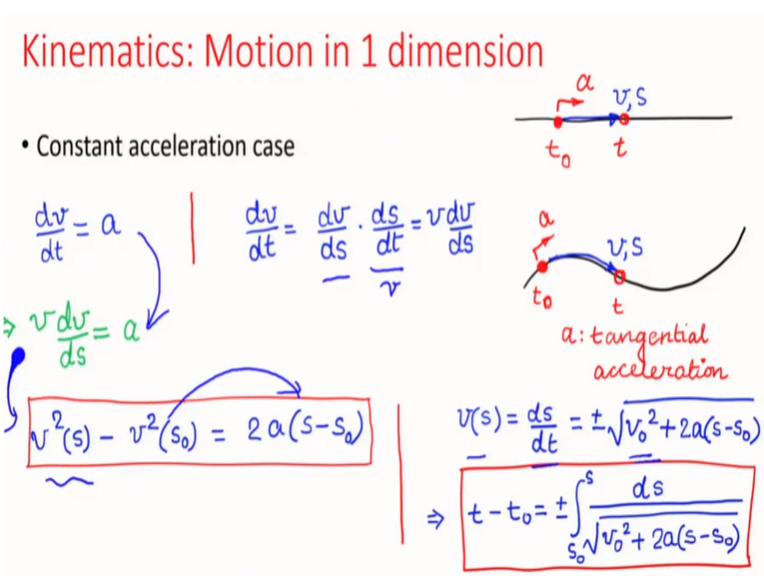

We proceed further and write dv, dt as dv, ds and ds, dt, dv, ds times ds, dt.

$$\frac{dv}{dt} = \frac{dv}{ds} \cdot \frac{ds}{dt} = v\frac{dv}{ds}$$

Therefore the acceleration can be written as

$$v\frac{dv}{ds} = a$$

Because acceleration is a constant I can integrate this equation

$$\boxed{v^2(s) - v^2(s_0) = 2a(s - s_0)}$$

Now we can obtain

$$v(s) = \frac{ds}{dt} = \pm\sqrt{v_0^2 + 2a(s - s_0)}$$

$$\Rightarrow \boxed{t - t_0 = \pm\int_{s_0}^{s} \frac{ds}{\sqrt{v_0^2 + 2a(s - s_0)}}}$$

This result should lead us to the distance travelled result that we just now discussed. Here it looks in a different form in terms of the time elapsed. The sign depends on the velocity at that point. If the velocity is positive/negative then you have to take the positive/negative sign

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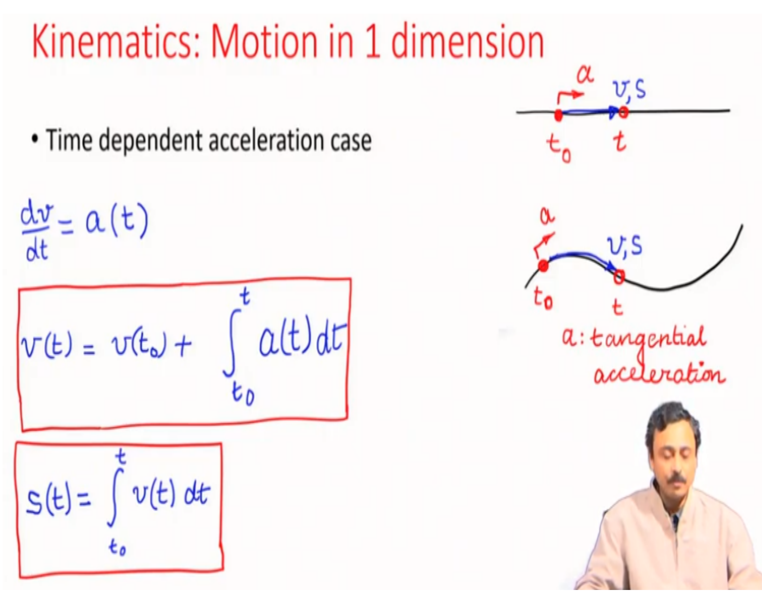

Next we look at the time dependent case. Here, dv/dt=a(t). Integrating, we can write

$$\frac{dv}{dt} = a(t)$$

$$\boxed{v(t) = v(t_0) + \int_{t_0}^{t} a(t)\,dt}$$

$$\boxed{s(t) = \int_{t_0}^{t} v(t)\,dt}$$

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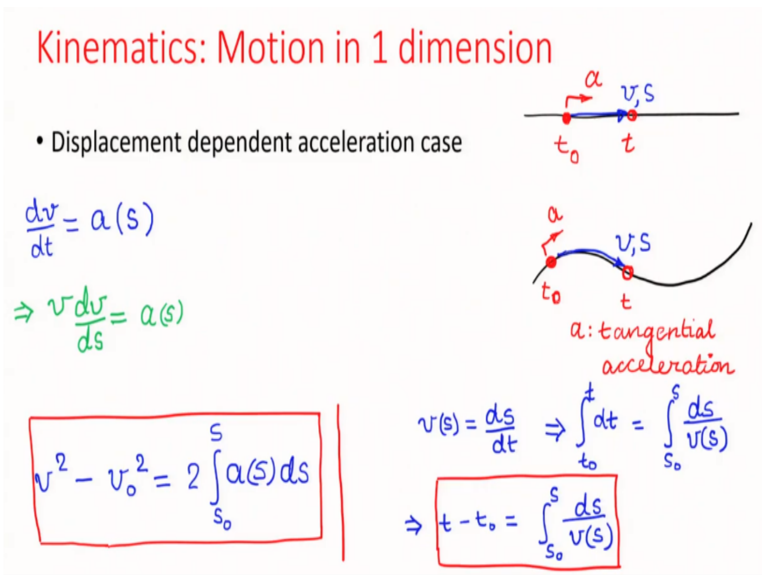

Next is the displacement along the path dependent acceleration case. In this case dv/dt=a(s). We recast this equation as

$$v\frac{dv}{ds} = a(s)$$

Integrating, we have

$$\boxed{v^2 - v_0^2 = 2\int_{s_0}^{s} a(s)\,ds}$$

Once we have velocity, we can write

$$v(s) = \frac{ds}{dt} \Rightarrow \int_{t_0}^{t} dt = \int_{s_0}^{s} \frac{ds}{v(s)}$$

$$\Rightarrow \boxed{t - t_0 = \int_{s_0}^{s} \frac{ds}{v(s)}}$$

And if you can perform this integral then you will get the flow of time. Again there was a sign to choose you have to be careful about.

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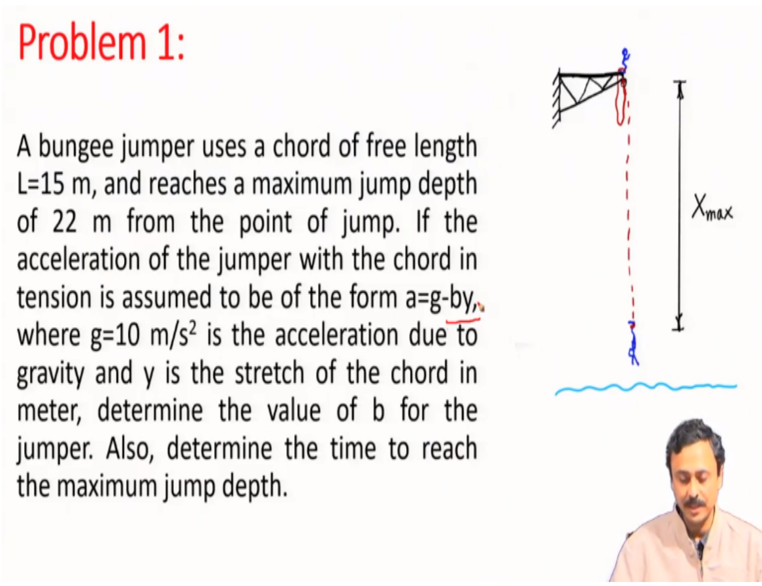

This example is about a bungee jumper who uses a cord whose free length is 15 meter and reaches a maximum jump depth of 22 meter. This depth is measured from the point of jump as I have shown X_{max}=22 meter from the point of jump. There are 2 acceleration regimes

I: $a_1 = 10\, m/s^2$ $\quad x \le 15\, m$ (free fall)

II: $a_2 = 10 - by$ (taut string)

$= 10 - b(x - 15) \quad x > 15\, m$

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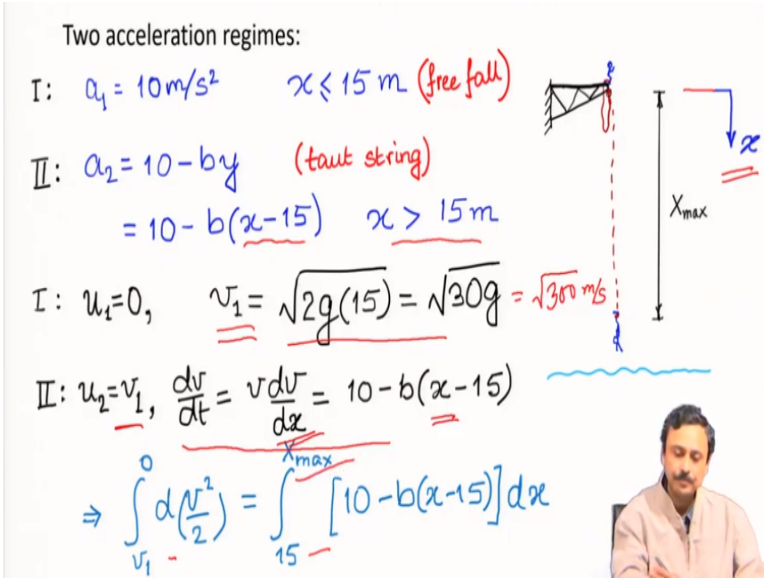

Now I start integrating the acceleration. In regime I, the jumper starts with zero initial velocity. Integrating, we have for regime I

$$\mathrm{I}: \quad u_1 = 0, \qquad v_1 = \sqrt{2g(15)} = \sqrt{30g}$$

Now in the regime II the cord starts getting stretched. The initial velocity for this phase is the final velocity of the first phase velocity. Integrating, we have for regime II

$$\mathrm{II}: \quad u_2 = v_1, \quad \frac{dv}{dt} = v\frac{dv}{dx} = 10 - b(x-15)$$

$$\Rightarrow \int_{v_1}^{0} d\left(\frac{v^2}{2}\right) = \int_{15}^{X_{max}} \left[10 - b(x-15)\right] dx$$

Performing this integration finally we arrive at the value of b as

$$\Rightarrow \int_{\sqrt{300}}^{0} d\left(\frac{v^2}{2}\right) = \int_{15}^{22} \left[10 - b(x-15)\right] dx$$

$$\Rightarrow \left.\frac{v^2}{2}\right|_{\sqrt{300}}^{0} = \left[10x - \frac{b}{2}(x-15)^2\right]_{15}^{22}$$

$$\Rightarrow -\frac{300}{2} = 10(7) - \frac{b}{2}(7)^2$$

$$\Rightarrow \boxed{b = \frac{440}{49} \ 1/s^2}$$

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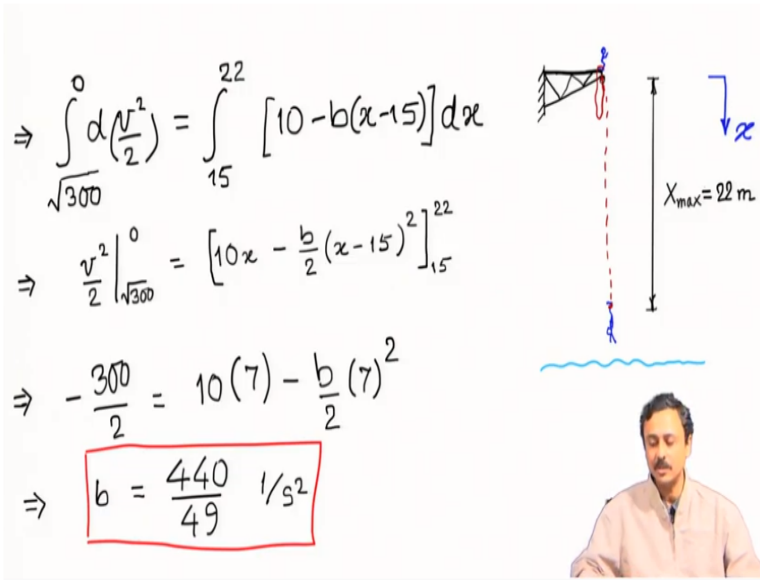

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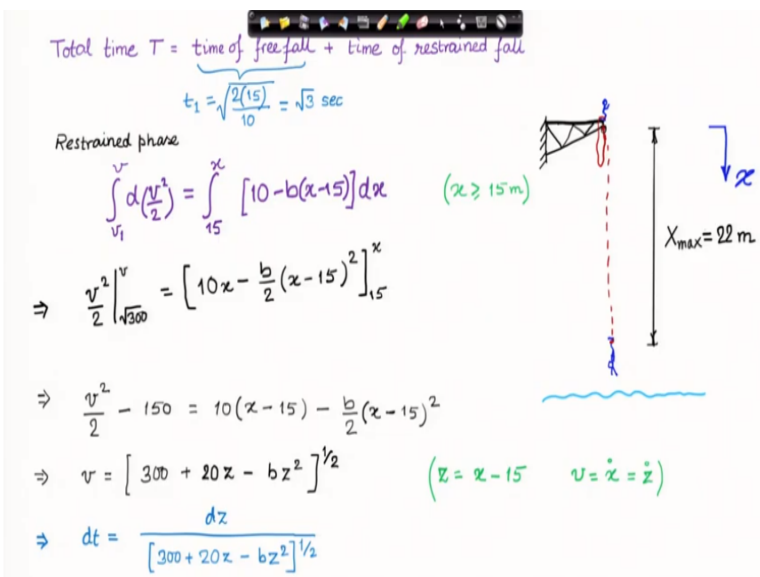

The next part of the problem is to determine of the total time it takes for the jumper to start from the top and reach the maximum jump depth X_{max}. Now the total time is equal to the time of free fall plus the time of the restrain fall. The time of free fall is very easy to find $t_1 = \sqrt{3}$ sec. The time of travel in the restrained phase is a little tricky. Here, we first obtain the velocity as

$$\int_{v_1}^{v} d\left(\frac{v^2}{2}\right) = \int_{15}^{x} [10 - b(x-15)]\,dx \qquad (x \geq 15\,m)$$

$$\Rightarrow \quad \frac{v^2}{2}\Bigg|_{\sqrt{300}}^{v} = \left[10x - \frac{b}{2}(x-15)^2\right]_{15}^{x}$$

$$\Rightarrow \quad \frac{v^2}{2} - 150 = 10(x-15) - \frac{b}{2}(x-15)^2$$

Now, we use the definition z=x – 15

$$v = \left[300 + 20z - bz^2\right]^{1/2} \qquad (z = x - 15 \qquad v = \dot{x} = \dot{z})$$

$$\Rightarrow \quad dt = \frac{dz}{\left[300 + 20z - bz^2\right]^{1/2}}$$

Integrating, we have

$$\int_{t_1}^{T} dt = \int_{0}^{7} \frac{dz}{\left[300 + 20z - bz^2\right]^{1/2}} \qquad (z = x - 15)$$

$$\Rightarrow \quad T - \sqrt{3} = \frac{1}{\sqrt{b}}\left[\frac{\pi}{2} + \sin^{-1}\left(\frac{10}{\sqrt{300\,b + 100}}\right)\right]$$

$$\Rightarrow \quad \underline{T = 2.32 \text{ sec}}$$

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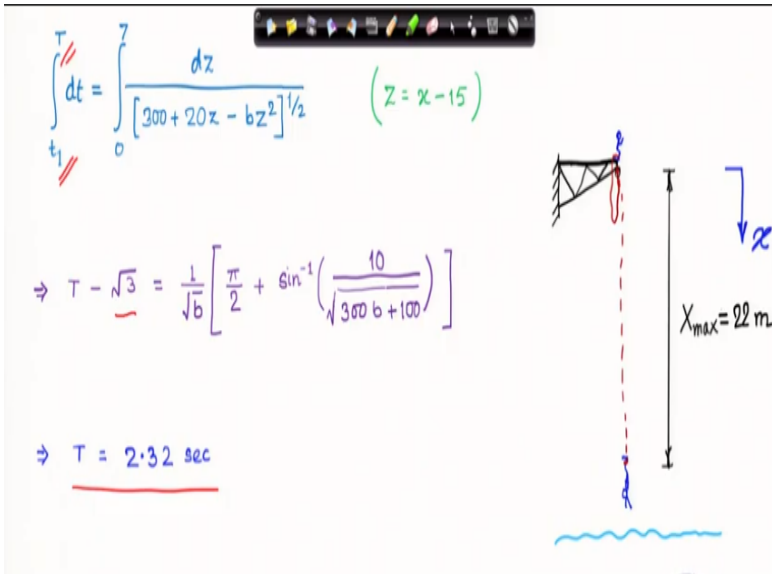

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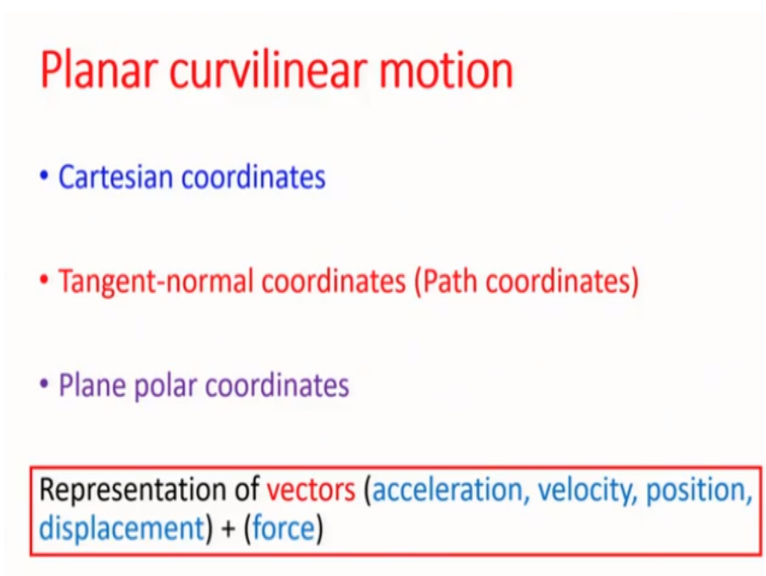

Next we are going to discuss some commonly used coordinate systems in planar curvilinear motion. We need coordinate systems to represent various vectorial quantities such as acceleration velocity position displacement etc. We start with the Cartesian coordinates, then move to tangent normal coordinates or what are also known as path coordinates, and then discuss the plane polar coordinates. In this lecture, we are going to start with the Cartesian coordinates.

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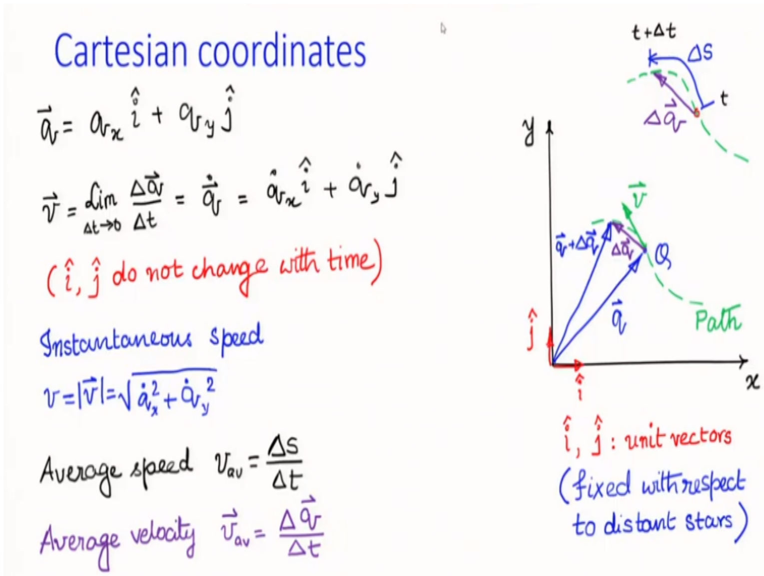

I will present a few important points here. Here, for the Cartesian coordinate system, I will keep the coordinate system fixed with respect distant stars. This is an important point because later, we are going to look at what happens when this is not fixed, for e.g., when it is rotating.

So right now I am fixing this coordinate system with respect to distant stars and that is because we are discussing kinematics, and I want to keep the kinematics correct. The Cartesian coordinate frame is defined by the unit vectors i-cap and j-cap which are the unit basis vectors along x and y directions.

Now imagine a particle q moving on a path as shown here in green. Vectors will be represented by putting a half arrow on top. Unit vectors will be represented by this caret sign or a hat sign. Therefore the representation of this q vector is

$$\vec{q} = q_x \hat{i} + q_y \hat{j}$$

Now velocity is defind as

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{q}}{\Delta t} = \dot{\vec{q}} = \dot{q}_x \hat{i} + \dot{q}_y \hat{j}$$

($\hat{i}, \hat{j}$ do not change with time)

Again I want to point out that the unit vectors i-cap and j-cap are fixed with respect to the distance stars, and hence, they do not change. The reason I fixed with distant stars is that they will not change direction they do not change magnitude of course since they are all unit vectors but they will not even change direction.

So when I fix a frame with respect to distant stars, I assume that their derivatives are 0 or negligible. We will consider that they are 0, and therefore they do not change direction. Now based on this, I can define the instantaneous speed which is the magnitude of the velocity vector, average speed and average velocity as

Instantaneous speed

$$v = |\vec{v}| = \sqrt{\dot{q}_x^2 + \dot{q}_y^2}$$

Average speed $v_{av} = \dfrac{\Delta s}{\Delta t}$

Average velocity $\vec{v}_{av} = \dfrac{\Delta \vec{q}}{\Delta t}$

Since speed is the magnitude of the velocity vector, it is always positive. Average speed is the ratio of finite distance travelled and the corresponding finite time elapsed. Average velocity is the finite displacement vector divided by the corresponding finite time elapsed.

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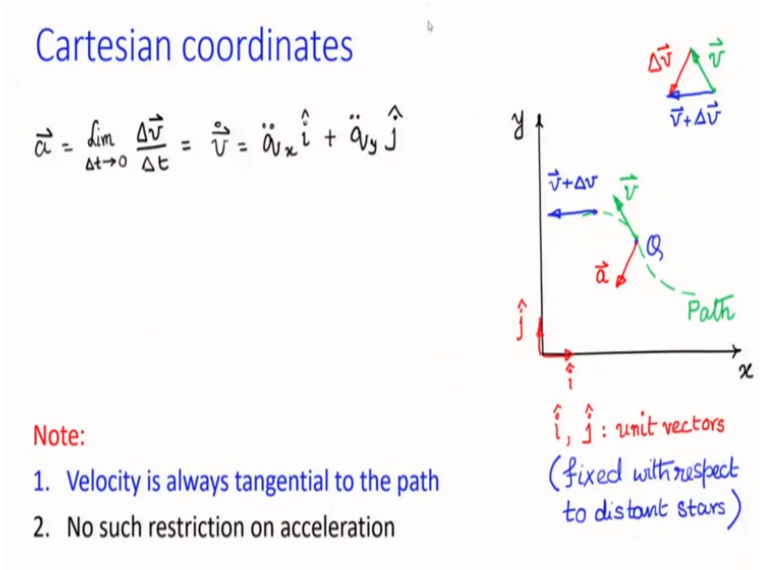

Acceleration is defined as the time derivative of the velocity vector. Using the velocity vector at two infinitesimally separated time instants, we can write

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \dot{\vec{v}} = \ddot{a}_x \hat{i} + \ddot{a}_y \hat{j}$$

Now you should note that very carefully that velocity is always tangential to the path but there is no such restrictions on acceleration; it can be arbitrarily directed.

Now let us look at a problem.

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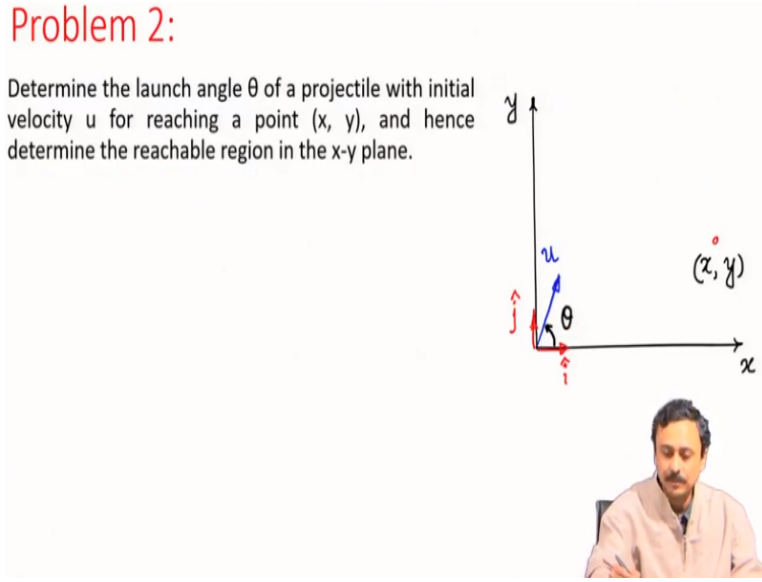

The reachable region of a projectile this is a very important calculation for artillery guns for example.

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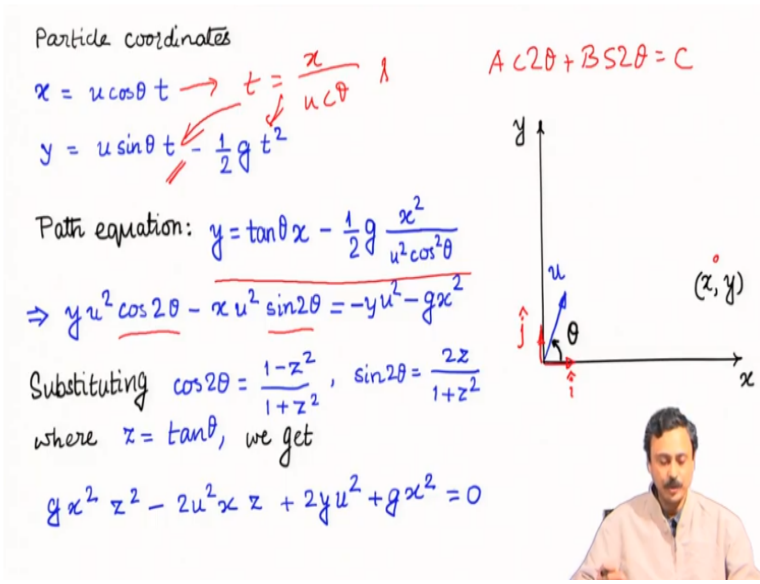

The coordinates of the particle can be written as

$$x = u\cos\theta\, t$$

$$y = u\sin\theta\, t - \frac{1}{2} g t^2$$

Eliminating the time t between the two, we obtain the path equation

$$y u^2 \cos 2\theta - x u^2 \sin 2\theta = -y u^2 - g x^2$$

Given the coordinates of the target point (x, y), we have to determine the launch angle θ.

Using standard substitutions

$$\cos 2\theta = \frac{1 - z^2}{1 + z^2}, \quad \sin 2\theta = \frac{2z}{1 + z^2}$$

we obtain

$$g x^2 z^2 - 2u^2 x z + 2 y u^2 + g x^2 = 0$$

where

$$z = \tan\theta$$

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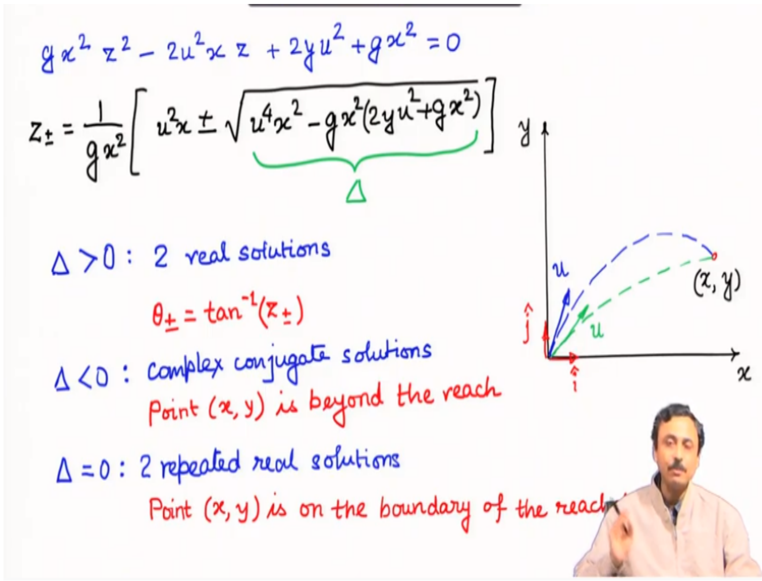

This quadratic equation has a solution of the form

$$z_\pm = \frac{1}{g x^2}\left[u^2 x \pm \sqrt{\underbrace{u^4 x^2 - g x^2 (2yu^2 + g x^2)}_{\Delta}} \right]$$

Now, depending on the value of delta, we can have 2 real solutions, 1 real solution (repeated), or no real solution. A complex solution means that the point (x, y) is unreachable. Only 1 real solution implies the boundary of the reachable space. Therefore, putting the discriminant to zero, we obtain the boundary of the reachable space as

$$\Delta = u^4 x^2 - g x^2 (2yu^2 + g x^2) = 0$$

$$\Rightarrow \boxed{y = \frac{u^2}{2g} - \frac{g x^2}{2u^2}}$$

which is a parabola. This concave downwards parabola defines the envelope of reach of the projectile.

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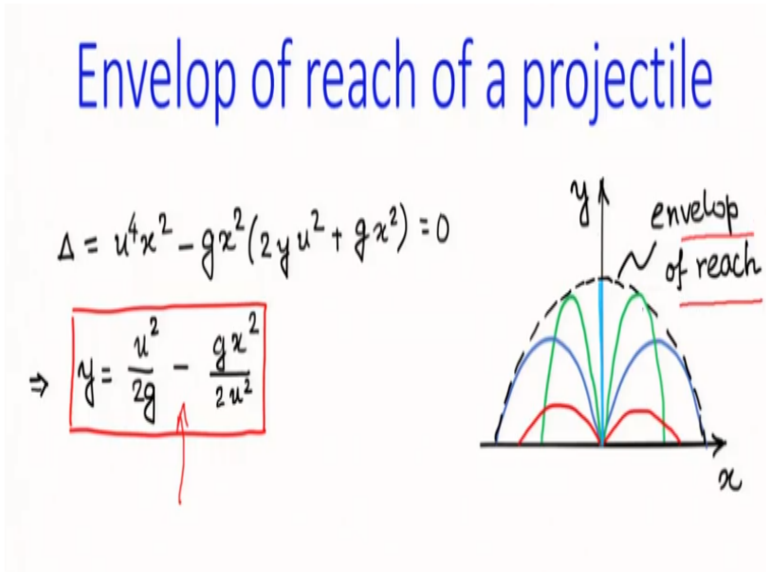

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To summarize we have discussed motion in 1 dimension, and we are looked at the Cartesian coordinate system.