**Tools in Scientific Computing**

**Prof. Aditya Bandopadhyay**
**Department of Mechanical Engineering**
**Indian Institute of Technology, Kharagpur**

**Lecture – 09**
**Geometric interpretation of ODEs**

Hi everyone. Welcome to this series of lectures on Dynamical Systems. Non-linear dynamics refers to those categories of equations, which are characterized by nonlinearities in some way or the other. And, while we have looked at linear equations we there is a host of methods with which we can solve linear equations, typically non-linear equations are not solvable.

And, it is very difficult to discern what kind of behavior such non-linear systems would have. And, while they are difficult to solve they are quite prevalent in nature. Almost every fluid mechanics problem or predicting weather, they involve some kind of equations, which have a very contrived feedback.

So, the evolution of the system keeps changing the system in such a way that it is becomes very difficult to keep a track of all the variables. Moreover various engineering systems, which involve lasers, microwaves, phase locked arrays. Things which are very common in communications they also rely heavily on how you can analyze these systems from a non-linear dynamics perspective. The original idea behind all of this was whether or not we can predict trajectories of planets.

If you have a two body system, we know from Keplers Law that you can find out the momentum the velocity the position all these things. But, the moment you have a three body system it becomes almost impossible to find out the exact dynamics at a given point in time in the future.

However, there are some other questions that we can pose, which can help us in understanding the system. Rather than knowing at what point will a planet be at a given time we can ask the question, whether this three planet system is stable, whether one of the planets will be flung off to infinity or whether the three planets will collide, these are the kind of questions that give us insight into such kinds of systems.

In this series of lectures we will first see how to analyze various problems geometrically? So, non-linear dynamics relies heavily on visualizing the solution rather than solving the equation. You will see that often you are not solving the equation, but you are trying to find out a way in which you can sort of classify how the system would behave? Based on that you can make various kinds of predictions.

So, after studying how to interpret equations geometrically, we will look briefly into what bifurcations are? Bifurcations refer to the situation where a parameter, when it changes it can seemingly cause the system to behave quite differently, than it was. Consider for example, the buckling of a beam.

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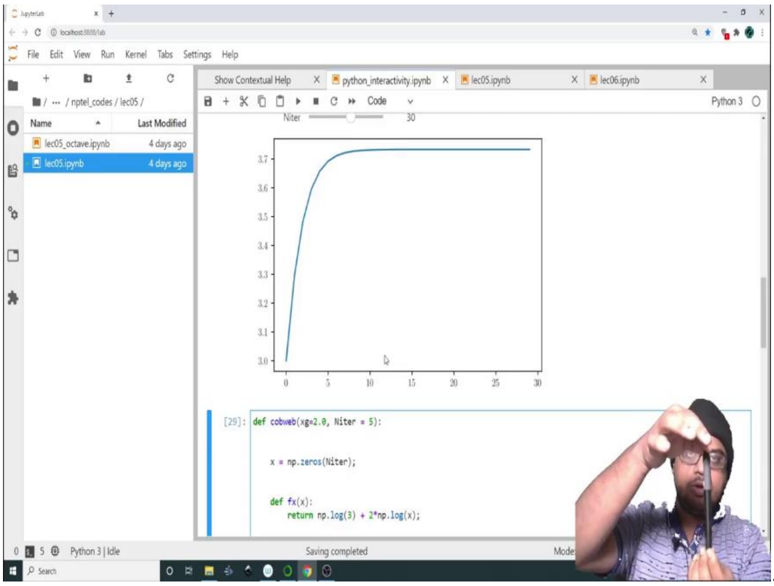

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When you take a beam like this and you put normal stress normal compressive stress nothing happens. But, at a certain point if you take a twig and try to compress it you will see that it, it bends ok.

So, that kind of a change in qualitative behavior occurs because of some parameters, which are related to not only the loading, but the material properties, the material shape and all these things. They end up dictating, how the system will behave and whether or not that behavior will change it is qualitative behavior dramatically.

So, let us begin our journey by analyzing things geometrically.

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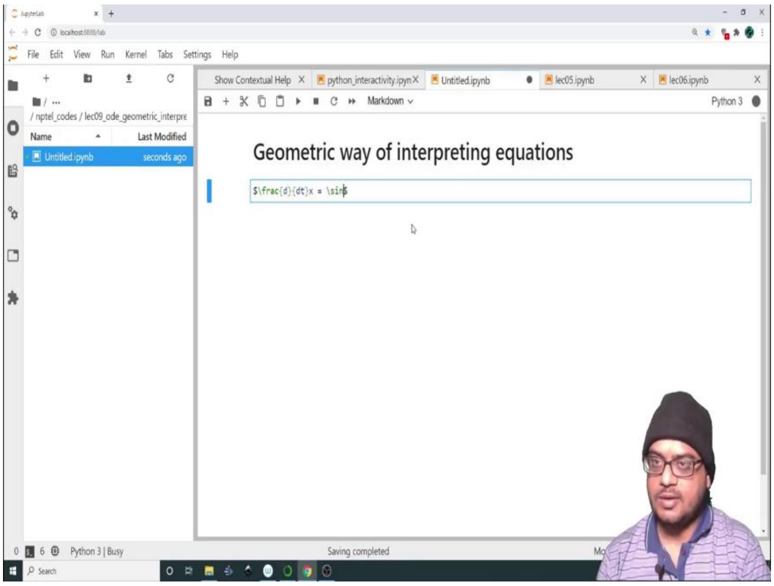


So, let me create a new file alright. So, let us consider a very simple non-linear differential equation. So, let us consider $\frac{dx}{dt} = \sin(x)$ .

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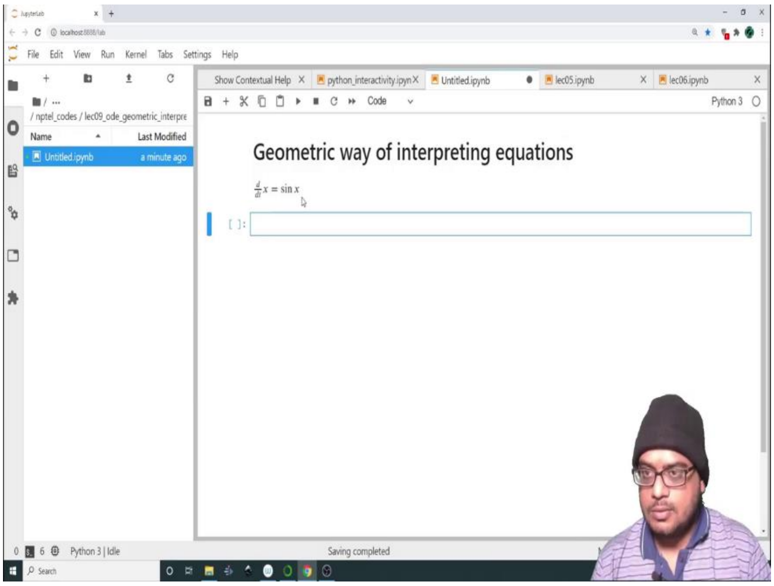


So, it is a very straight forward equation. And of course, it is a non-linear equation, because sine makes it non-linear and it is an autonomous equation that is the right hand side it does not depend on time.

Now, we can integrate this particular equation in fact, the solution to this is.

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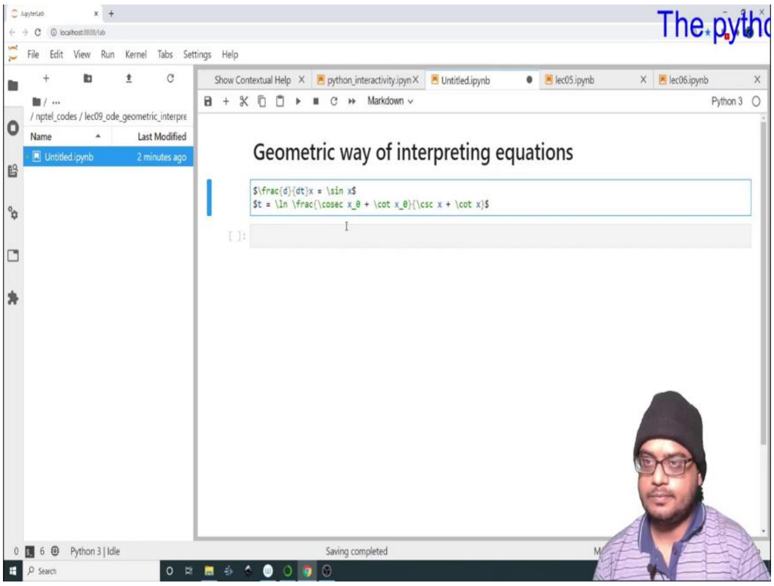


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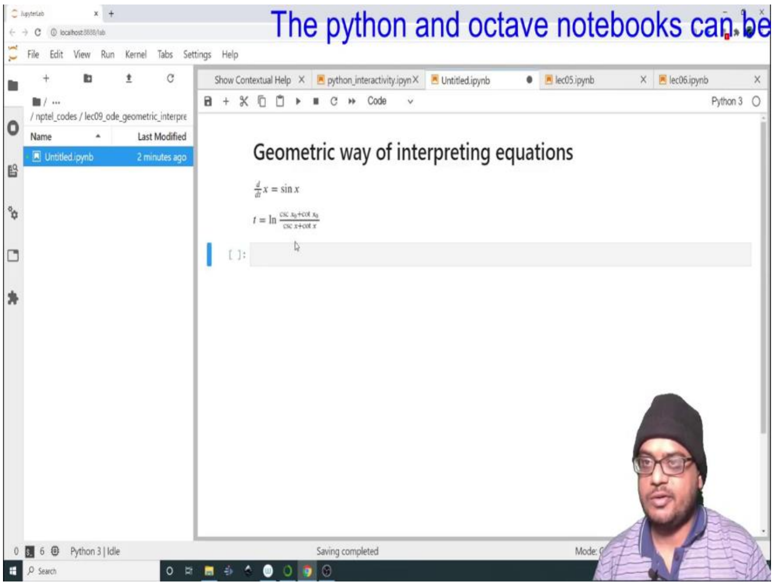


So, the solution is given by log of this entire thing and; obviously, it has to be positive. So, now, what do we do with this solution? Obviously, you cannot it is not easy to tell what the solution means, I mean you have a bunch of cosec cot, I mean just looking at it you cannot really tell what happens at long times. First of all it is a implicit way of representing the solution that is you would expect x to be a function of time.

But, here you have been given the inverse; you have been said that given this x, this is what the time you expect it to be. So, first things first let us look at how this function looks like keeping in mind that it is it is not explicit function, where x is given as a function of time.

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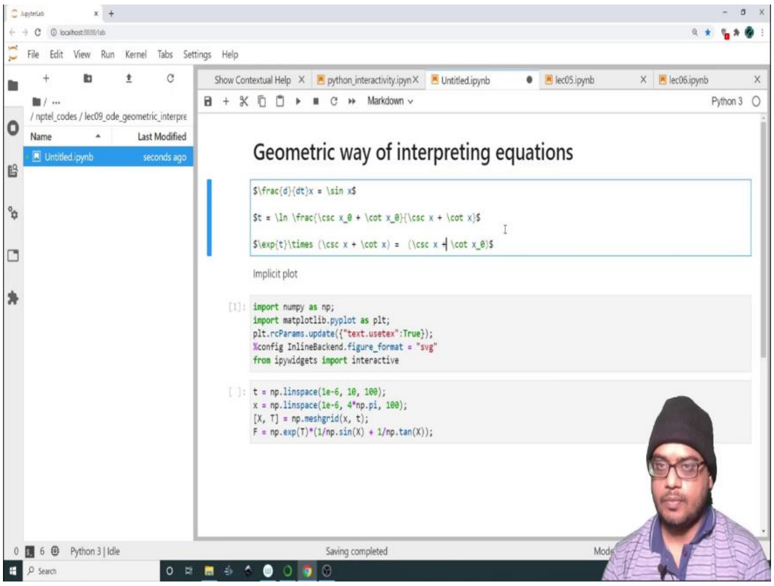


So, we; obviously, must make an implicit plot ok. So, let us create the domain on which we want to make the implicit plot. So, let us import in fact, let me copy this bit of code we will require it in this study as well.

So, we are importing numpy matplotlib we are updating the matplotlib parameters, we are making the in inline figure to s “svg” and we are importing ipywidgets ok. So, let me execute this; let x, rather let t be a linspace from $1\times e^{-6}$ to 10.

And, let me take 100 points let x be a linspace from $1\times e^{-6}$ to $4\pi$. Let me take 100 points in this and the reason why I am choosing $1\times e^{-6}$ is because I do not want this to be 0, because log of so, 0 is fine for time, but once we have 0 appearing in the inside the log it will, it will blow up.

So, I want to avoid dealing with those and you can later on play with this and see, what happens? Ok. So, I have defined the two things; let me create a mesh grid. So, [X, T] = np.meshgrid(x, t). So, I am denoting the mesh grid by capital X and capital T, let me define the implicit function.

So, it will be np dot exponential of time minus this entire expression. So, or in fact, I can multiply it as well times. (1/np.sin(X) + 1/np.tan(X). Essentially, what I am trying to do is to recast this equation as t exponential of t times.

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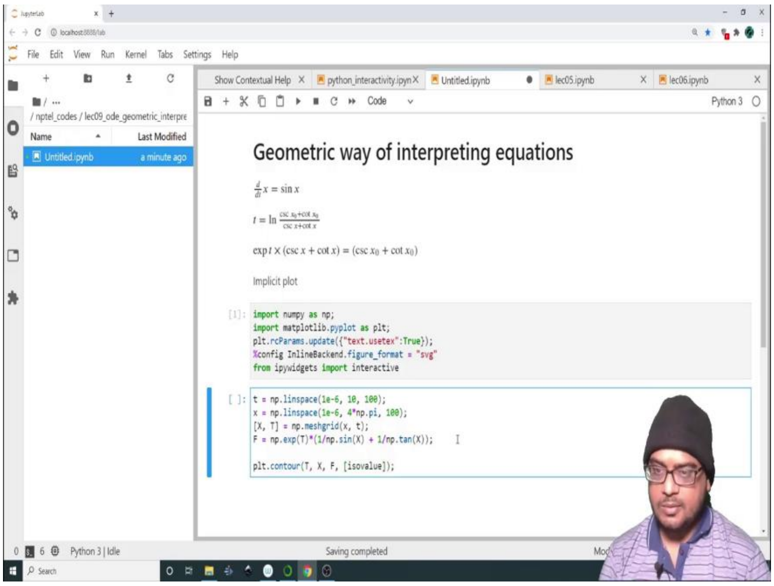


So, I am trying to cast it in this particular form where x0 is; obviously, the initial condition for this equation. So, now let us see what happens? So, I am defining a function F as this the left hand side of this. And, I want to plot the isoline of F, which has this particular value. This is what would be the meaning of creating an implicit plot ok.

So, this is the function I repeat, this is the function and I want to plot the isoline which corresponds to this value. If, this were to be 0, I would be plotting 0 isoline of this particular equation.

But, in this particular case I would select what x naught would be. And, depending on the value of x0, I would have a different solution ok. So, let me do this. So, I will do plt.contour and I will say time on the x axis, x on the y axis, F and here I would define the isovalue. So, isovalue is a variable, which I will define over here.

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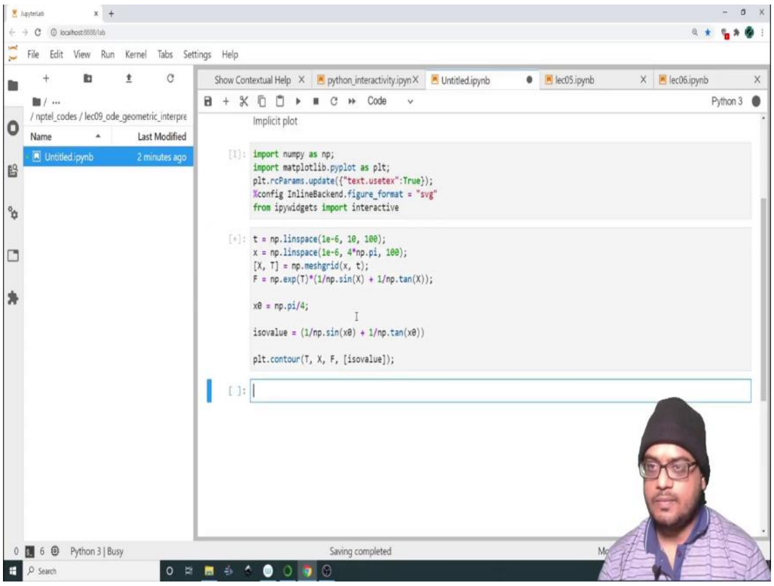


And, thus this will be equal to this particular thing, but instead of x we will have x0. So, something like this and I will define what x0 is. So, let x0 be equal to $\pi/4$ ok. So, let me run this and let us see, what happens great?

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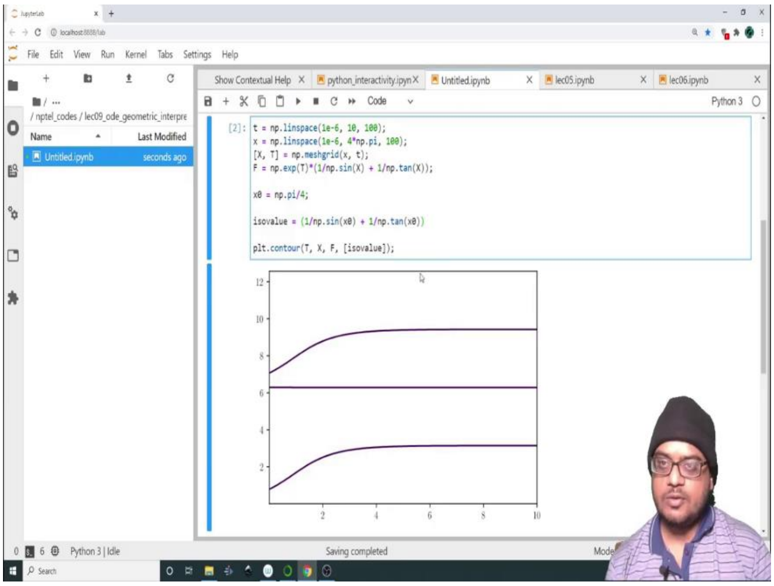


So, when $x0=\pi/4$. So, disregard this because periodicity would dictate that this is also a solution in this space. Corresponding to this particular isovalue, this is how the evolution would look like? But, now whatever this isovalue is.

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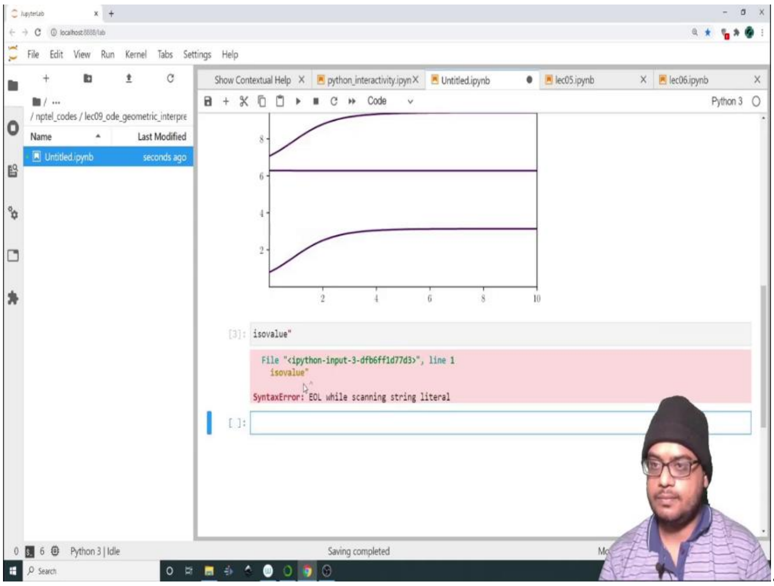


So, let us print out what the iso value is?

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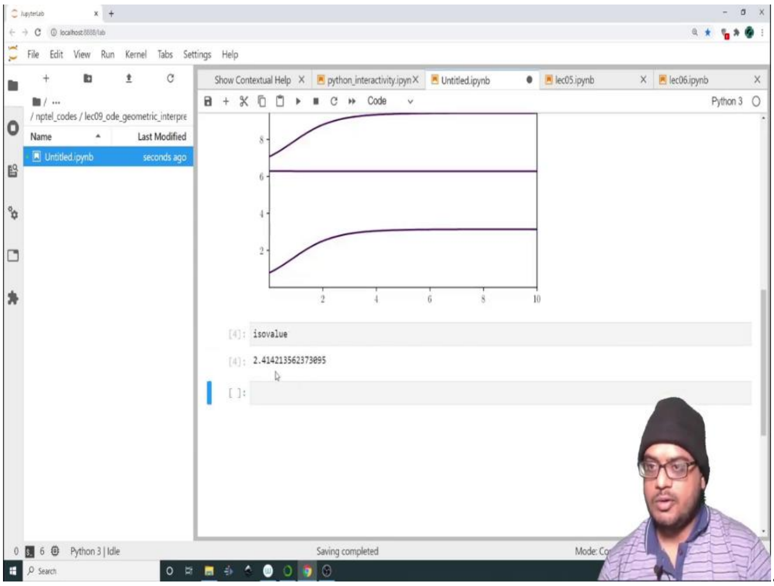


It is 2.41.

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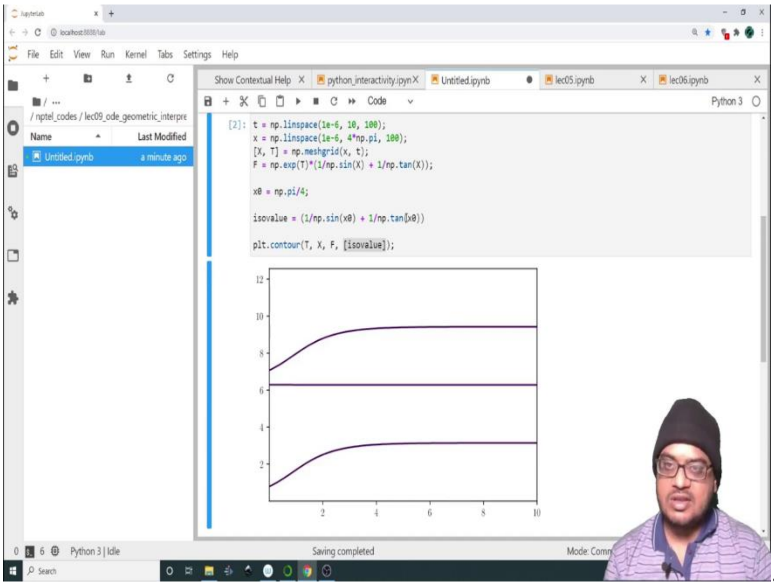


So, these three curves, they are the plot they are like the curves of F in the X T plane which correspond to this particular iso value and $2\pi$ and this particular curve.

So, this particular curve is quite, obvious this particular curve is this particular curve plus $2\pi$. Because, sin and tan they are periodic after a period $2\pi$ that is $\sin(2\pi + x0) = \tan(2\pi + x0)$.

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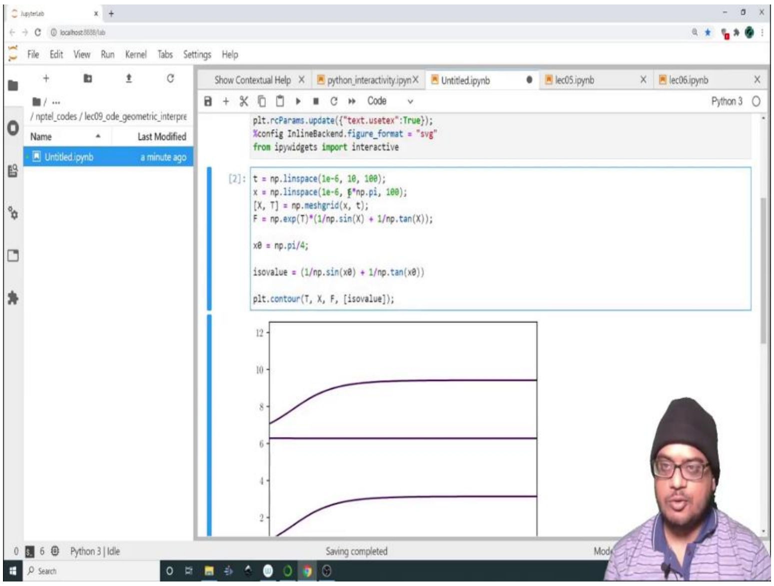


In fact, if I increase this to $6\pi$ we should be able to see more curves ok.

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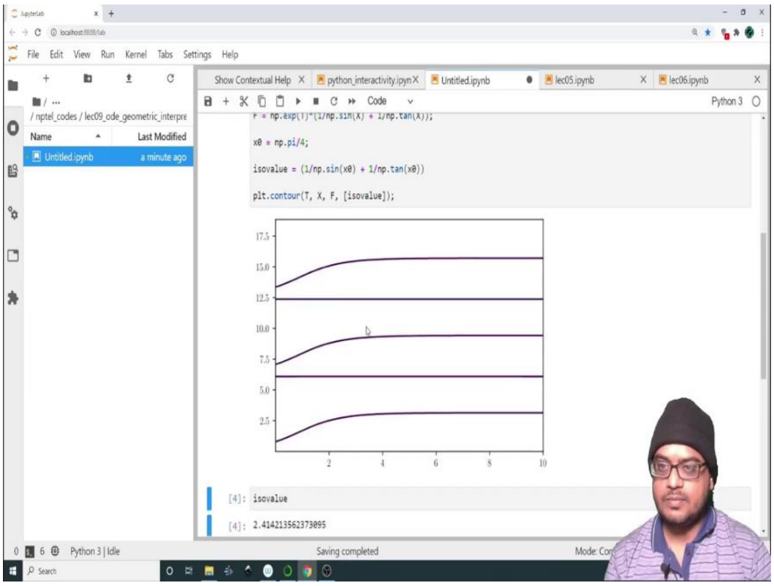


So, it is just an artifact of the fact that everything repeats after $2\pi$ .

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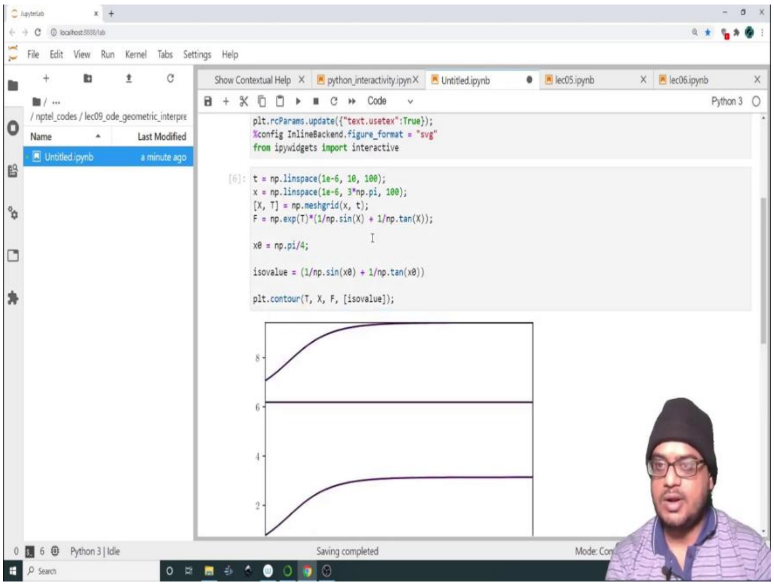


So, in fact, let me keep it till $3\pi$ . So, that or in fact, let me keep it till $2\pi$ itself.

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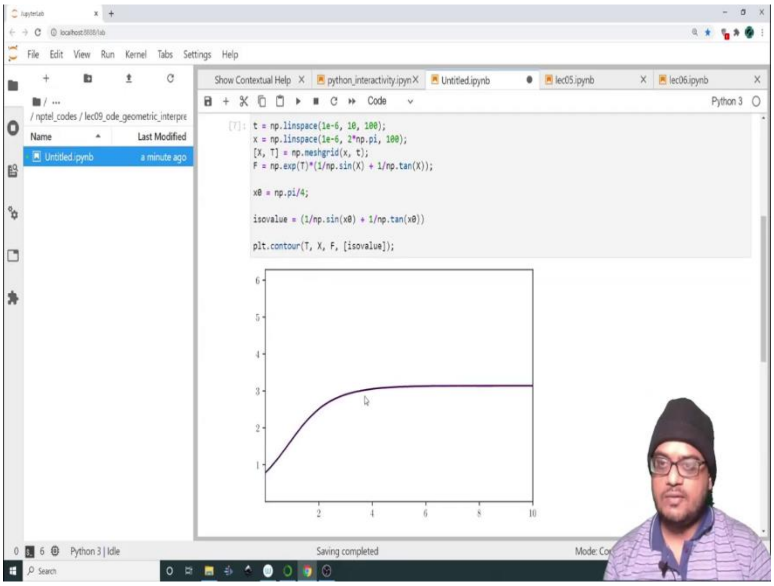


So, we have just one curve ok. Do not be confused it is just a repetition. So, now, because of this we have this particular curve, this is the solution.

So, essentially this curve is the solution that we are looking for. So, let me put some labels.

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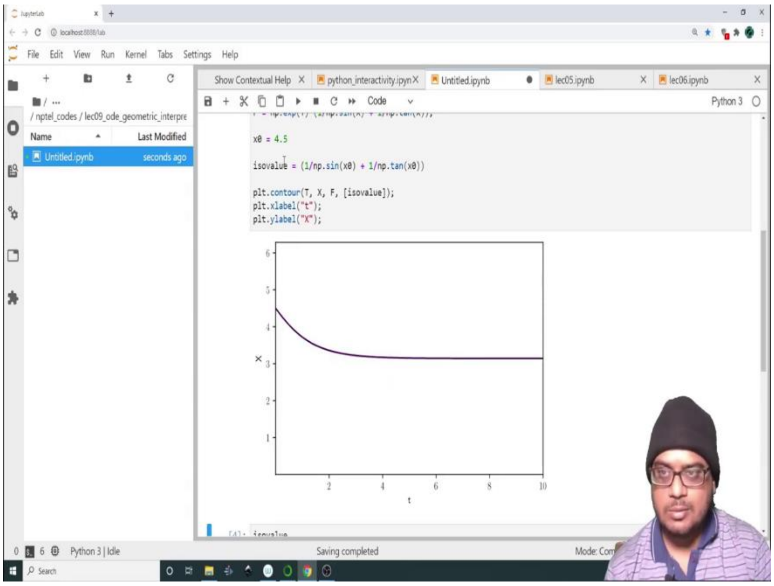


And, for x axis is time and this is x. So, when we start at a value of $\pi / 4$ over here in time the value grows and it settles to this value. Now, if I change the initial condition will this evolution in time also appear to be something like this; obviously, not because when I change the initial condition it will start from some other point.

Let us say that x naught is somewhere over here it is suppose 4.5. So, it starts at 4.5 and in time it decays and it reaches this value. So, there has to be something very specific about that value.

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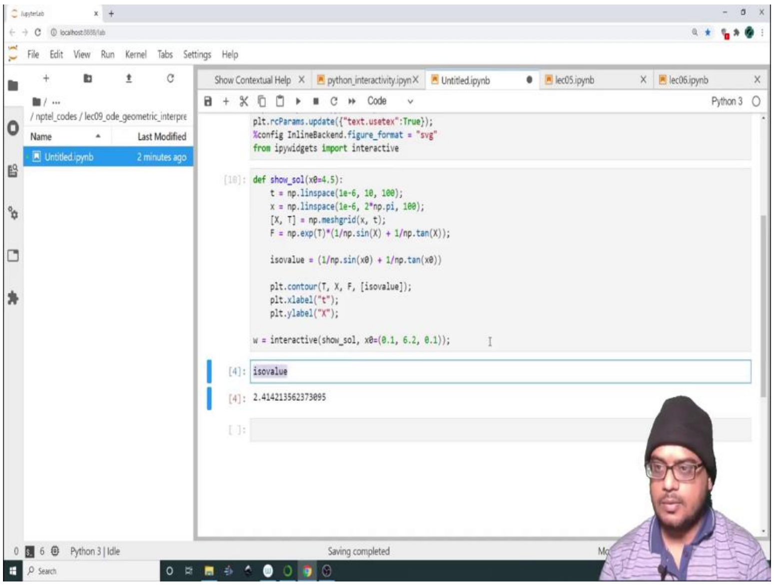


So, let me use the interactivity or in python and pass x naught as the parameter. So, def show solution and the input will be x naught equal to let us initialize it at 4.5.

Let me put everything inside this function and let me define the interactive digit. So, w equal to interactive. Let me pass show sol to this and x0 will take values from 0.1 all the way to 6.2 in steps of 0.1. So, let me run this.

Now, let me display what w is ok.

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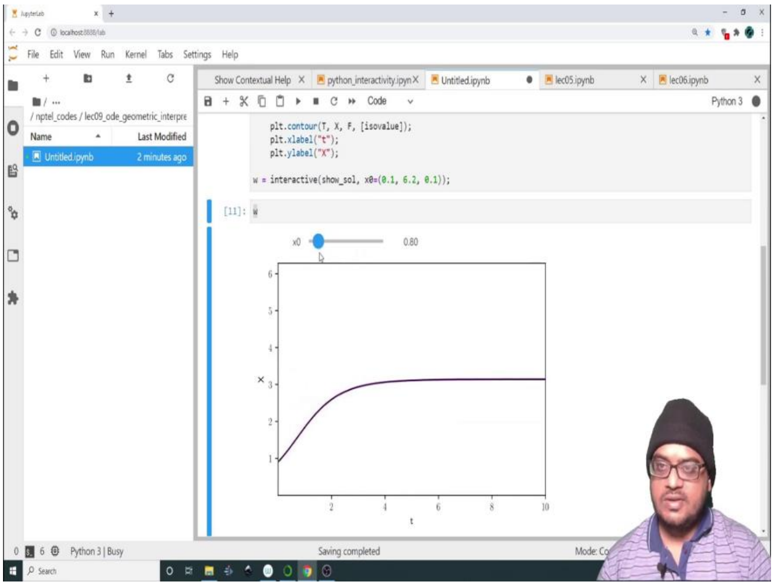


So, let us not change the slider. So, we see that even though we are changing the initial condition. The curve seems to settle down at this value ok. And, this value appears to be something like pi alright.

So, but there is a change in behavior so, here it is increasing monotonously until this value, it is asymptotically reaching that value. Over here also it is asymptotically reaching that value, but when we are far away from this value, there seems to be a s like shape. So, this is sigmoid like shape coming into the picture, similar over here.

So, can we geometrically would we have predicted the presence of that sigmoid kind of function let us see. So, let us now just plot this, let me just plot x and sine x let us try to understand, what actually is going on.

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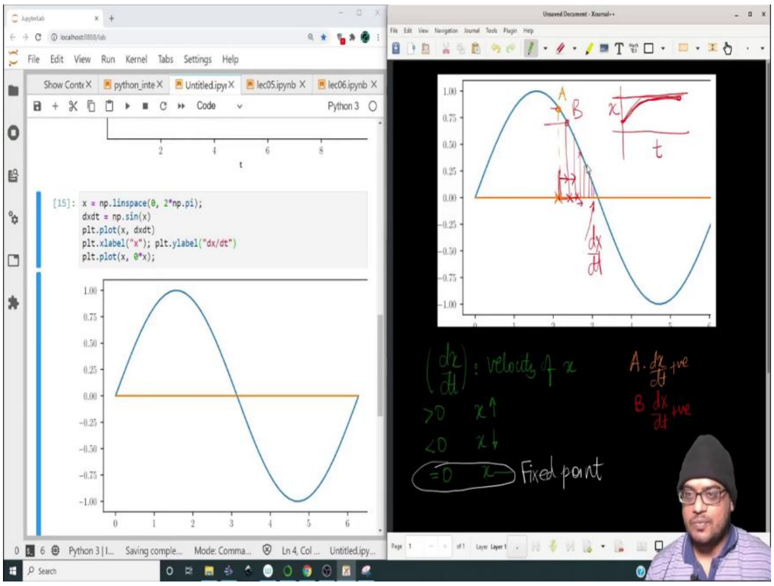


So, let me define x as np.linspace in fact, we have defined it already, but it is inside it is wrapped inside a function, so, we can define it outside. So, x = np.linspace(0, 2*np.pi) and dxdt = np.sin(x), let me plot it ok. So, let me plot a vertical horizontal line as well alright. So, for so, what do we have on the x axis we have x, and on the y axis we have dxdt ok.

So, I have kept the plot of sin(x) over here. So, on the y axis we have dxdt and on the x axis we have x ok. So, what is dxdt? dxdt is like the velocity of x ok, if, we consider x to be somewhere on the axis.

So, dxdt positive would mean, so, when dxdt is greater than 0 it would mean the value of x would increase in time, when it is negative it would mean x would reduce in time and when dxdt is equal to 0, it means x would not change ok. So, what this is? So, this point where dxdt equal to 0 it is called as a fixed point, because the value of x does not change ok.

So, let us analyze whatever is going on over here. So, let us consider this as the initial condition. If, this is the initial condition the velocity which corresponds to this initial condition is over here. So, if I call this point a at point A, we have dxdt positive. So, at the next time instant it would go the particle would move to this point ok. Let me call this point B.

So, at B dxdt is again positive, but the magnitude of dxdt is less than the magnitude of the velocity at this point. So, the magnitude is reduced. So, the velocity has reduced. So, in a given time instant if it travel this distance at the next time instant it will further travel along this, but the distance it would travel would be less. Again when it would travel the next time instant the distance it would travel would be less.

So, if I were to plot x versus time. If, it started at this position x would increase, but at each point the velocity. The velocity is dxdt that is the slope it would keep on reducing. So, you expect this kind of behavior ok. And, it is sort of reaching the asymptotic point.

So, when it reaches more and more towards this point where dxdt equal to 0. The velocity is reducing continuously. So, that is why that asymptote behavior is there.

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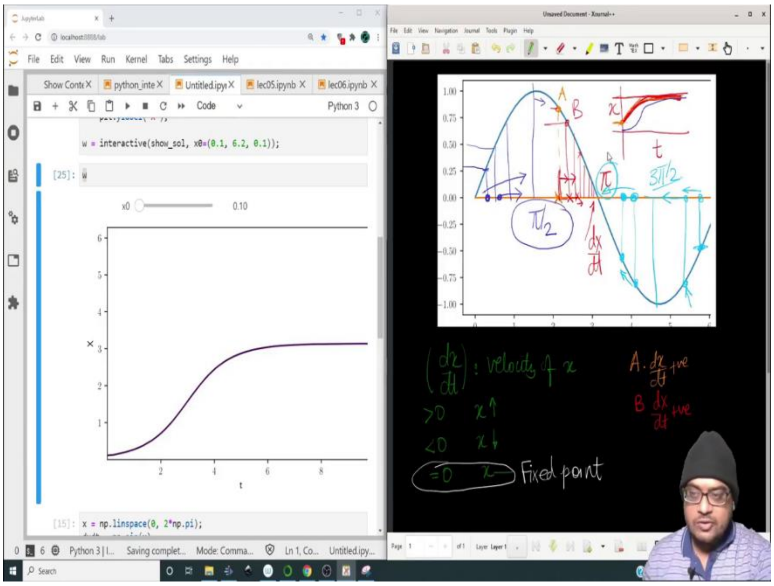


So, if I if I go over here that is why this asymptote behavior happens ok. It is trying to reach the value of $\pi$, but it will reach there after infinite time ok. So, that is why that asymptote happens. Because, velocity over here is large, over here is less. So, if you want to interpret the velocity from this curve, it is simply the slope of this curve, it is simply the slope ok.

So, it is it is taking curve like this. It is increasing initially fast ok, but not increasing initially fast. It is the highest value of the velocity is at the initial point, then the velocity keeps on reducing up until the point it reaches the asymptotic value. Now, what happens, when you start of an on an initial condition over here? And, start over here over here the value of the velocity is something like this. But, it is positive.

For the next time instant it will go over here. When it reaches this point, when it reaches a value of higher x, the velocity has increased. So, it will move faster towards this region ok. Because, each time you increment x ok, each time the particle moves towards a higher value of x, the velocity will increase.

So, there will be a rapid increase in x after this point. So, this point corresponds to $\pi/2$. So, after $\pi/2$ it will increase x, but the velocity will not begin to reduce. So, we will have an initial increase ok. The value of x will increase and then peak out and then it will reach the asymptote let us see, whether that is true ok.

So, initially the value the slope is increasing, as we are moving towards as we as we spend more time as time increases the slope increases, but after a point the slope then begins to fall. So, there is a qualitative change at $\pi/2$. Similarly, if we look at this point, this is this point is $3\pi/2$, if we have an initial point over here ok, over here the velocity is negative dxdt is negative.

So, the particle has the x value will reduce. So, it will go towards this then the magnitude of velocity has reduced further. So, it will very slowly asymptotically reach the value of $\pi$ the value of x will tend towards the value of $\pi$ asymptotically, but if you start over here, the magnitude of velocity is small, but and the direction is negative.

So, x will try to reduce. As it reduces it will encounter a higher magnitude of velocity. So, it will move faster initially, after it crosses $3\pi/2$, the velocity will again start to reduce and it will asymptotically reach the value of $\pi$. So, in order to see this in our interactive plot, let us plot $\pi/2$ and $3\pi/2$ as well ok.

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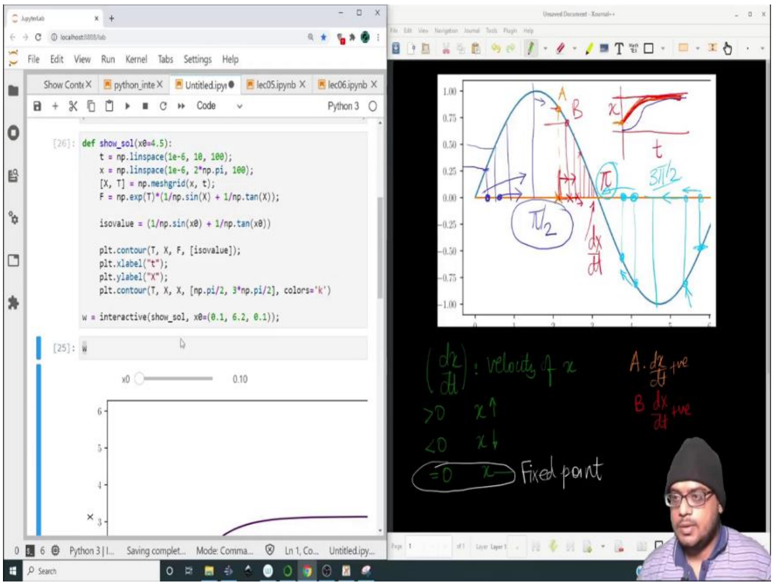


So, let me draw these two contours. So, plt.contour(T, X, X, [np.pi/2, 3*np.pi/2], colors='k') colors let us set it to be black.

So, let me plot this ok.

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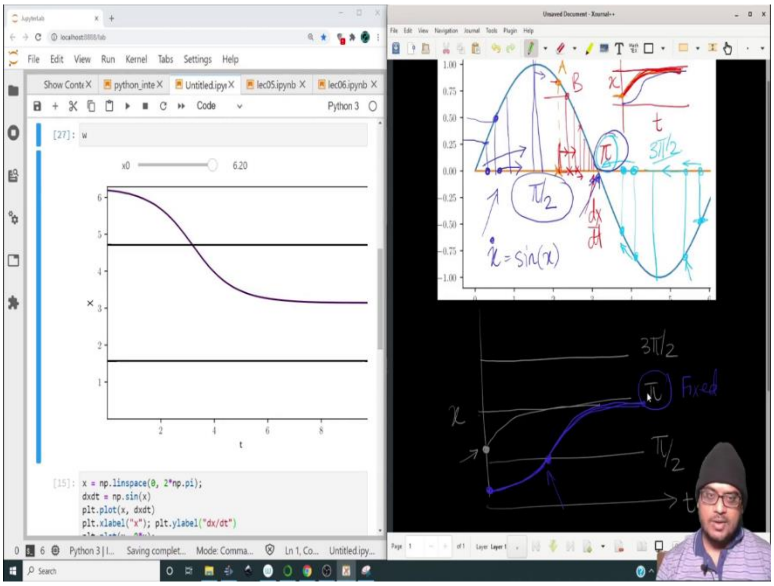


So, these are the two lines across, which we expect a fundamental change in behavior ok. So, this line is $3\pi/2$, this line is $\pi$ this is $\pi/2$, this is $\pi$ and this is the time axis. So, if we start over here. So, this becomes the initial point ok.

So, we are starting between $\pi/2$ to $\pi$ ok, let me remove all this, let me bring this plot over here ok. So, when we start over here essentially, we are starting at some point over here. So, the velocity will continue to decrease up until we reach the equilibrium point. So, it will be something like this.

Now, if we start at a point somewhere over here, over here, that is the velocity will initially increase then it will fall. That is x will rapidly increase in time, x will rapidly increase in time up until it crosses $\pi/2$ after which it will slowly reach the asymptote. Let us see, whether our physical intrusion is correct or not ok.

So, it rapidly increases in time. So, for a small time duration, we have a rapid increase. So, the slope increases after which it asymptotically turns towards the value of $\pi$, but when we start off with an initial condition over here, it monotonously tries to reach the asymptotic value of $\pi$.

Similarly, on the other side it monotonously decreases to a value of $\pi$, while if we cross this initially there is a in there is a high velocity. The velocity is increasing because you can see the slope is increasing as we go over here, up until this point after which the slope starts to reduce and asymptotically we are tending towards the value of $\pi$.

So, just like that without even solving the differential equation, we could have estimated these curves ok. As time increases the value of x is increasing and the slope is increasing because of these reasons that we have discussed.

After, it crosses $\pi/2$ that is this point, the velocity begins to reduce despite x increasing and you will try to asymptotically reach the value of $\pi$. So, this is how we could interpret this particular equation $x = \sin(x)$, or rather $\dot{x} = \sin(x)$, without actually formally going into the solution.

Of course, we could make sense of everything, because we had the exact solution with us. Had we not had the exact solution we could have still predicted these occurrences. We could have said that the slope increases, then slope reduces, and asymptotically we reach $\pi$. Because, pi is the fixed point we must once it tries to reach $\pi$ the velocity will become 0 and x cannot change anymore. So, $\pi$ is the fixed point.

So, just like that I hope I have shown you a way of interpreting certain equations geometrically.

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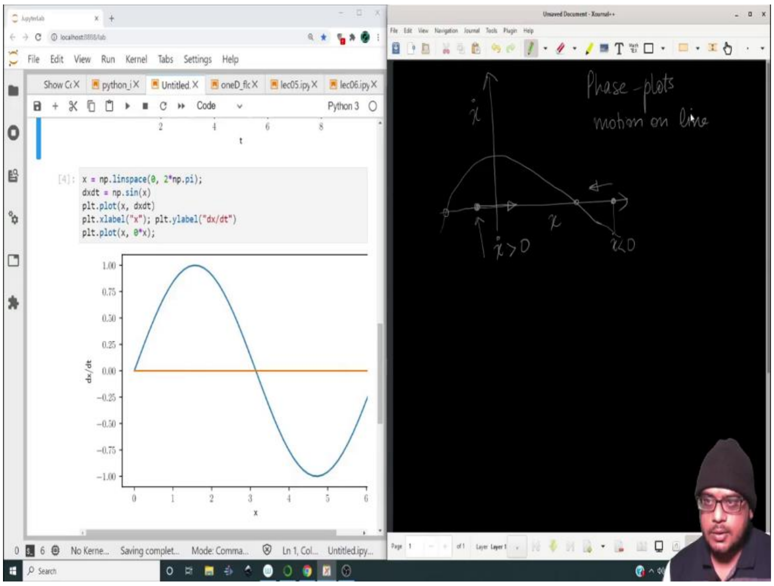


These kind of plots where we are plotting x dot versus x for autonomous systems, they are called as phase plots and the reason is the nature of the plot.

If, $\dot{x}$ is positive over here you know that if this is an initial condition, it will move towards the right, because, this point by virtue of this particular curve as $\dot{x}$ greater than 0. If, you have an initial condition over here, you know that the point will try to move towards the left, because over here $\dot{x}$ is less than 0. So, it gives you an idea about how a particle will move on a line? So, whatever we are doing is motion on a line.

So, let us move forward to what is called as the population growth model.

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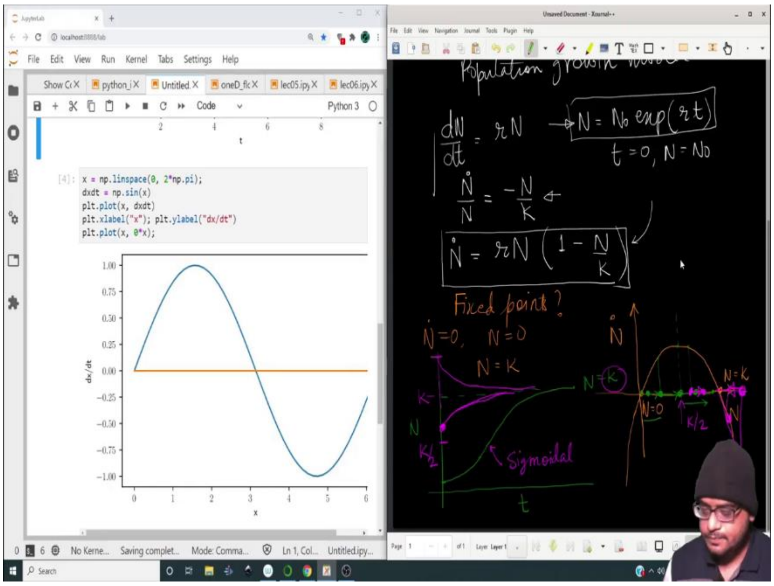

Also it is called as the logistic equation. So, in this particular equation we are trying to model how a population grows? So, consider the following equation, $\frac{dN}{dt} = rN$ . The solution to this is; obviously, $N = N_0 exp(rt)$ , where at t equal to 0, N is equal to $N_0$ . So, it predicts an exponential growth in population.

But, we all know that such kind of an exponential growth cannot be sustainable, because there is a finite amount of resource that you always play with. So, in that case for large values of N, we can say that $\dot{N}/N$ should then decrease. So, $\dot{N}/N$ is given by - k yeah so, as or rather 1 sec as $-N/k$ .

So, this is kind of a feedback loop. That regulates that if N grows quite large there is a feedback which would try to reduce the population ok. So, combining these two things so, $\dot{N}$ is; obviously, the time derivative. We can write down a single equation as $\dot{N} = rN\left(1 - \frac{N}{k}\right)$.

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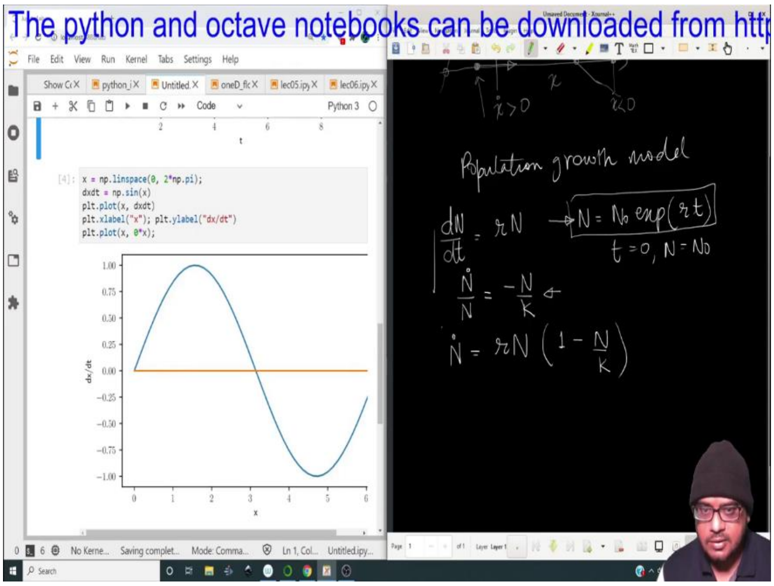


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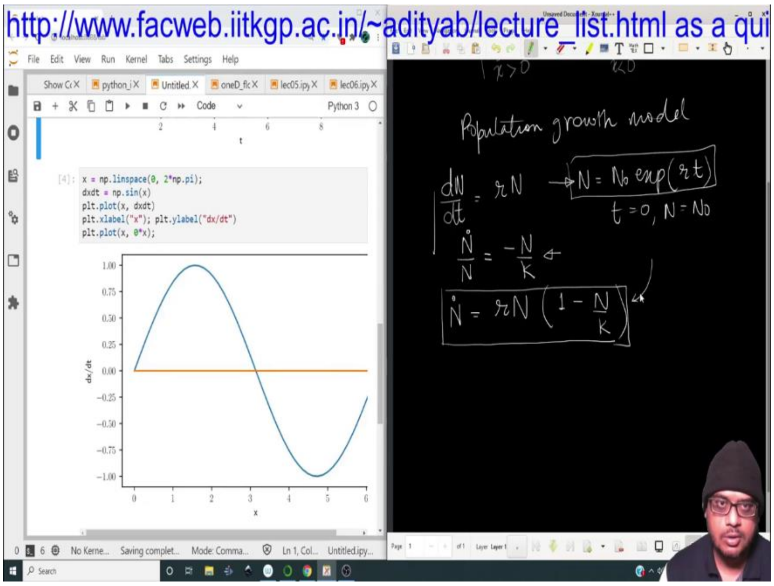


So, this is the governing equation which so, we sort of estimate that this will be the governing equation. And, once we try to look at look into this equation from a geometrical viewpoint, we will see whether or not this equation can model certain population growth or not.

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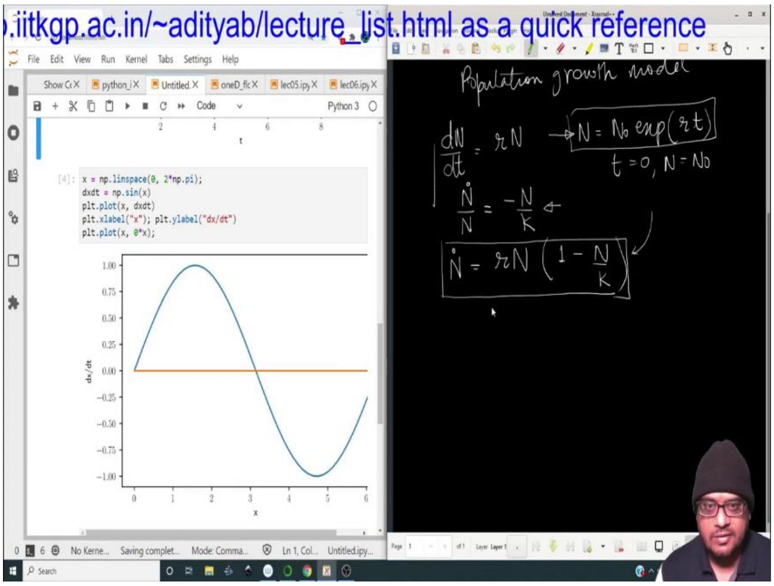


Let us see, what the fixed points are? So; obviously, the fixed point is where $N_0$ is equal to 0. So, when N is equal to 0; obviously, it is a fixed point when N is 0 this entire thing becomes 0. And, the other condition where $\dot{N}$ becomes 0 is when N = K. So, when N equal to K this becomes 1 and $\dot{N}$ becomes 0. So, N = 0 and N = K are the 2 fixed points.

How does this equation actually look like? So, if we draw it like this $\dot{N}$ on this axis and N on this axis. Obviously, it will pass through the origin and it will be a parabola which will look something like this. And, this is the point N = 0 and this is the point N = K ok.

So, it is a parabola like this I mean what I have drawn is not a parabola something like more like this. So, let us see, what we can infer from this diagram. So; obviously, there is a maxima over here fine. And, if we are starting off at 0 population, let us see what happens? I mean of course, 0 is an 0 is a fixed point meaning at N = 0. There is no change in N, $\dot{N}$ is identically 0 so, nothing.

So, let us start very close to the origin. At this green point $\dot{N}$ is positive meaning the population will grow. So, it will move on the line towards this direction. At this particular point the $\dot{N}$ has increased. So, the population will grow quite fast in the same time it will grow faster because $\dot{N}$ has increased. Once it reaches this point then $\dot{N}$ is still positive.

So, this point will still move to the right once it reaches this point $\dot{N}$ has begin to has begun to reduce. So, the point will slowly move towards N = K. So, what we expect. The behavior in time is that if you start at low species number.

Initially there will be a fast rise, then at certain point there will be a change in the behavior and there will be a slow gradual increment of N towards N = K, something like this. So, this asymptote is N = K. Of course, it will take a long time to reach N = K. Now, if we start if we have an initial condition over here, somewhere over here. Then, what happens?

We are already at a point where $\dot{N}$ is positive and we are moving towards N = K, but now what happens is as we start moving towards N = K, we are in a region where $\dot{N}$ has reduced. So, it will slowly move towards N = k.

So, if we start off with the concentration somewhere over here. So, this point is K/2. So, suppose this is K/2. So, if you start over here, there will be a monotonous function which will go something like this. We would not have this initial increase and then reduction ok. So, this sigmoidal kind of behavior will be true when we are starting with low populations.

When we already start with population greater than K/2, we will have a monotonous function like this. So, K is called as the carrying capacity that is the ultimate population. Now, what happens when we start at a larger concentration? Somewhere over here, if we start somewhere over here $\dot{N}$ is negative, so, the point will move towards this direction.

When we move toward this direction we have an $\dot{N}$ which is still negative, but it is reduced in magnitude. So, it will slowly start moving towards N = K. So, if we have a population larger than K, then we expect a monotonous reduction in the population until it reaches, the carrying capacity ok.

So, this is what we can infer directly from the geometry of the problem, without even solving the problem ok. So, now, let us see whether whatever we are predicting is true or not. So, let us go to the computer and so let us first yeah. So, in order to do this we must integrate, we must sort of find a solution to this initial value problem.

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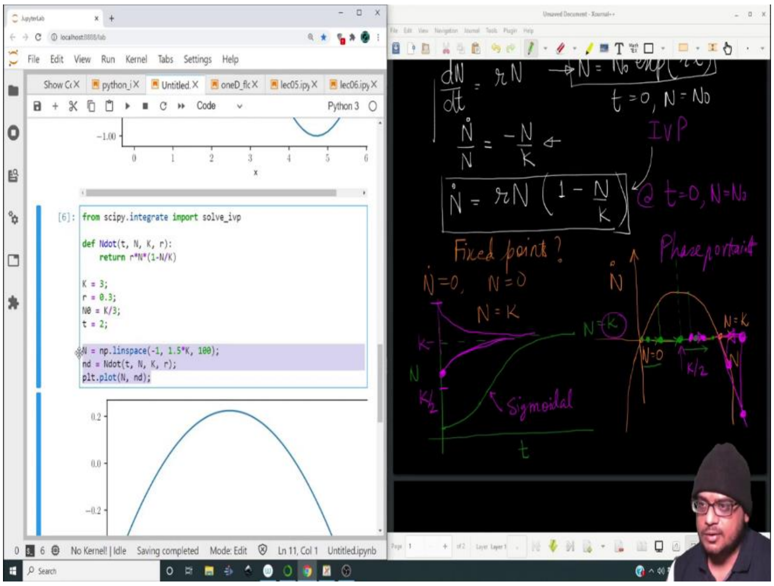


So, this kind of a problem is called an initial value problem, where you must also specify some initial condition at t = 0, suppose the population is N = $N_0$ and then we can vary $N_0$ and see what happens? Ok.

So, let us first import the relevant function from scipy. So, from scipy dot integrate import solve ivp. So, ivp so, solve ivp is the name of the function which resides inside the sub module integrate inside scipy ok. So, we will need this and that is pretty much it I mean, now we can go ahead and use the function to solve. So, I will talk slightly more about the function once we are done with this ok.

So, let me do this. So, let us define what K and r are. So, let K = 3, let r = 0.3, let $N_0$ = K/3 ok. We can define it in terms of K fine and let me define the function handle. So, define $\dot{N}$ and it will take as an input t, it will take, it will output rather it will take as an input N also and the 2 parameters K and r.

So, these will be the inputs. So, based on this it will it should return us the right hand side. So, it should return as r*N*(1-N/K) alright. So, now, that we have this, let us check whether $\dot{N}$ returns us something so, nd = $\dot{N}$ , so, 2 $N_0$ K r.

Now, let us print nd. So, nd return something. So, we are in the green. And, in fact, we can easily check this by plotting the whole thing, we can define N = np.linspace - 1 to 1.5 K, let us take 100 points and nd will be $\dot{N}$ so in order yeah.

So, it will be $\dot{N}$ t comma, so, we have to define what t is although the function is autonomous we still can pass some time let us say 2. And, actually this plot is independent of time we are not plotting the time dynamics on this plot we are just plotting the face portrait, that is a very important thing to appreciate and this is simply a face portrait alright. So, we will pass time will pass N will pass K and r and let us then plot ok.

So, we do have that curve that we are looking for and of course, this is the x axis ok. So, once we are convinced we can move ahead and solve this.

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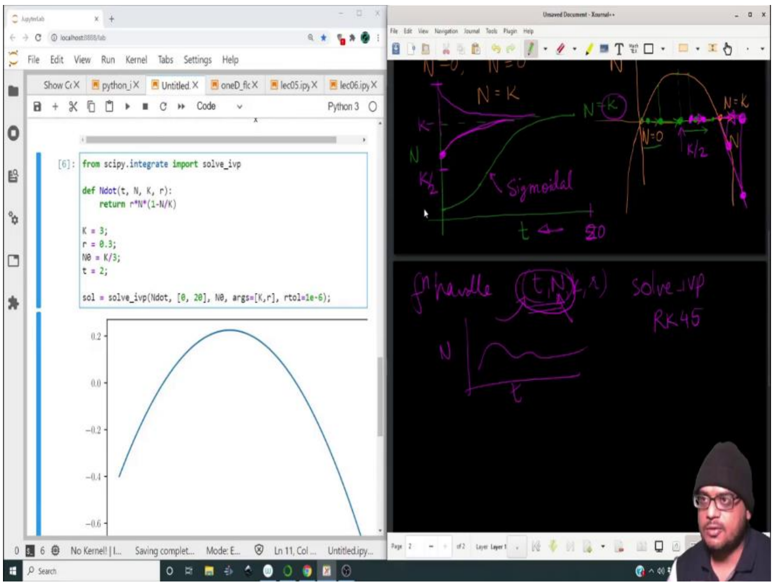


So, let me write it at write the solution as sol is equal to solve_ivp, we have to pass the function handle $\dot{N}$ we have to pass the time span. So, let us say we want the time dynamics.

So, we are looking at the time dynamics over here. So, let me plot it till t = 20 for example. So, time span is the initial time 0 and the final time 20. We will pass the initial condition $N_0$ and we will pass the additional arguments to the function handle. So, here is the thing if you have additional arguments.

So, ideally a function handle could simply have t and N. And, you are actually solving the initial value problem, where the independent variable is t dependent variable is N ok. So, essentially you will have N versus t, but here our function handle has two additional arguments K and r.

So, solve ivp ideally would accept a function, which will give output as this, but since we have additional arguments K and r we need to do the following, we need to pass args equal to and then we need to pass K, r. In the same sequence as the function and we will pass the relative tolerance. So, rtol=1e$^{-6}$ no problem.

So, so far we have not specified what solver it should use and we will discuss initial value problems later on in the course, but for now I just want to show you that, this tool exists and you can make use of it to understand things quickly. Just for those of you who are interested the solve ivp uses the Runge Kutta 45. So, 4th 5th order accuracy. And, let us in fact, have a look at the contextual help ok.

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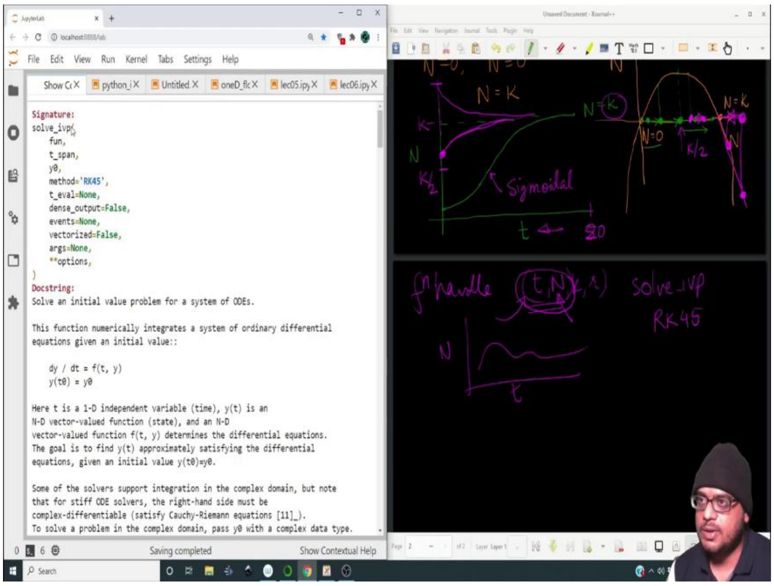


So, solve ivp it accepts the function, we have to give the time span, initial condition we can additionally specify the method, we can additionally specify at what times you want the output to be dense output these things will discuss later on in the course, events also perhaps if we have time we can discuss it and args. So, these are the arguments to the functions.

Additionally there are options, which we can pass to the solver. So, over here we can look at all this as well.

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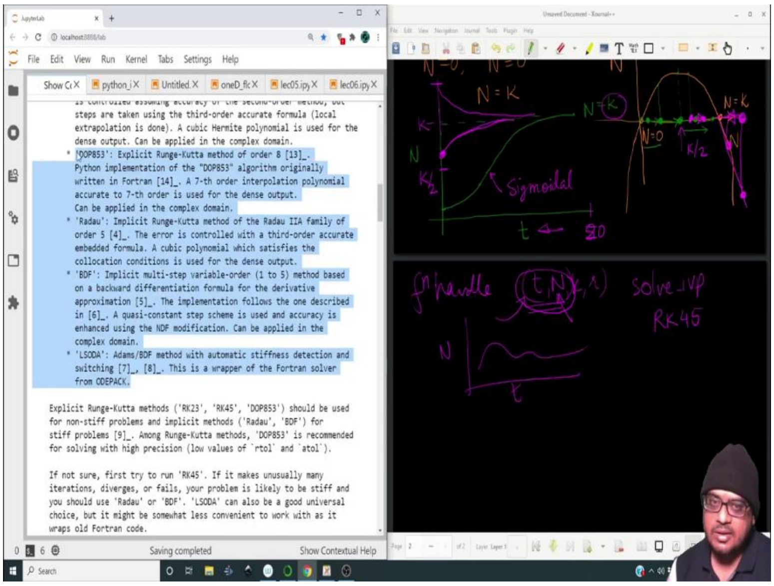

So, method consists of RK 45, explicit Runge-Kutta RK 23 explicit Runge-Kutta of reduced order, the dormand prince algorithm the ‘Radau’ integration BDF solver, LSODA solver.

So, these are various solvers that have been written by very proficient people, who spend their lives making things more not only accurate, but very efficient to solve on a computer ok. And, I mean you can spend some time to have a look at it. So, we are more interested in the relative tolerance.

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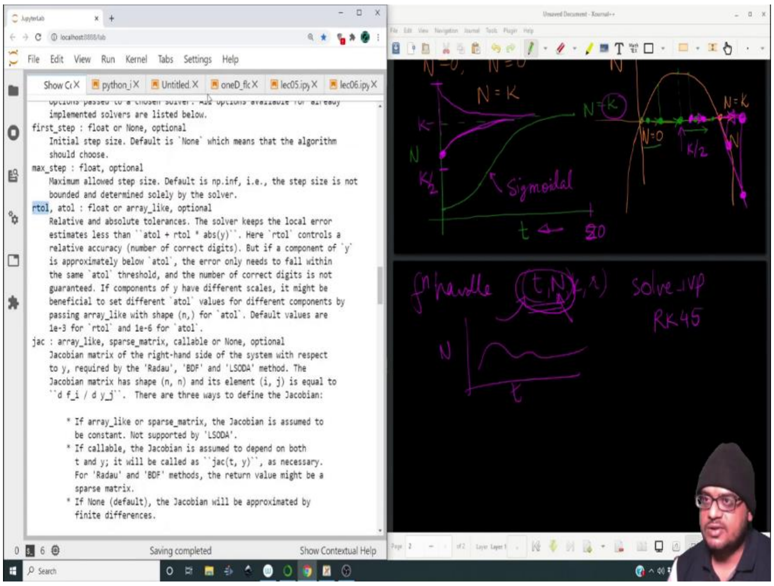


So, we need accurate solutions ok. So, let us get back to our file, this is how we would call the function. Let us run this an error.

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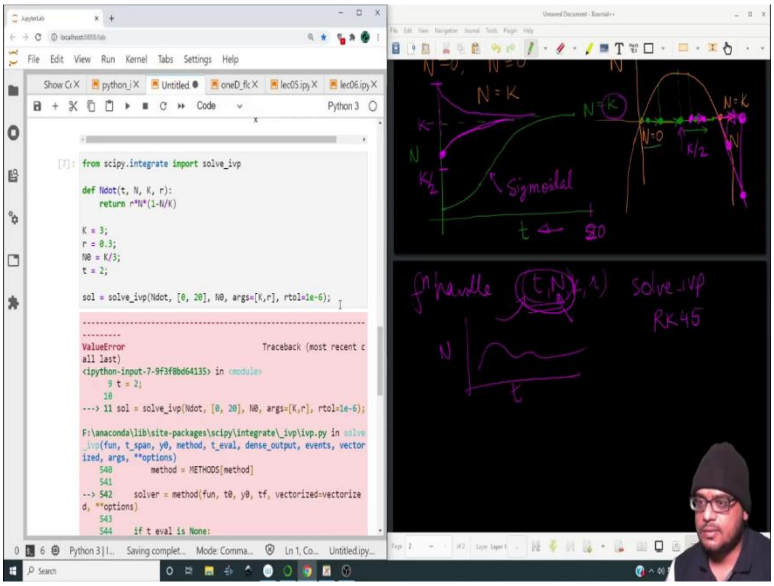


Let us see, what has happened, $y_0$ must be one dimensional ok.

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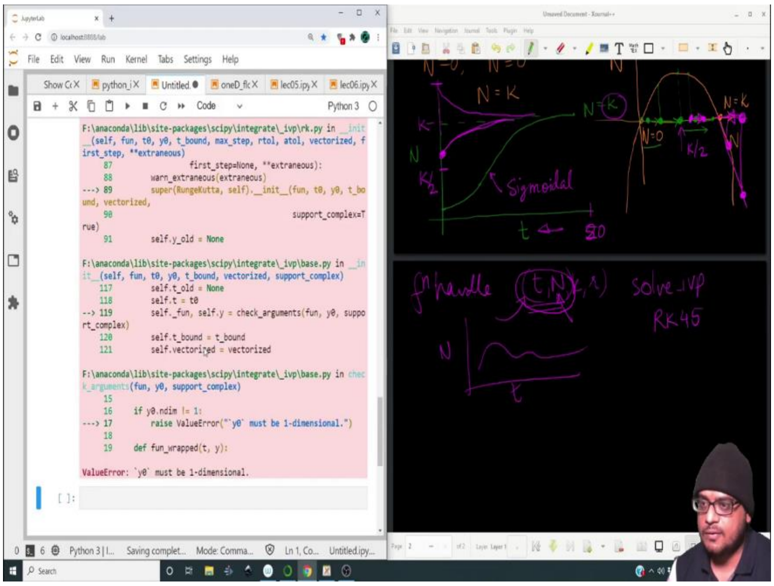


So, we must pass it inside as an array.

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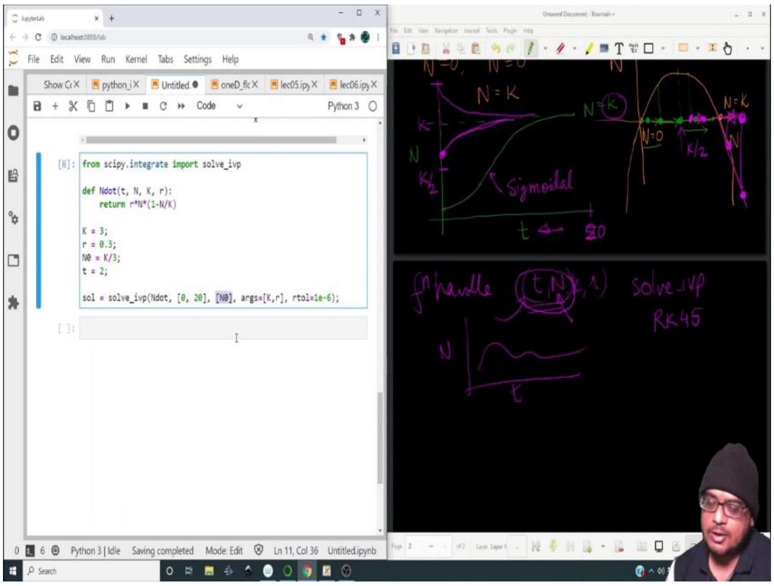


We cannot pass it as a scalar; we must pass it as an array ok. So, now, that we have the solution, let us see what the solution structure actually contains.

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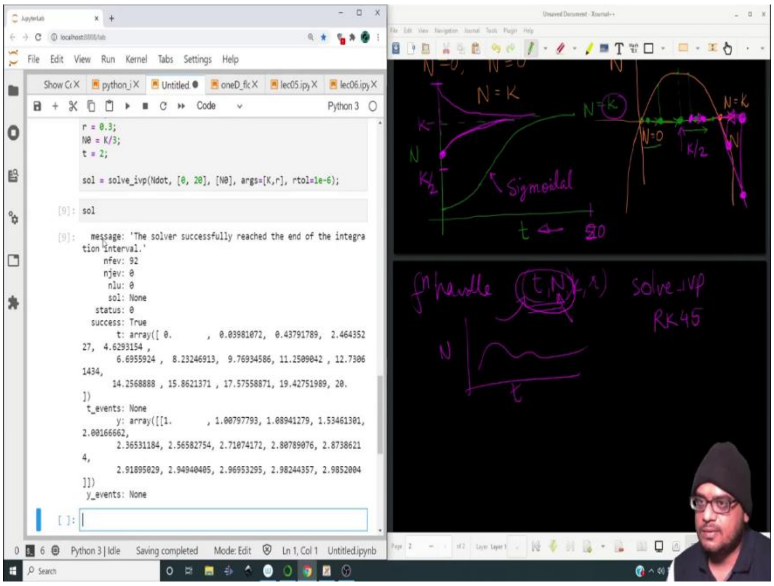


So, solution structure contains solver reached the end of integral. So, which is fine the number of evaluations is ninety 2 t array is this the y array is this.

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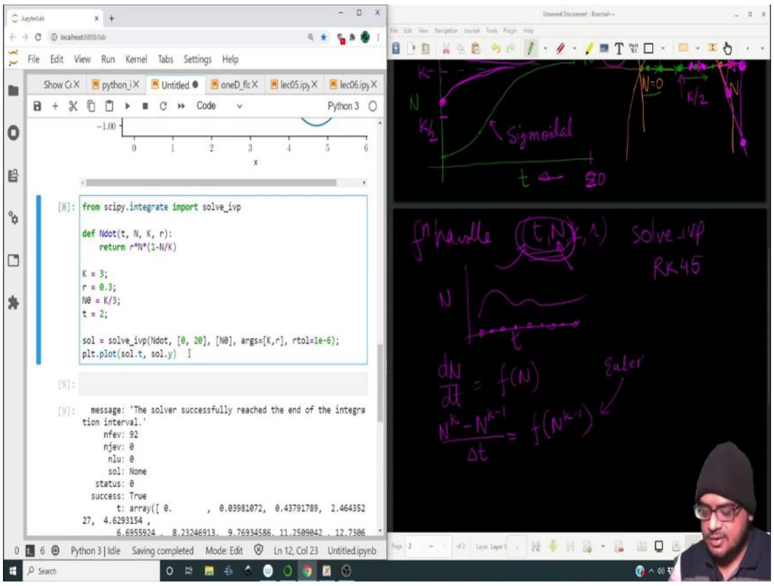


So, we are more interested in the structure sol.t and sol.y. So, let us do that let us plot sol.t and sol.y. So, we will do plt.plot. Sol.t and sol.y.

So, the structure sol contains the points the solver has chosen on the x axis. So, we have not specified what times do I want the solution. Given this relative tolerance, the solver will automatically choose the time steps on which it will perform the integration steps ok.

So, the simplest way of doing it I mean if you are interested. So, $\frac{dN}{dt} = f(N)$, you would discretize this as some kind of as a as a forward step. So, $\frac{N^{K} - N^{K-1}}{\Delta t} = f(N^{K-1})$ ok, this is what you would do? But, there are more sophisticated algorithms of doing this, because this method which is the Euler's method it suffers from a variety of truncation issues ok.

It's not very accurate; it is only order $\Delta t$ accurate. So, we use the default solver which is RK 45, 45 and let us plot this. Let us see, how it looks? Ok is ok.

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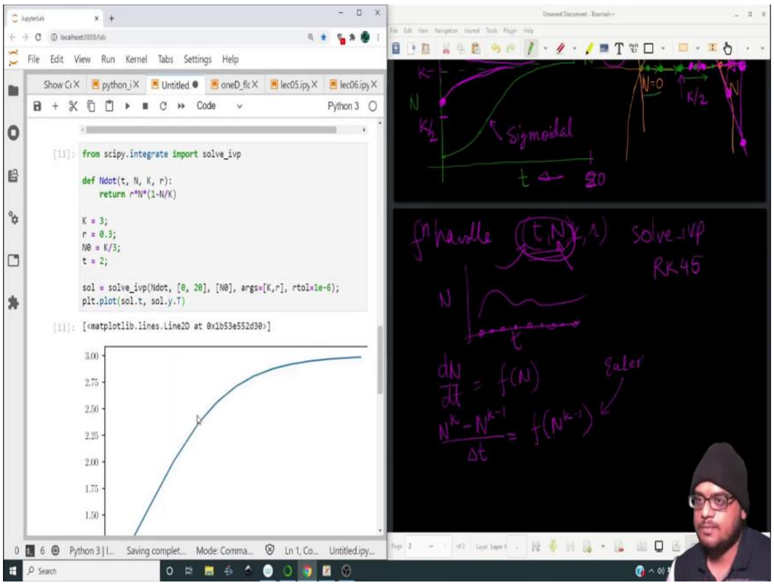


So, we have to take a transpose ok. So, here is the plot.

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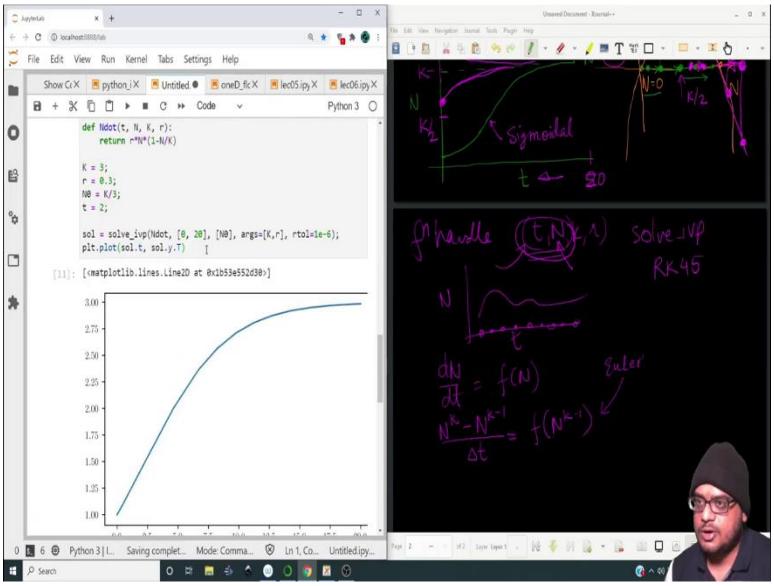


So, let me put some labels on the plot.

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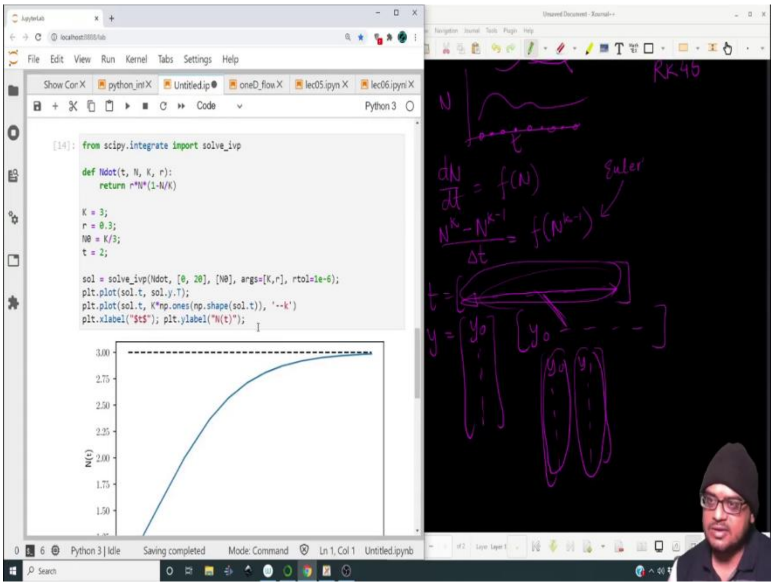


So, this is going to be time, this is going to be N as a function of time. And, you might be wondering why I took a transpose of this object, because when you solve this it gives. So, these are small idiosyncrasies of the output. So, t is a single array like this, but y so, by default it assigns the solution to y. So, y is something like this. So, if you have one variable, $y_0$ is something like this.

So, you must take a transpose of this, so, that you can plot these two. Once you take a transpose of this it will give you something like this and then you can plot these two. But, you cannot plot a row vector with a column vector that is not possible, that is why this transpose is necessary.

So, in case you have a system of equations, which we will encounter later in the course, you will have $y_0$ all the values over here, $y_1$ all the values over here. So, you have to plot t as a function of this, t as a function of this and so on. So, that is why you need to take a transpose. So, there is nothing to do with the techniques at all it is just some something which is inbuilt into the function ok. So, this is how the plot looks like. Let me enlarge this yeah.

So, we do see that there is a growth and then saturation towards K = 3. So, in order to see the carrying population, let us plot K = 3 line as well. So, let us do plt.plot(sol.t, K*np.ones(np.shape(sol.t)), '--k'). So, it will initiate an array which is all ones multiplied by K ok. So, it should be np.shape ok. And, let me make it as a dotted black line. So, this is the carrying capacity alright, this is how it varies?

So, let us now exploit the interactivity in python to vary this initial condition and see how the plots look like? So, for doing that we need to wrap everything inside a function we have already done this. So, let me copy this bit, let me keep this cell as it is let me delete this particulars or let me use this cell in.

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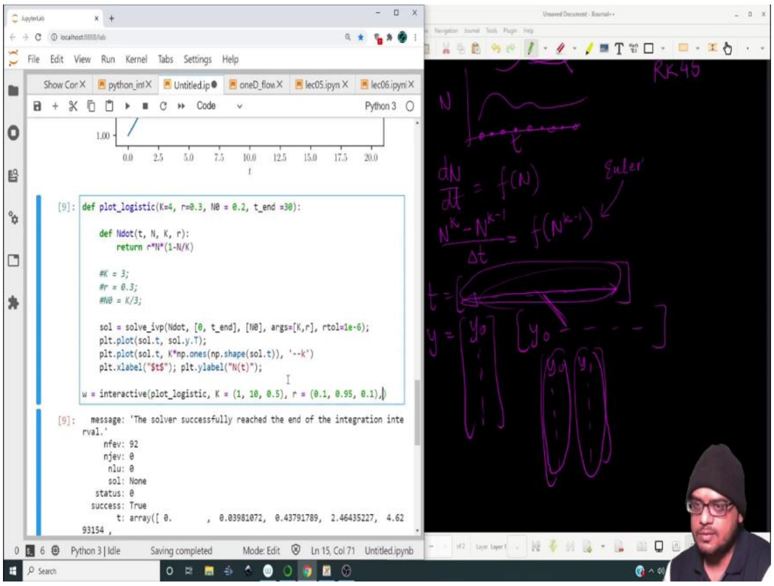

So, we need to wrap everything in another function. So, let us say def plot-logistic and let us pass K and r as the two variables ok, or other inputs to the function. Let us indent everything inside the function, because we want everything inside the plot logistic function.

Let me comment out K and r. In fact, let us take as an input $N_0$ as well. So, let me comment out this particular thing as well, t it makes no sense. So, let us in fact, remove this line. So, so we have this is going to be t and let us not hard quote this, so, t_end and $y_0$.

Let us pass t_end as an input to the function as well. Let us put default values of 30 $N_0$ as 0.2 r as 0.3 and K as 4. So, these are the default values. So, once we are done with this, let us call w = interactive then we will pass the function handle which will call the interactive thing.

So, plot logistic we will say K goes from 1 to 10 in steps of 0.5, r goes from 0.1 to 0.95, in steps of 0.1 $N_0$ should go from 0.1 to 20 in steps of 0.5.

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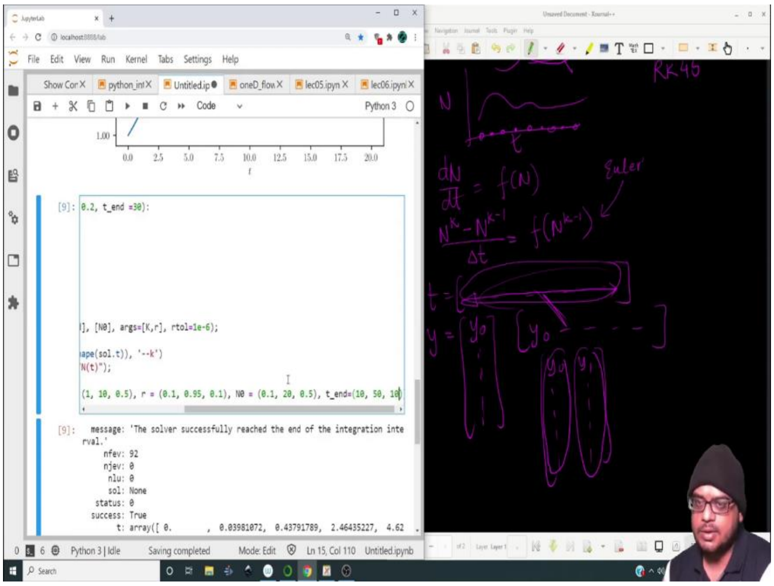


And, t_end should go from 10 to 50 in steps of 10 that is fair enough, let us display the widget we have to display the widget as well.

So, let us have a look, let us run this and see what happens good.

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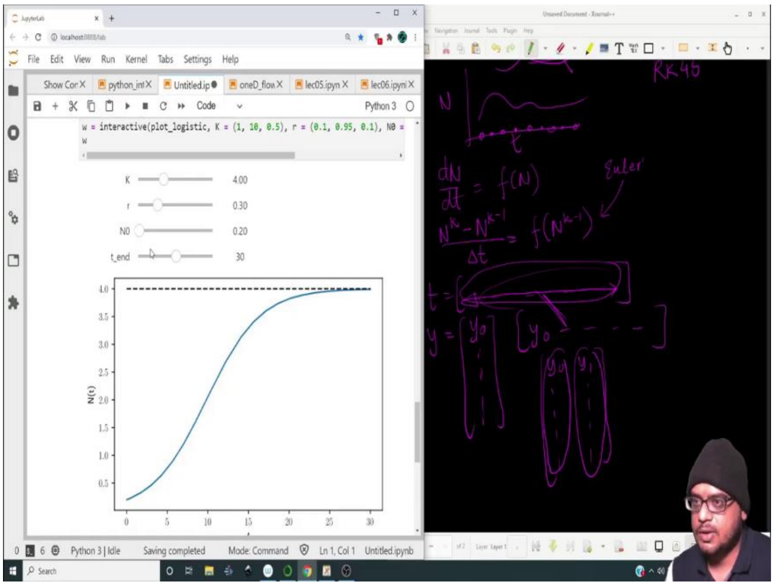

So, we do have the plot. So, right now K is 4 and $N_0$ is smaller than K/2.

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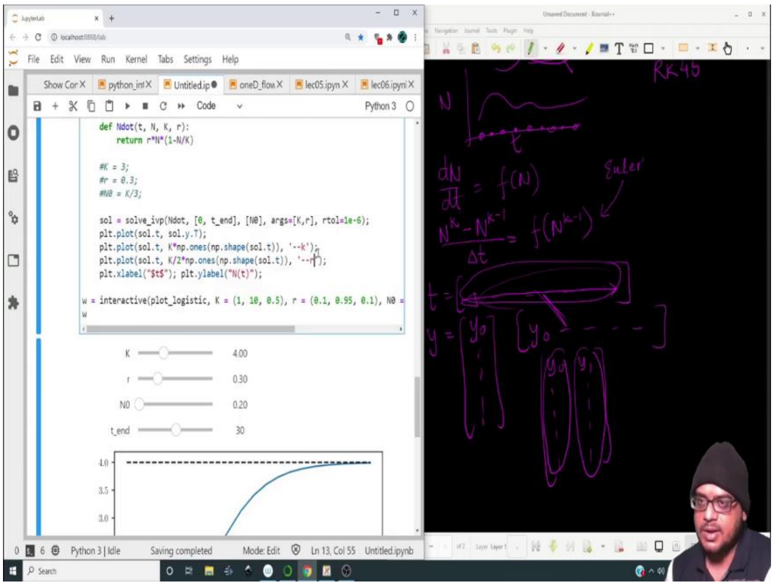

So, let us in fact, plot K/2 as well, that is a good indicator of the midpoint of the parabola. So, let us plot K/2 and let me make this a red line alright.

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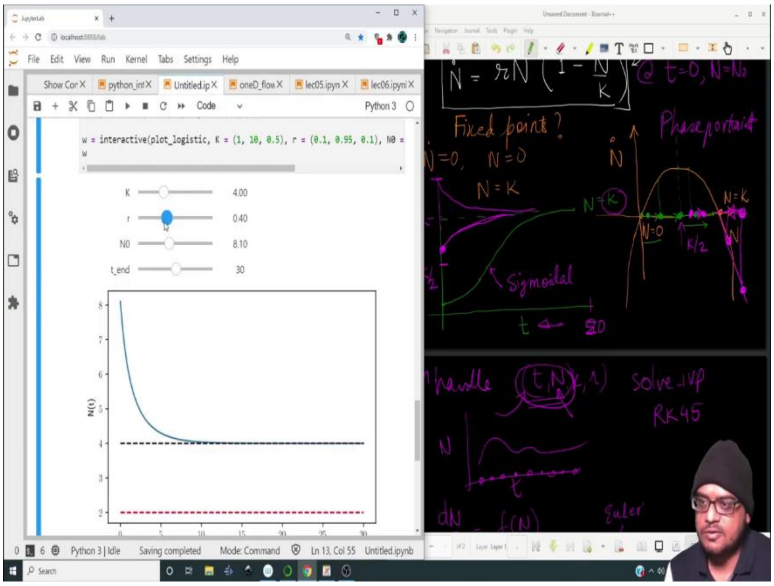


So, when $N_0$ is smaller than K/2, we do have a sigmoidal kind of curve when we increase this. So, we start getting that monotonous increase which we had predicted over here. And, once we exceed we do have a monotonous decrease as we had predicted as well alright.

So, let us change r and c. So, if the rate increases it falls quickly or slowly. So, r is like the time constant of this particular plot alright. Let me reduce this ok.

So, if I reduce this we get a flatter curve because it would take more time for the population to grow. If, I do this it would take less time for the population to grow. So, r is like the rate constant, K is like the carrying population; t end is how long you want the simulation to be ok.

So, this is how we can model the population growth using a very simple model; obviously, this model has various limitations, we are not going to discuss the limitations over here. But it provides a nice way, so, if you think about the problem geometrically.

It provides you a very convenient means of getting a lot of information without actually solving the problem ok. Without actually arriving at how N actually varies in time, we could make a fair estimate of how it would actually tend towards the carrying population, or we could see that the population would tend towards the carrying population in the first place ok, because it is a fixed point.

So, we could not only predict the fixed point. But, also we could predict how the slope would change as we move towards the fixed point.

So, with this particular example, I will end this particular lecture and I hope you enjoy doing the homework problems. I will see you again next time with a new lecture until then it is goodbye from me bye.