Tools in Scientific Computing
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Lecture – 18
Bifurcations and 3D Flows

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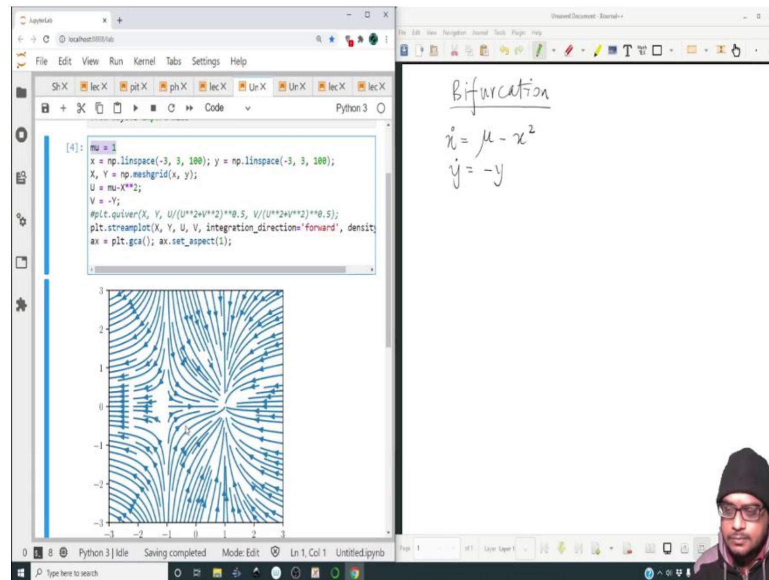

Hello everyone to this last lecture of week 3, in which we are going to study Bifurcations in 2 dimensional systems. And lastly we are going to conclude with Poincare section and 3 D vector flows. So, coming to the point of bifurcation; I have already created a file in which I have imported all the necessary modules and created a small stream function plot of the following vector flow.

So, $\dot{x} = \mu - x^2$and $\dot{y} = -y$. So, let us first look at how the vector flow looks like before analyzing the given set of equations, ok. So, when $\mu = 1$, the vector flow looks something like this, ok. So, at $x = 1$, we have trajectories which are attracted towards it.

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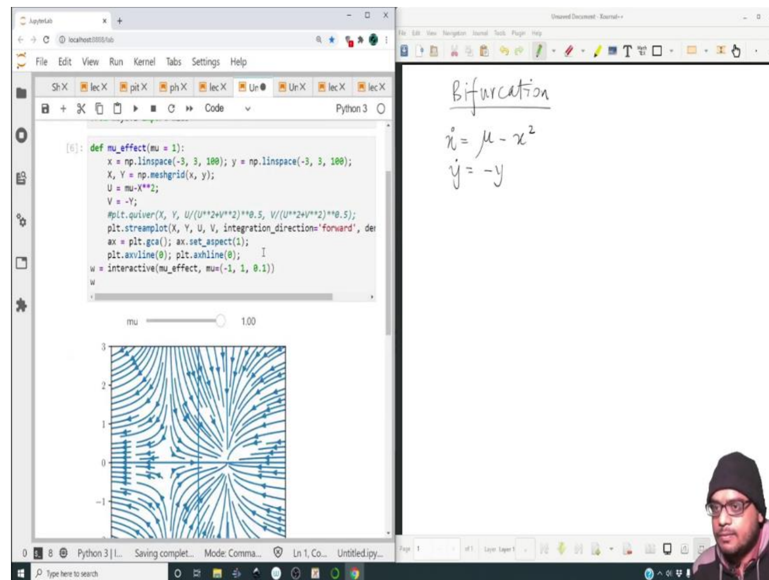

Let me also plot the y axis, on the x axis and y axis, alright. So, with reference to these axis it is clear that $x = 1$ is attracting trajectories; $x = -1$ is sort of attracting trajectories towards it along the y direction and its repelling trajectories away from it along the x direction. Meaning, the y direction at $x = -1$ is the attracting manifold, the stable manifold; whereas the x axis is the unstable manifold.

Now, why is this why is, why does it look like this? And what happens when this control parameter μ is changed, ok? So, in order to assess the effect of the control parameter, let us wrap everything inside an interactive widget. So, let me remove this. So, def mu_effect and let us create a default value of $\mu = 1$; let us indent everything, so that it is all inside the function and w = interactive , let us pass that function handle and mu let us say it goes from -1 to 1 in steps of 0.1. Let us then display the widget ok.

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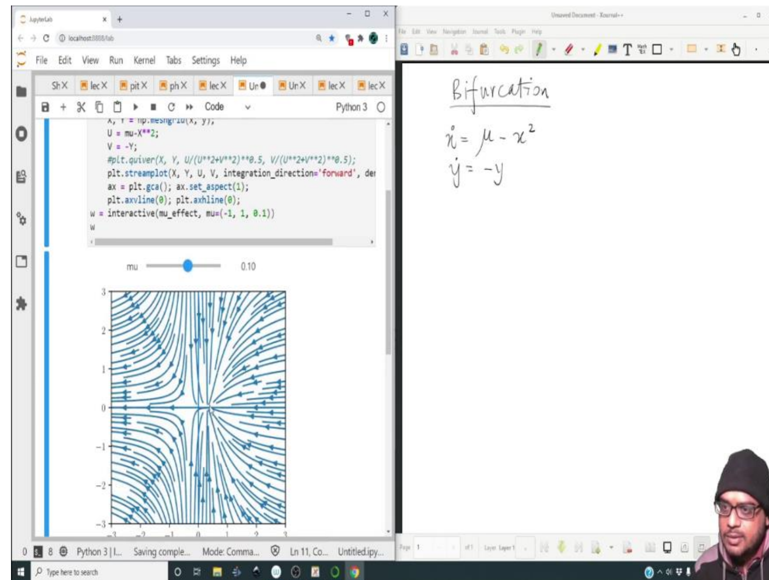

So, now let us change the value of μ at 0.9, at 0.6. So, it appears that the points are coming closer ok; at this point the unstable manifold, which is still the x axis. So, if you look carefully in this region, you can see that the trajectories are still repelled towards the negative x direction. But there are still attracting trajectories in the middle and still this point is like the stable node, alright. Let us see what happens when μ becomes negative.

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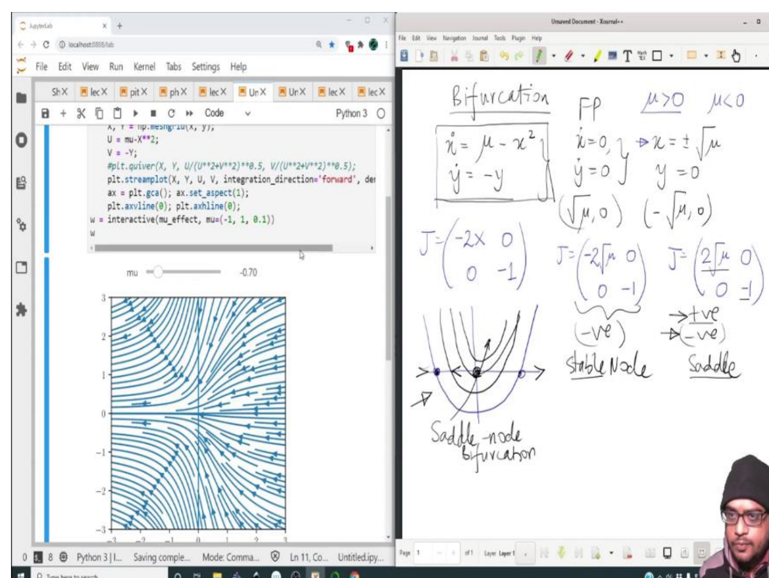

So, when μ becomes negative, the trajectories while they are attracted towards some point; they are not really getting attracted towards a specific point on the x axis, they seem to be

flowing away and away, ok. So, they are attracted and they seem to flow away. When μ is made further negative; you see that all the trajectories are driven away towards - infinity.

So, let us now analyze this particular equation for this change of behavior as μ is changed. So, what are the fixed points of the system? So, the fixed points will be where $\dot{x} = 0$ and $\dot{y} = 0$.

So, this happens, because these equations are sort of decoupled; meaning $\dot{y}$ depends only on y and $\dot{x}$ depends only on x, it is rather easy to find it out, and $x = \pm\sqrt{\mu}$ and $y = 0$. So, these are the solutions of this; meaning that the fixed points are $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$

Let us now assess the stability of the trajectories near these fixed points; for that we will have to evaluate the Jacobian. So, the Jacobian of this set of equations is $\begin{matrix} -2x & 0 \\ 0 & -1 \end{matrix}$. Now, the Jacobian at this particular point, it will be $\begin{matrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{matrix}$ and the Jacobian at this point will be $\begin{matrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{matrix}$, alright.

So, when does this fixed point exist? It exists on this real plane, when μ is larger than 0. In fact, when μ becomes less than 0, there is no fixed point; because the curve does not intersect the x axis. So, now, if you recall the one dimensional case; so if we have a curve like this, if we have a control parameter which allows two intersection points right and if we change continuously the value of the control parameter, so that this convex shape is moving upwards, there will be a point, where there will be no intersection.

So, it will have two fixed points; one of which is stable, and one of which is unstable and eventually you will have no fixed points. And while doing that, we will pass through a through a certain control parameter, which you will have one point which is half stable. So, this means that, this analogy between this particular one dimensional flow and this two dimensional flow is similar to a saddle node bifurcation.

And why is it so? Because both the eigenvalues are negative over here; while both, rather one of the eigenvalues is positive and one of them is negative. So, it is trivial to tell this, because it is a diagonal matrix; the Jacobian is a diagonal matrix in this case. So, the eigenvalues are directly the values at the diagonal value at the diagonal locations. So, one is positive and one is negative, alright.

So, this point is a stable node; well this point is a saddle node, as a saddle, I mean it is not a saddle node. So, it is a saddle, because one direction is attracting; so this is the repelling direction, this is the attracting direction and hence when mu is changed, so that we have the presence of a stable node and a saddle.

So, as μ goes from positive to negative, you no longer have the roots and so you are causing annihilation of the fixed points. And hence this is prototypical of a saddle node Bifurcation in two dimensions, alright.

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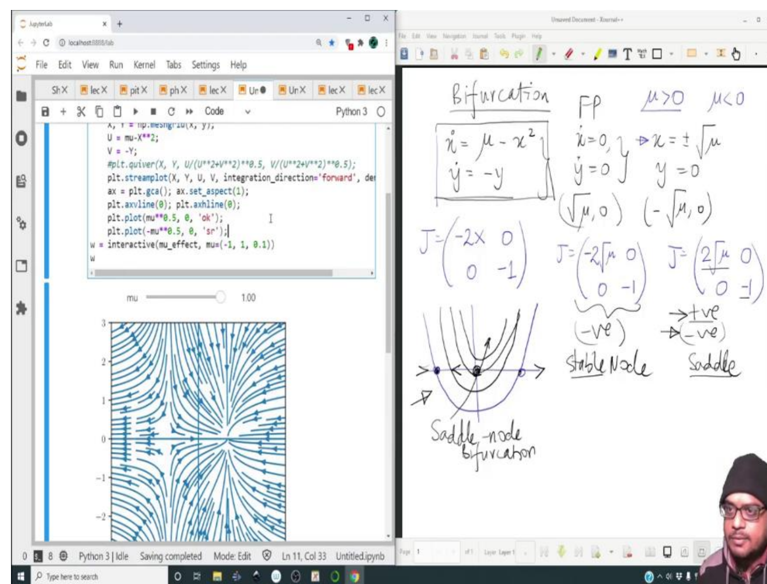

So, what are some other characteristics of this kind of flow? We see that when μ is positive, we have this fixed point; in fact let me plot the two fixed points also. So, one of the fixed point is this; let me mark it with black marker, and one with a red square marker.

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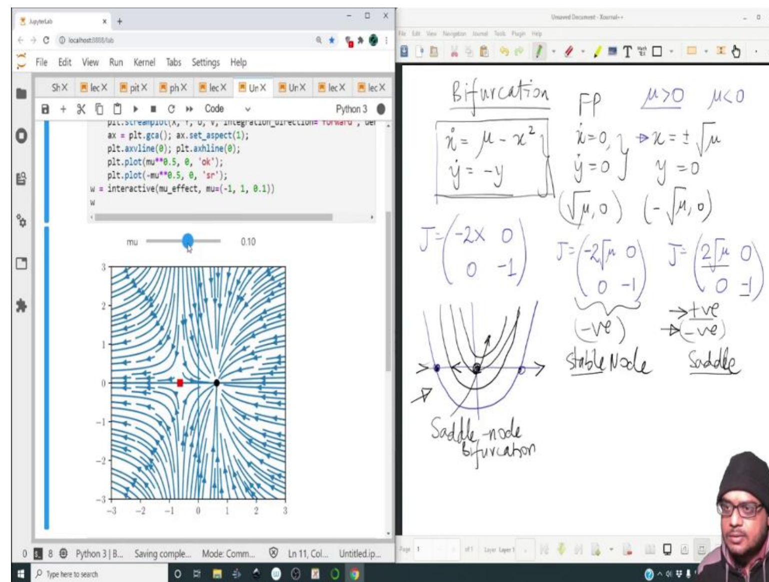

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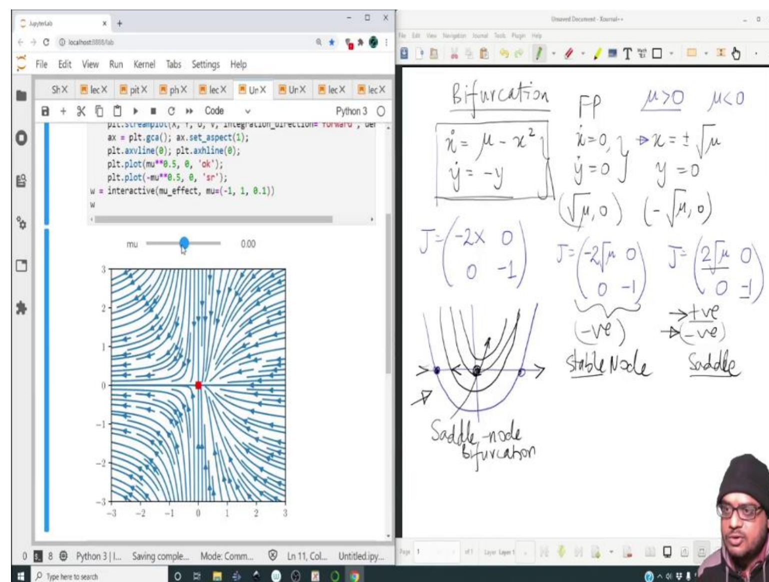

So, the black point is the in this stable node, while the red square is the saddle, alright. So, as we change the value of μ, we see that the two points are coming close, at 0 they merge, alright.

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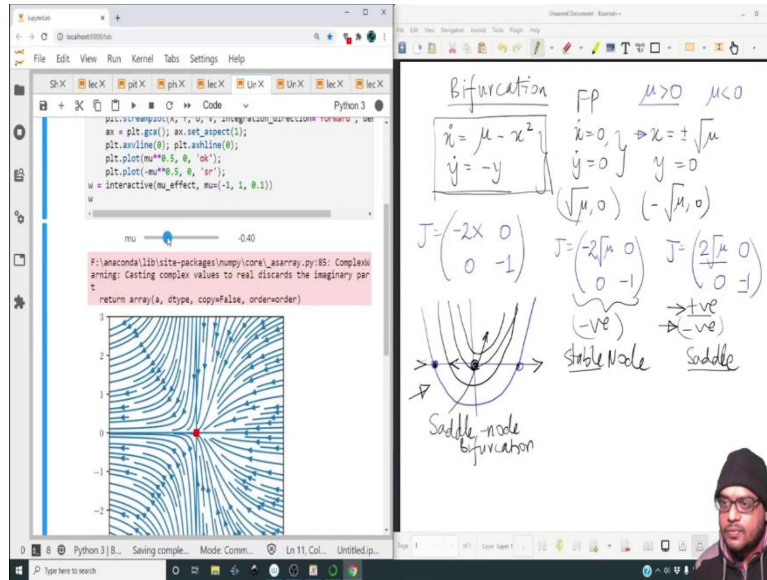

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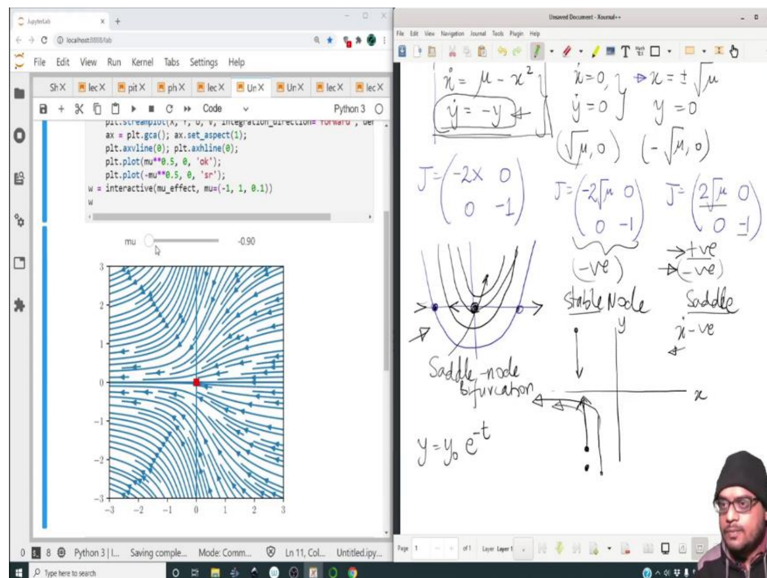

And after that, we are no longer able to plot the fixed points; because the value of μ becomes negative, but that is ok, the flows all tend towards the left. So, why does the floor tend towards the left?

So, let us try to figure it out. It is not that difficult and probably you might have guessed it by now. So, if this is the plane; this is the x axis, this is the y axis. So, when μ is negative, $\dot{x}$ is negative for all values of x and as x becomes larger in magnitude; the flow is stronger towards the negative direction. Actually similar case will happen when x is larger in magnitude on the negative side as well.

Whereas all the trajectories are still uniformly attracted towards the x axis, because of this equation; this equation says the trajectory over here will be attracted towards $x = 0$, ok. It is trying to make or rather $y = 0$; it is trying to reduce it, because this equation will be tantamount to $y = y_0 e^{-t}$. So, as the trajectory progresses in time, the point will try to approach the $y = 0$ line that is the x axis.

Now, as it does this, it is sort of repelled in this direction as well. Now, depending on the value of μ; the repulsion appears to be such that, it is first fast in this direction and then it this particular flow takes over, ok. If you look at these regions; it is coming in from a large distance quickly and then smoothening out to go in the - infinity direction.

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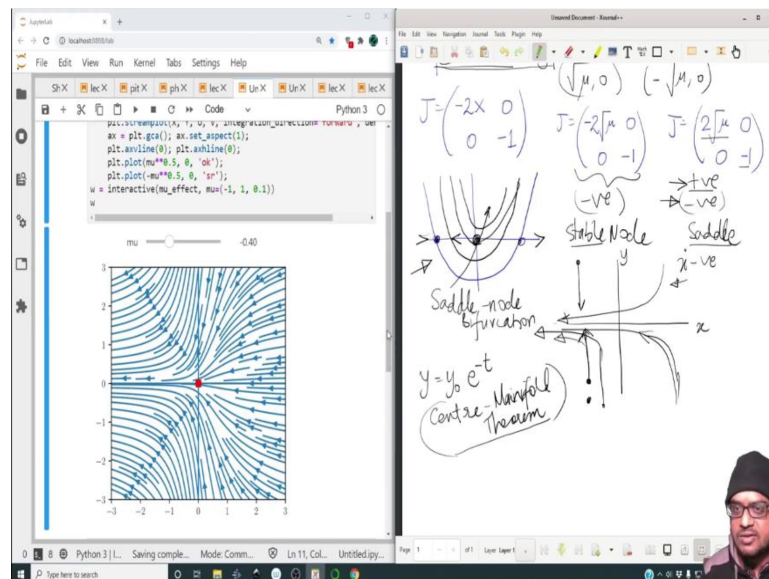

So, something like this; it is coming fast and then going like this, it is coming fast and then moving like this. So, all this indicates that depending on the relative magnitude of the eigenvalues, we can have a fast dynamics, a fast dynamics and then a slow dynamics. So, it quickly homes into a region near the x axis and then moves merely along the y axis, that is the unstable manifold.

So, the annihilation of fixed points still gives rise to a flow, but a flow which is devoid of any fixed points. And once you learn advanced concepts in this field of mathematics; you will see that such kinds of systems can also be analyzed by means of the centre manifold theorem.

And I will link some; I will put some links in the description as well. So, this is what I wanted to show you about saddle node bifurcation. Before we move on to the next kind of Bifurcation bifurcation, let me them to show you another example from Strogatz.

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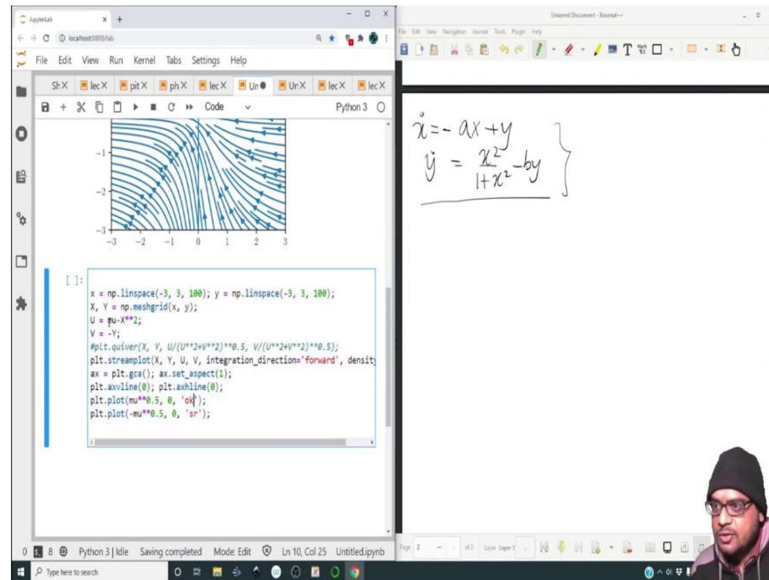

The example is $\dot{x} = -ax + y$ and $\dot{y} = x^2/(1 + x^2) - by$ this particular example will really help you in assessing under what circumstances will such a kind of Bifurcation exist, ok. So, suppose you have this particular set of equations.

Now, can we say directly, whether this particular set of equations has; I mean will experience a saddle node bifurcation, that is the question. So, let me copy this particular code and paste it over here; let me remove this for now, ok. So, let me change the flow, let me get rid of these points as well, ok.

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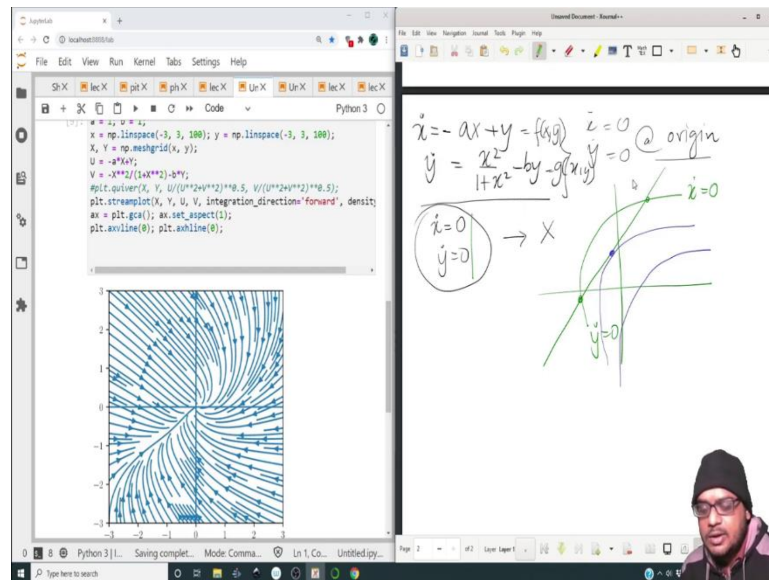

So, let me show how the flow looks. So, quite obviously, there appears to be an attracting direction like this and trajectories are attracted in general towards the origin. Is the origin of fixed point? The answer is yes; because when we plug in $x = 0$ and $y = 0$, $\dot{x}$ and $\dot{y}$ are indeed 0 at the origin. Now, what you can do is, find out the Jacobian of all of this and try to figure out what happens.

But really speaking that sort of work in order to assess, whether this set of equations will undergo saddle node bifurcation. So, for a saddle node Bifurcation to exist, we must go from a situation where roots exist, two roots no longer exist, ok. So, when will roots exist? When the null clines will have some intersections; meaning if this is f(x) or rather f(x, y) and if this is g(x, y). So, now if we have two nullclines.

So, this is the x y axis. So, suppose I am just drawing two null clines. So, this is the nullcline of x = $\dot{x} = 0$ and this is the nullcline of $\dot{y} = 0$. So, we need both of these to be 0 simultaneously and they are 0 simultaneously at this point and this point, ok. So, meaning these two are like fixed points; but if the control parameter causes this particular curve to shift in this manner.

So, there is just one root and then there is no roots, ok. So, if it shifts in this particular fashion; it means that we are going from a situation where we have two roots, then only one root, and then no roots. So, such kinds of evolutions of the nullclines is an indication whether or not the system will undergo a saddle node bifurcation. Let us plot the null clines of this particular system.

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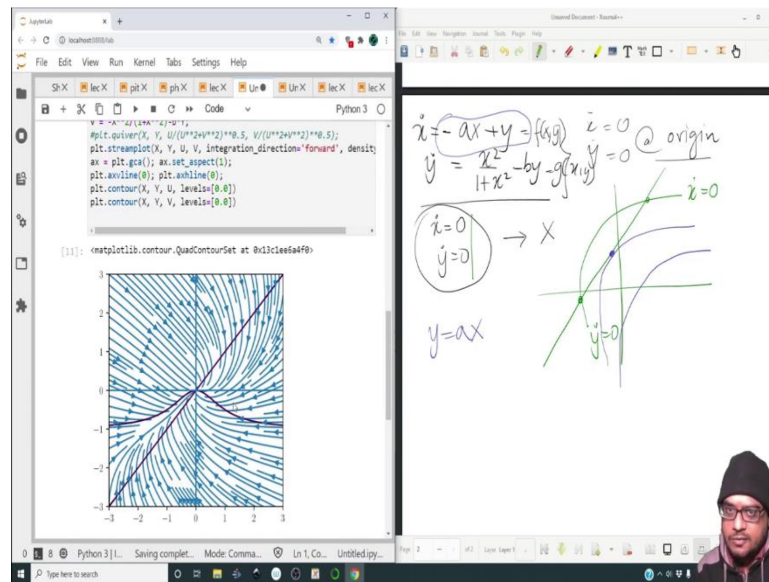

So, we want the nullcline of U. So, this is the nullcline of this particular flow that is going to be a straight line obviously; because when $\dot{x} = 0$, y is simply going to be a x, it is a straight line passing through the origin. Let me also plot the nullcline of the function g(x, y), that is V ok. So, the nullcline is obviously intersecting over here.

Now, let us change values of a and b, in order to assess whether such kinds of intersections will exist for all values of the control parameters a and b. So, obviously we have two control parameters now.

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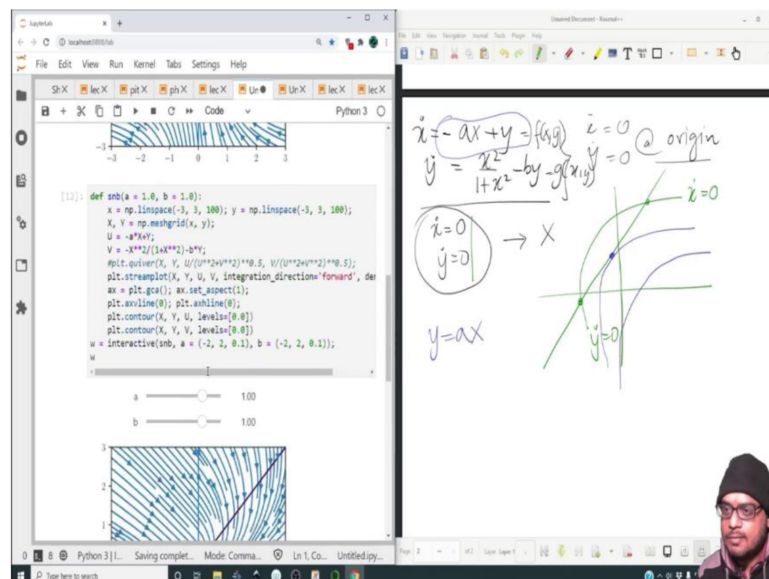

So, let me wrap this inside a function. So, we will make a function saddle node Bifurcation sorry. So, $a = 1.0, b = 1.0$; see these are just the default values that I intend to use when the function is called, you do not need to do that necessarily. So, w = interactive(snb, a = (- 2, 2, 0.1) , b = (-2,2,0.1)); let me then show the widget, alright.

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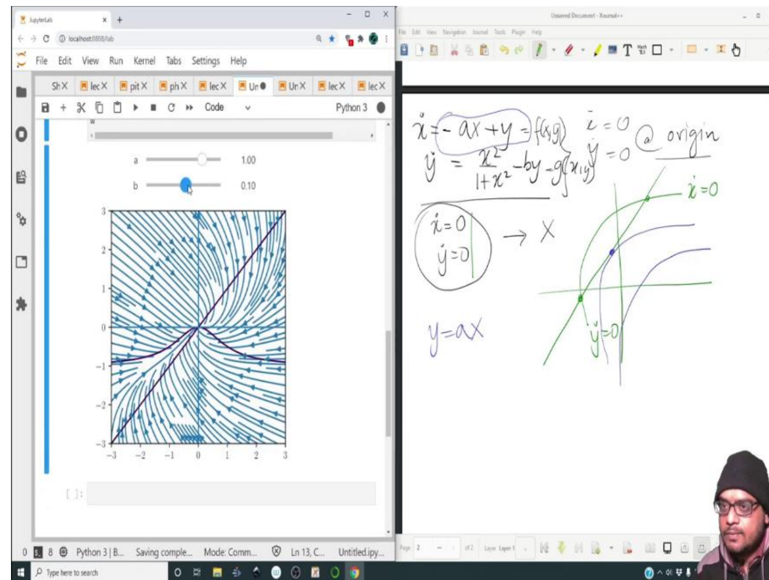

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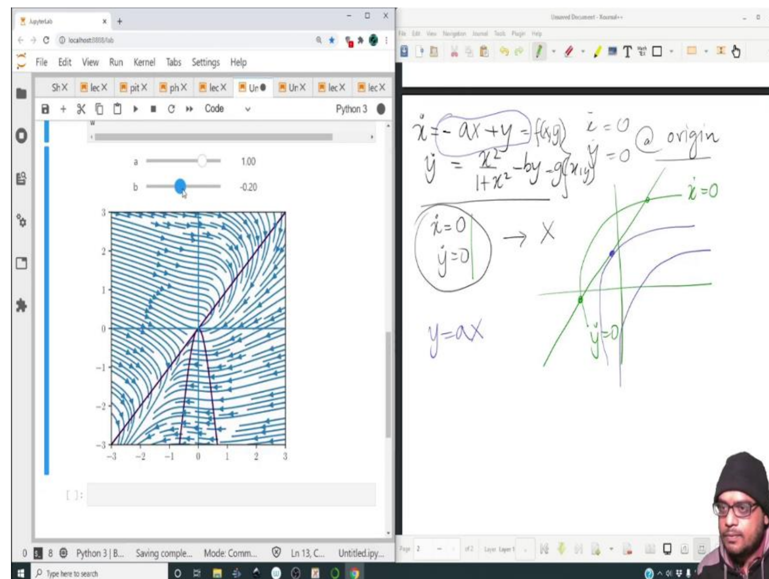

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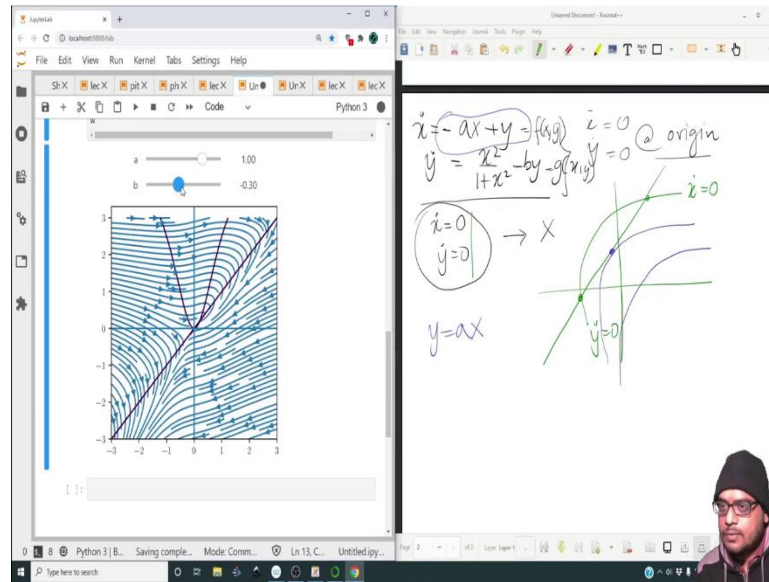

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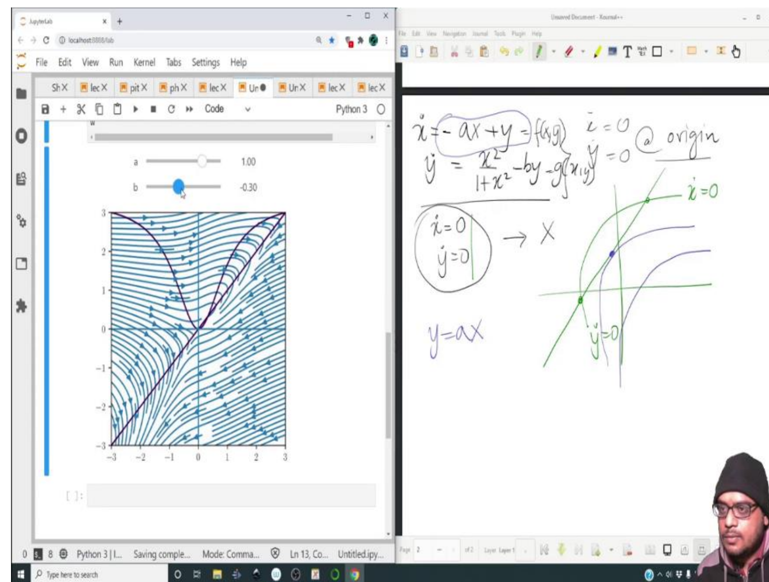

So, let us focus on the region near the origin in order to understand what is really going on, ok.

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So, let me change the x limits to - 0.2 to 0. yeah 0.5 to 0.5, ok.

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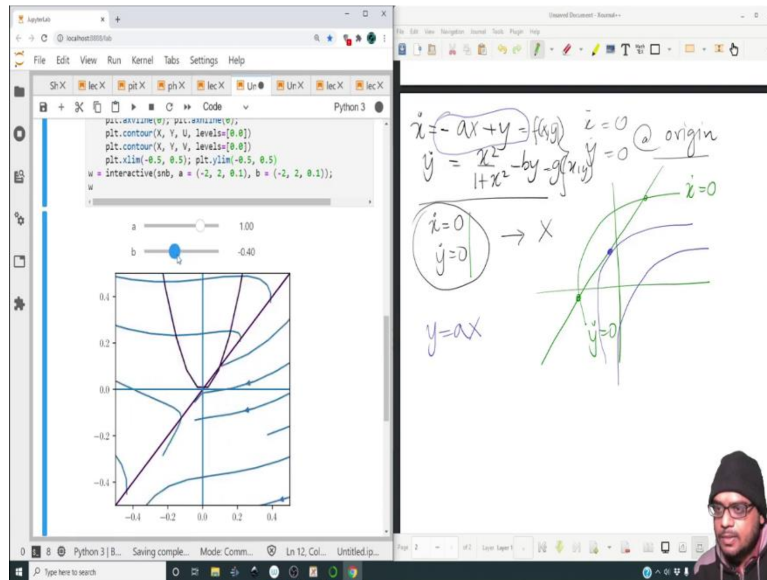

So, this particular case has only one intersection at the origin.

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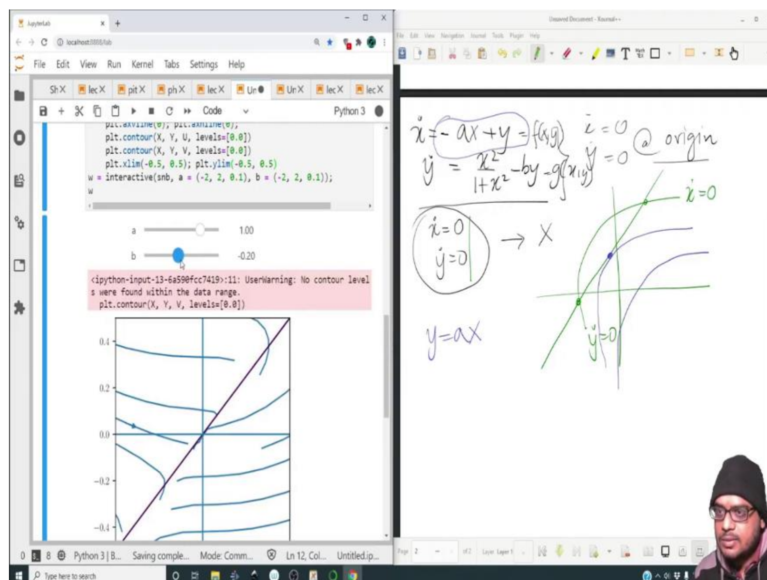

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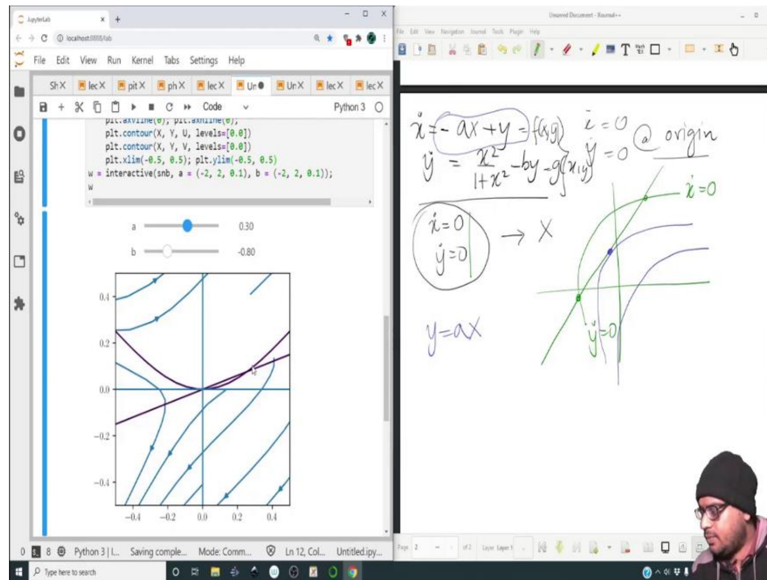

So, now, as I change the value of b, we have two intersections ok; this will have two intersections, but changing the slope will cause only one point of tangency and this particular value will vanish.

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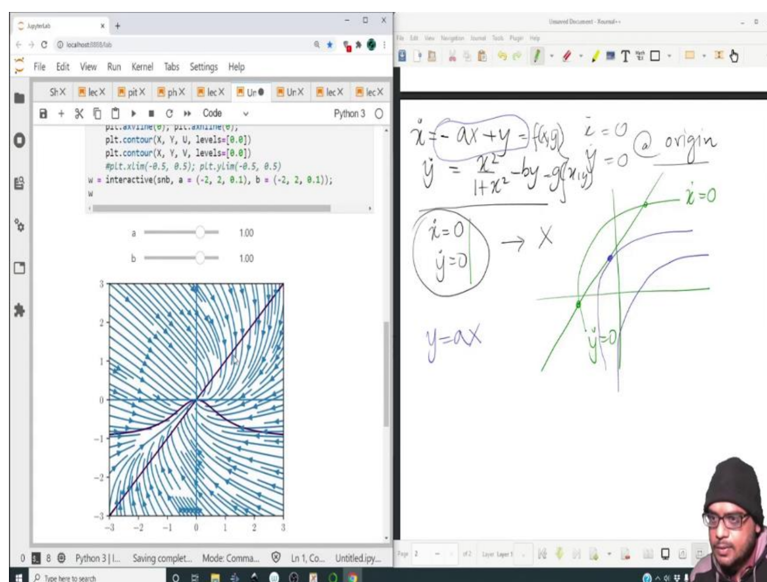

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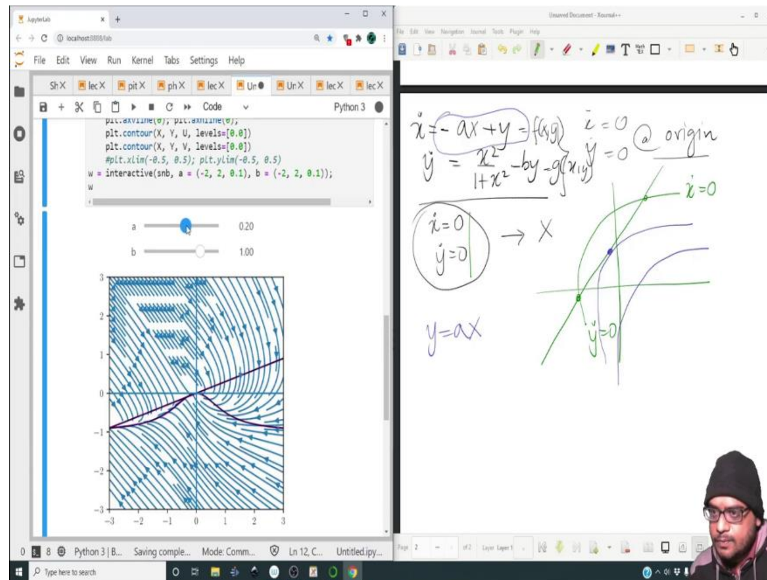

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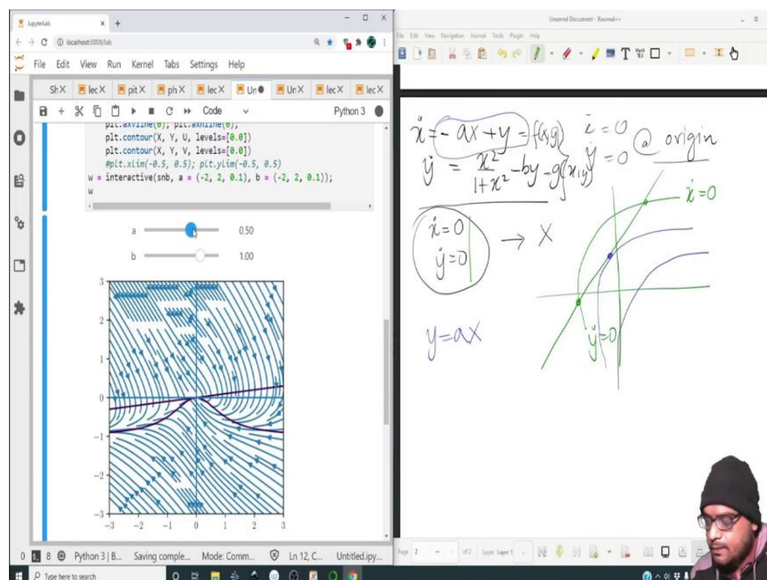

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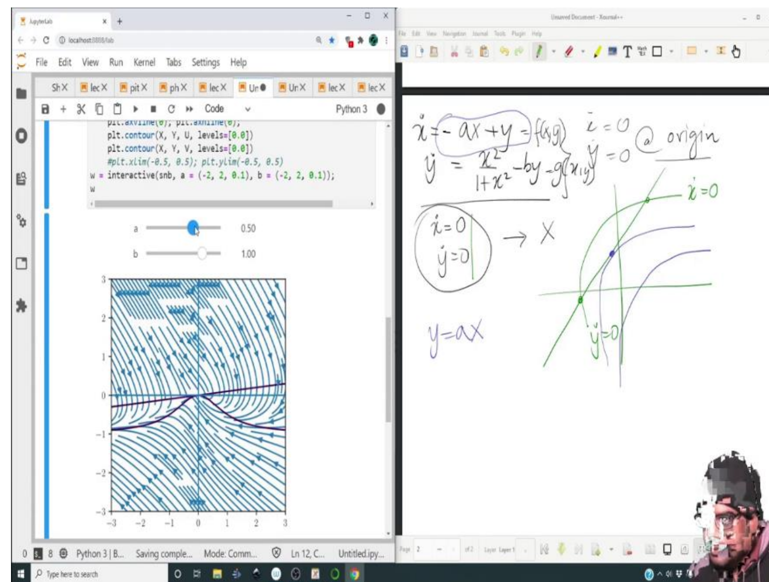

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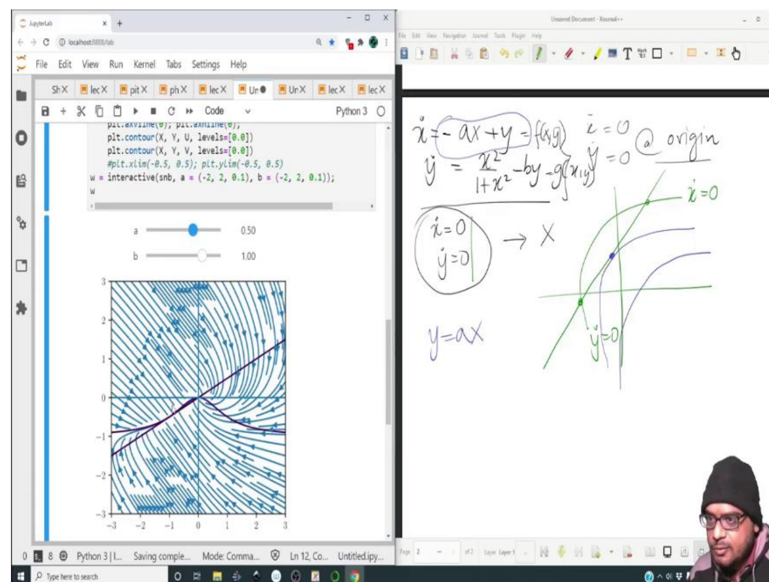

But is it the entire picture? Let me change the slope; at this particular point you can see that there is a possibility of multiple intersections over here. Let me change the focus to this particular point, ok.

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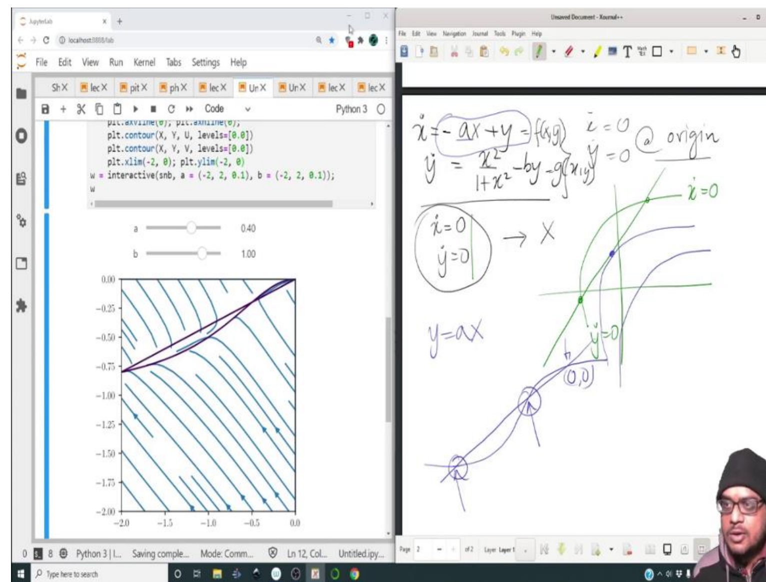

So, I am changing the slope of the fixed line, alright. So, over here we have one point of intersection over here, one point of intersection over here, and one point of intersection at the origin. So, while the trivial fixed point was the origin; there are other fixed points, one near - 0.5, x = - 0.5, and y = - 0.25, and another fixed point at x = - 2, and y = - point between something like - 0.75 and - 0.8, ok.

So, depending on the slope of the line, what we have is a curve something like this and the slope of the line will cause in intersections of the nullclines like this. So, this is the origin, this is another fixed point, this is another fixed point. And using the Jacobian, you can easily determine what the fixed points will behave like and you will see it is a saddle node kind of behavior.

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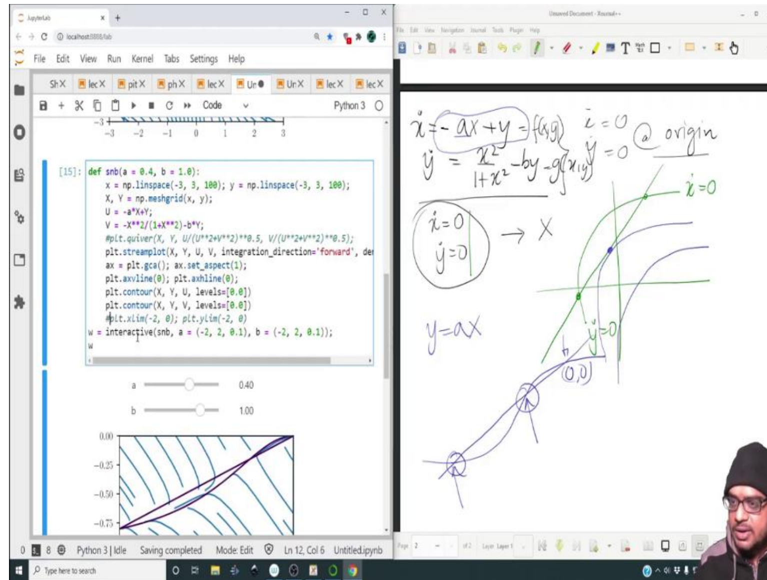

So, let me hard code this value of 0.4 in the default value, so that when we rerun the cell, we will directly begin at that particular point, ok.

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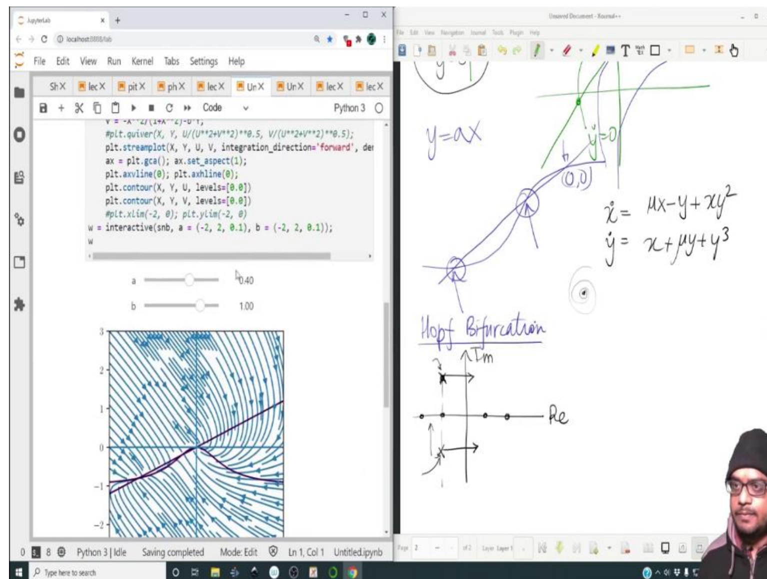

So, you see over here, they are intersecting over here and over here and this is an attracting manifold, ok. This is also an attracting manifold in one direction; but it is unstable in this along the straight line and that is the saddle and this is the node, the stable node, while the origin appears to be like a spiral inward.

So, you can find out the Jacobian, you can find out the eigenvalues and eigenvectors and you can try to justify why the plot looks like it does. Now, we can discuss about the

pitchfork bifurcations that can occur in 3 dimensions, but in 2 dimensions; but I will I will leave them as home tasks for you to really work through and that will give you a lot of confidence and that will allow you to see for yourself, how these things can be analyzed.

So, now, we go on to Hopf bifurcation, there is something which does not have an analog in one dimension. So, Hopf Bifurcation refers to situations where stable points become unstable by crossing the y axis in the imaginary plane. So, imagine you have a stable point; it means that the eigenvalues must be on this side. So, this is the real part, this is the imaginary part of the roots.

So, the two roots if they lie over here; which means that the real parts are negative, the imaginary part is 0. So, this will act as a stable node; if one of these points lies over here, it will act as a saddle. If both the points are over here, it will act as a repelling node. So, now, apart from having real roots, we can have a pair of complex conjugates like this.

Now, this implies one of this is turning in one direction, one of this is turning in the other direction; but the real part is negative regardless and so they will eventually spiral into the fixed point.

But now, when they cross this particular axis in the argand plane; it means that the nature of the spiral is changing from attracting spiral to an unstable spiral. So, now, let us do a very simple example and it is one of the examples of Strogatz. And hopefully that will allow you to study this on your own with the help of these numerical tools that python provides.

So, let us try to find out how this particular system behaves. So, $\dot{x} = \mu x - y + xy^2$ and $\dot{y} = x + \mu y + y^3$. So, these are all very, I mean you will you will find such examples appearing all over the place if you read textbooks on this topic; but it is a very convenient example to show the effect of the control parameter. Let me grab this particular snippet and let me change r.

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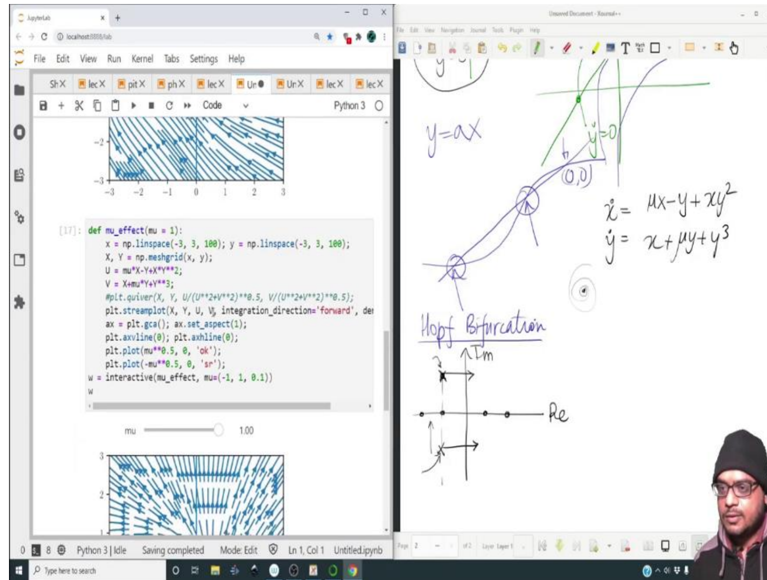

Let me grab this, because there is only one control parameter and let me change the function.

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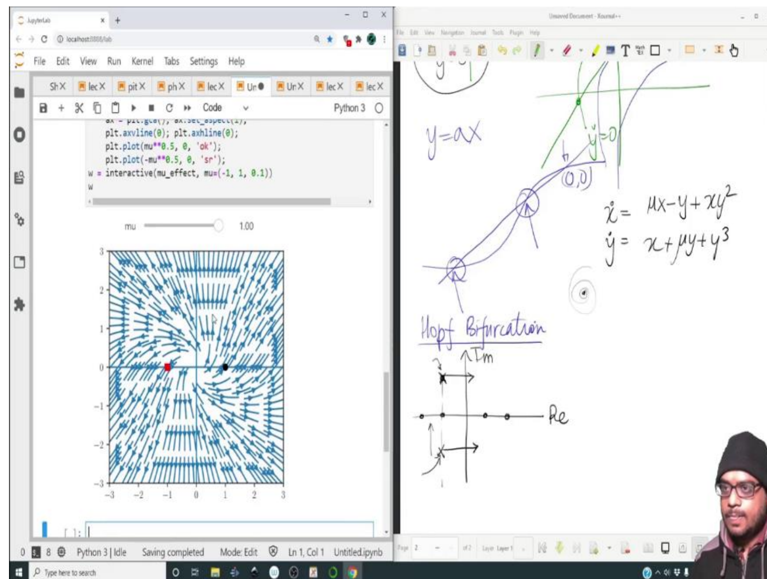

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So, let me plot this; let us see how it looks, we have to get rid of this, ok. So, it appears to be an outward spiral; but why is it an outward spiral, that is the question. So, quite obviously, this particular set of equations, the fixed point is at the origin. You can easily verify that this is a fixed point, but what about the stability of the fixed point, alright?

So, let us find the Jacobian of this real quick. So, this will $\begin{matrix} \mu + y^2 & -1 + 2xy \\ 1 & \mu + 3y^2 \end{matrix}$, ok. So, at the origin this Jacobian will be $\begin{matrix} \mu & -1 \\ 1 & \mu \end{matrix}$. And so, what are the eigenvalues of this particular Jacobian? So, it will be $(\lambda - \mu)^2 + 1 = 0$ and that implies that $\lambda = \mu \pm i$, so ok.

So, you take this - from the right hand side, take a square root; so you get $\pm i$ and then you do this. So, these are the two eigenvalues and clearly depending on the value of μ, we will have either an unstable spiral or a stable spiral. So, when μ is positive, obviously the real part of this eigenvalue is positive and hence it will be an unstable point. So, the origin has an unstable spiral, alright.

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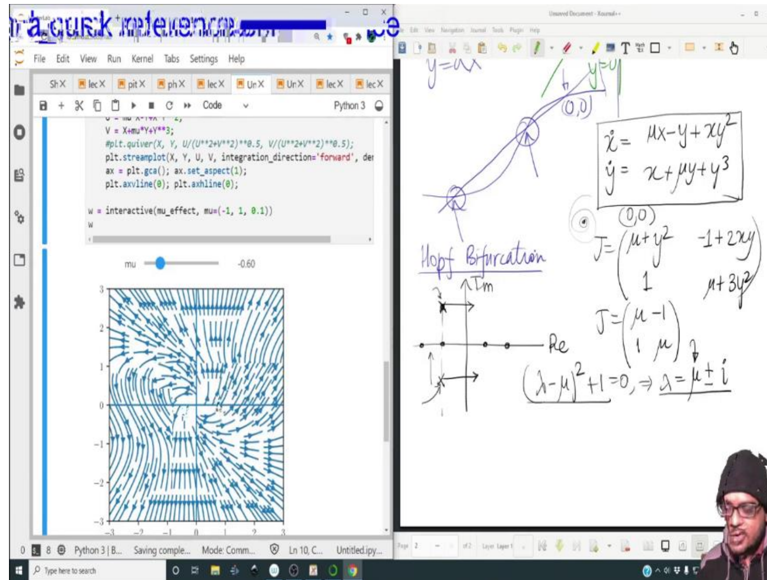

But what about, when μ is negative? When μ is negative, we do have an attracting spiral, alright.

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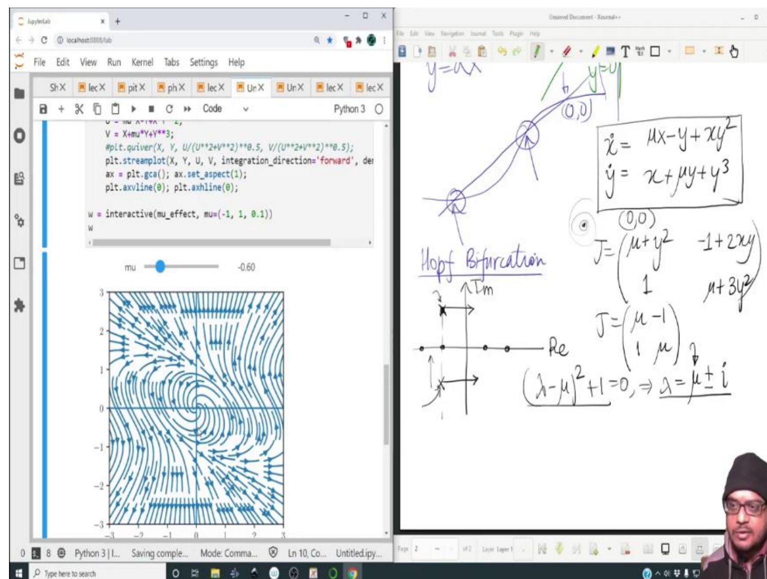

So, we have fundamentally changed the behavior of the set of equations by altering the control parameter. And when it goes from positive to negative, we are changing the fundamental behavior of the origin from being an unstable spiral to a stable spiral, alright.

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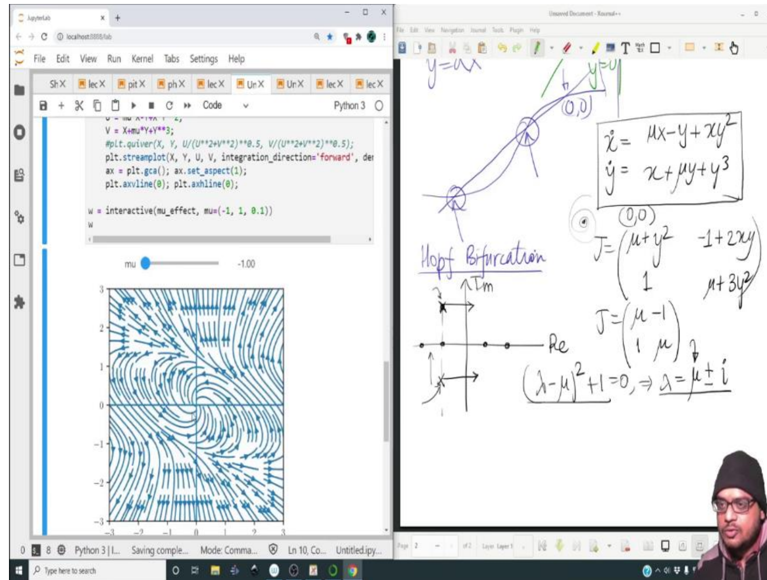

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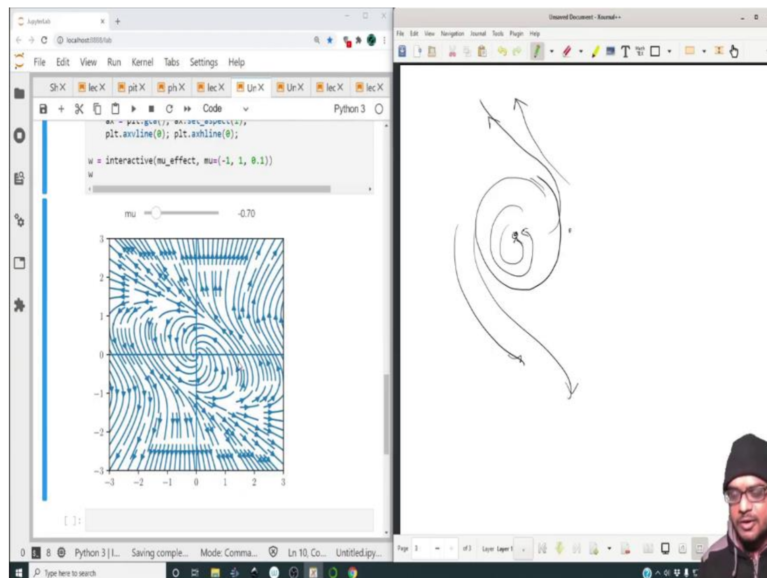

So, as we turn it up, we do see that it tends to become more and more enlarged the zone of dominance and look at how the trajectories appear. So, these trajectories appear to be spiraling outwards, while. So, what do we have? If you look carefully, we have a bunch of trajectories moving outwards into infinity, right.

And we have a bunch of trajectories moving into the origin; so obviously there has to be a limit cycle somewhere, because you cannot have the behavior, where a set of trajectories are being attracted towards the origin, while another set of trajectories are repelled from the initial point towards infinity, ok.

So, when these trajectories are all repelled, these are all attracted; somewhere in between there will be a limit cycle. And I request you to find out how you can find out that limit cycle; as to how you can precisely pinpoint that that limit cycle exists.

So, all these are some advanced questions, which are not in the scope of this present course; it is present course I am trying to show you some of the tools with which you can inquire into the nature of the problem, while not losing a lot of time and energy into getting things done, ok.

So, Hopf Bifurcation is often associated with the presence of a limit cycle and is something you should go back to the previous lecture on limit cycles and try to figure out. One small point that I forgot to tell; that when $\mu = 0$ the origin behaves like a center. So, if I have put the slider at $\mu = 0$ and you do see closed orbits appearing near the origin. So, you are going from repelling spirals to a center, but the outside still looks like it is repelling everything.

So, if you have a center at the origin and you have repelling trajectories outside; when you change the parameter, you still going to have the essence of the closed orbit somewhere, and that is sort of a very informal way of saying that a limit cycle will exist.

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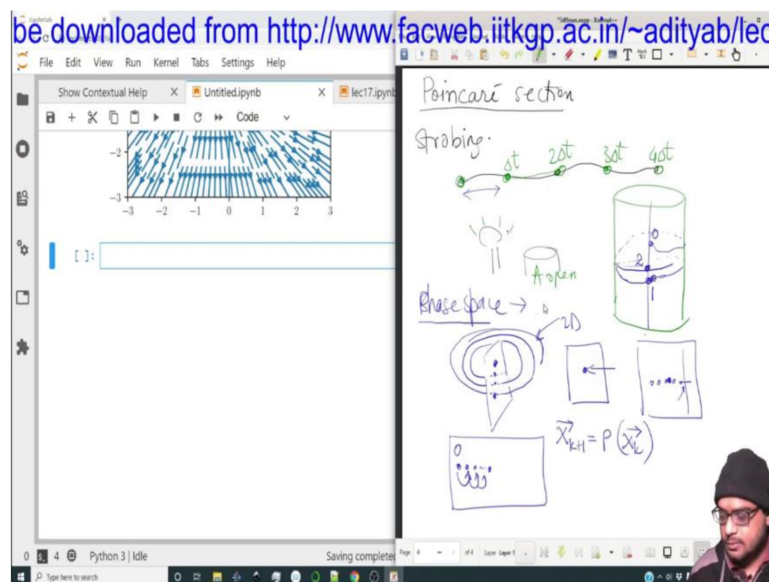

Let us now move into 3 dimensional close but before that, I would like to discuss an important concept that of a Poincare section. So, typically trajectories look very

complicated in 2 dimensional space and they look even more complicated when you look at 3 D maps.

So, instead of that, if we would like to strobe the system at certain times, where the strobing would reveal the position. So, what is strobing? So, suppose an object is moving continuously like this, it is moving in the dark and you have camera and a light.

So, everything is dark, but the aperture of this camera is open ok; it is open and this light is flashing at every say Δt. So, after Δt, the particle goes over here; then the particle goes over here at $2\Delta t$, then at $3\ \Delta t$, and $4\Delta t$. So, when the aperture is completely open and things are dark, we get nothing on the film; but as the light is flashing every Δt, once we develop the film, we will see that the particle has appeared at these locations.

Now, imagine the same dynamics, but it is happening periodically; meaning we imagine a cylinder and the cylinder is more of a appropriate phase space for periodic systems. So, as a particle begins over here, it goes passes this and so, it comes over here. So, this is like t = 0; then the next period, then it goes like this, say it crosses this and arrives like this. So, this is at t = 0, t = Δt, t = 2 Δt and so on, where Δt is like the time period of this particular system.

So, then when we visualize the things, we can see how the point is iterated by the non-linear flow over each time cycle. So, we are more interested in the state space or the phase space rather. So, the phase space at each time interval, but not the trajectories that the particle has taken to reach that point, ok.

So, imagine you have a closed orbit. So, a closed orbit in this case, it would go around in circles ok, it would go around in circles. So, if I have this as my photographic plate right and each time the particle is crossing, so the frequency of crossing is fixed.

So, each time I take a photograph. So, when I develop this, I will simply see that the particle was at this point, nothing has changed. So, this the fact that the location is not evolving; means that the orbit is a fixed orbit, over several cycles it will not change. But if the particle is having an inward spiral alright; so what happens when the particle has an inward spiral? Then initially it will cut over here, then it will cut over here, then over here.

So, at each time instant, it is sort of going towards the limit cycle; eventually if so, if it is a just attracting spiral, it will go towards the center point. So, the film would look something like this. So, this is like the equilibrium point, ok. So, this is the Poincare map of an attracting spiral. If things are tending towards a limit cycle; then it will lock at a certain position, it will. So, if you develop this, you will see that it is going and then slowly it is locking on, the spacing between these will reduce.

If it is an unstable spiral, similarly you will get a set of points which are going away from this. So, the phase space is like a reduced order map. So, it is what is actually happening? So, this is at t = 0; then the non-linear system is sort of iterating or advancing the particle to this state, then it is advancing the particle to this state, then to this state and so on.

So, it is like X k + 1, X vector k + 1 is being iterated upon with the help of the previous state, ok. So, this can include the location, the velocity and whatever; I mean it need not even be location, velocity, it can be anything, but it has to be the set of variables that you are looking at, it can be x y for example, ok. So, this is how we can sort of reduce the order of the system.

So, this entire thing if it is a 2 D system, then we are essentially analyzing everything on a one dimensional plane and we are looking at the entire system over only times Δt, nothing else. We do not care about the trajectories in the middle.

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So, a very very popular system to get you familiar with all this; it is the Duffing equation ok. So, the Duffing equation can be written as a set of linear equations as follows.

So, $\dot{x} = y$ and $\dot{y} = x - x^3 - \gamma y + d\, cos(\omega t)$, ok. So, the reason I had written v over here is because, you can look at the analogy with an oscillator and this becomes the damping term, ok. This is the damping term, this is the forcing term, and this is the double well potential that you know, ok. So, this defines an oscillator and there are three parameters to play with one is γ, one is d, and one is ω.

So, let us first see how this particular system of equations looks like, ok. So, let us first have a look at the phase space trajectories. So, let us go and copy some snippets to reduce our efforts; I think we can use this, no, let us use this, ok. So, over here we need to change the forms of the equations. So, let me quickly do it, alright.

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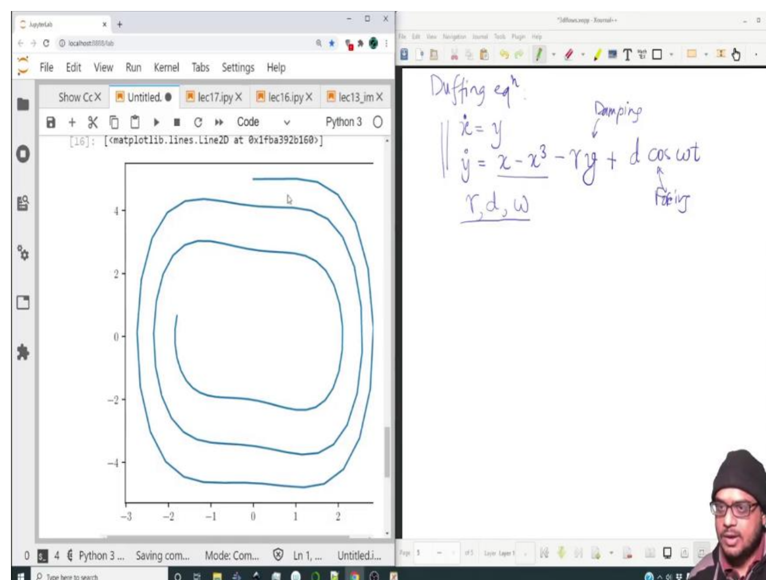

So, I have coded this up and the figure looks something like this. So, the lines look a bit jagged, so that is because the array on which we are solving it has only 100 points.

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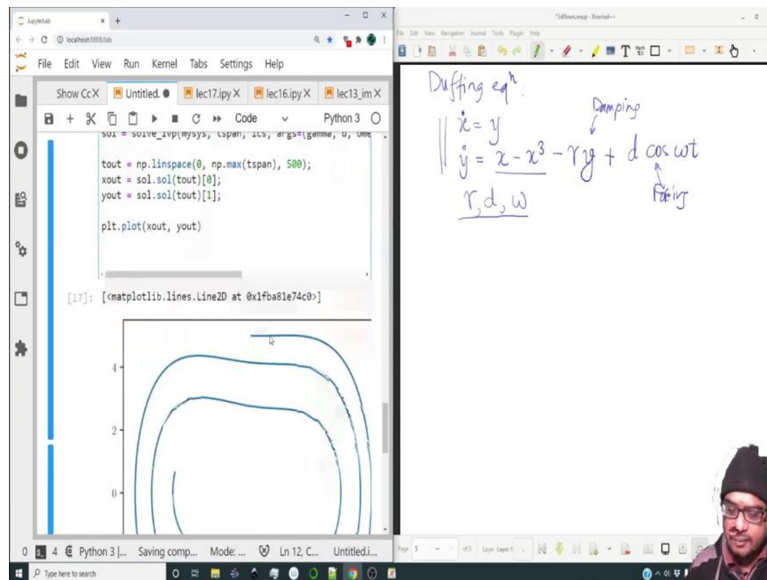

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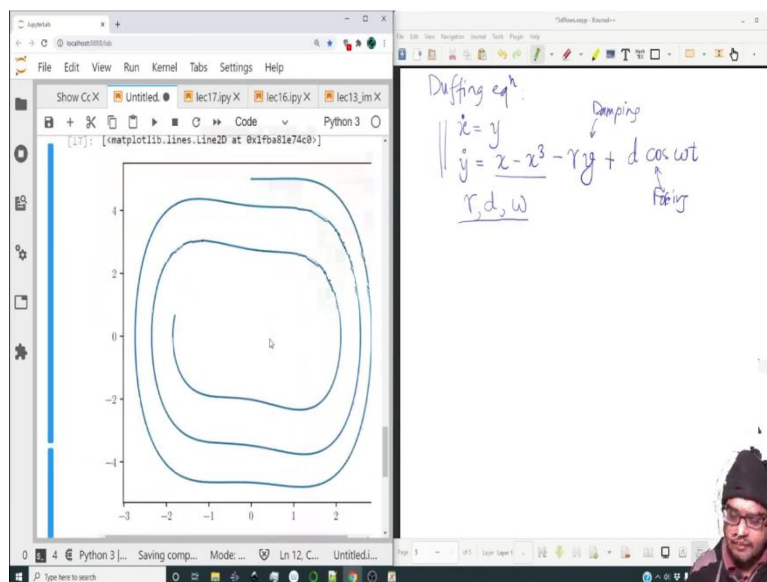

Let me make it 500 points and then everything looks much smoother. Let us in fact carry on the integration for a larger time.

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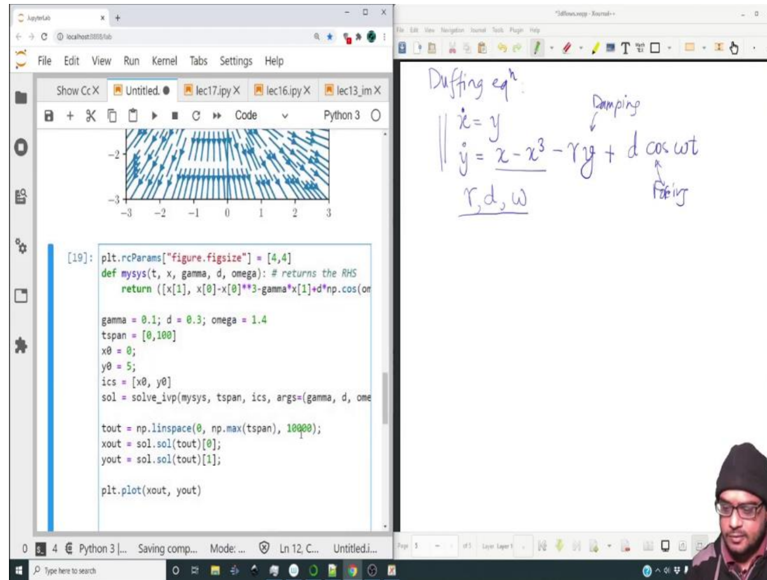

So, let me increase the tspan to 100. So, something weird happens; let me increase the number of time steps.

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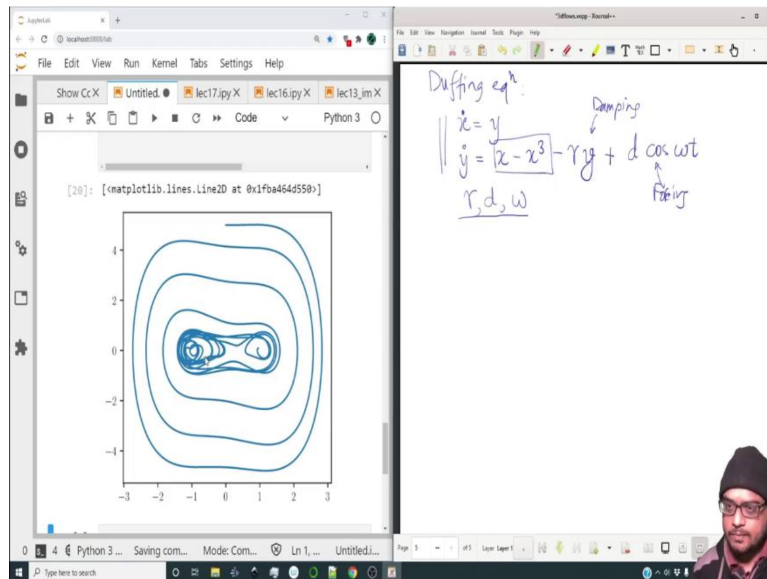

So, it starts over here, then it evolves, it decays and then it is trying to hunt to one of the two potential wells, ok. So, this particular potential well has two humps; one at 1 and one at - 1 as you might have noted from one of the previous lectures. And it is not entirely clear where it wants to settle down, ok. But let us do one thing, let us. [FL] So, let me decrease the value of d, ok.

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Let me reduce the forcing term to something like this, let us see what happens, ok. So, it starts over here, then it gets attracted towards one of these points, ok. So, in this case it is, it is an attracting point and it ends up; whether or not whether or not it ends up on one of the points which remains to be same, but it gets attracted towards one of the stable lobes of the potential well, ok. So, now, let us in fact plot only the large time dynamics of this, ok.

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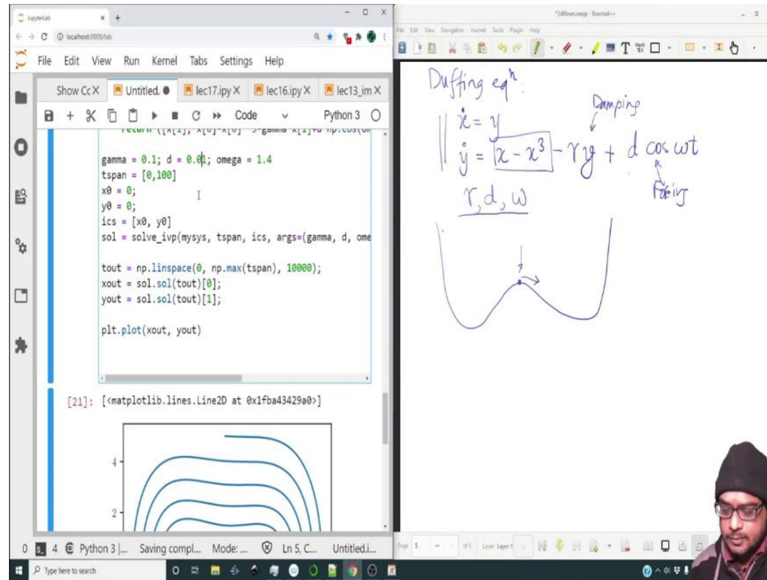

So, this initial transient I want to avoid. In fact, let me give it an initial condition like this; let it start at the unstable point, because the double well hump and it looks something like this.

So, it is starting at a very unstable point and because of the driving force, there will be some, it will be driven to one of these parts and then the entire dynamics will follow. If the driving force was not there, this would have been an equilibrium point, ok. So, let us see what the figure looks like.

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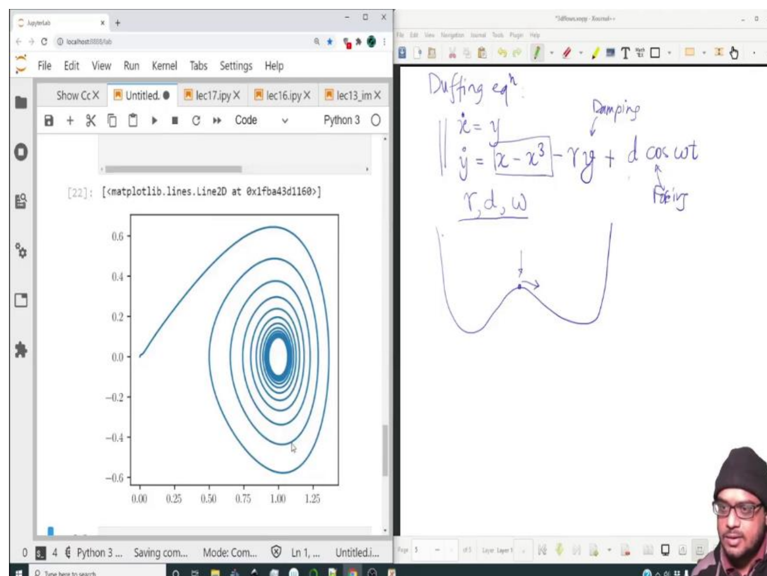

So, it starts over here and then it appears to go into a limit cycle; but is it really going to limit cycle? Let us plot only the last 10 seconds of it or 10 seconds or 10 units of it.

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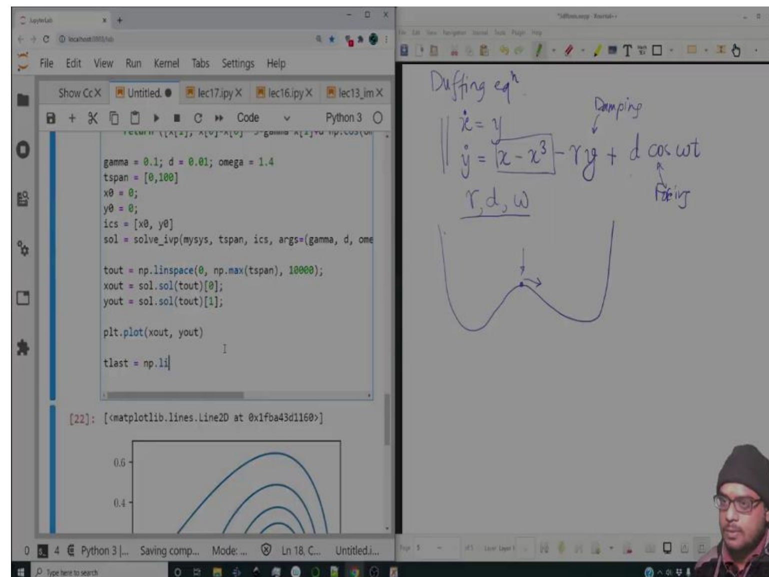

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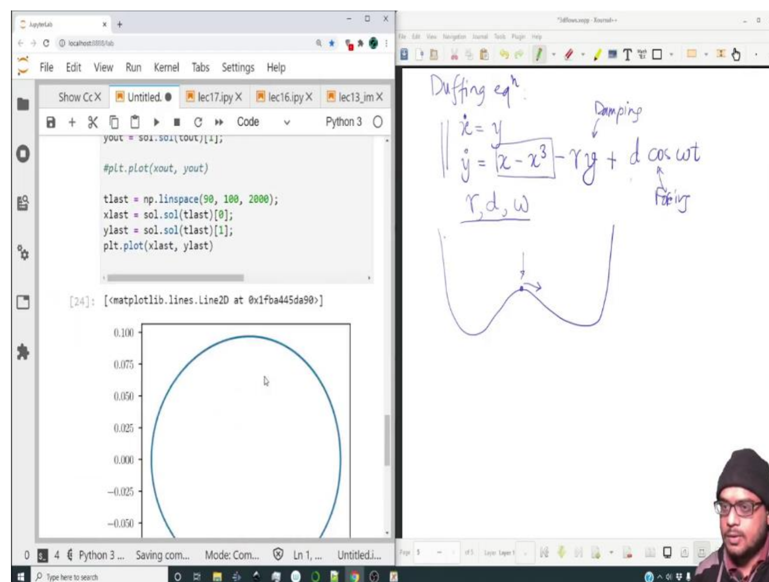

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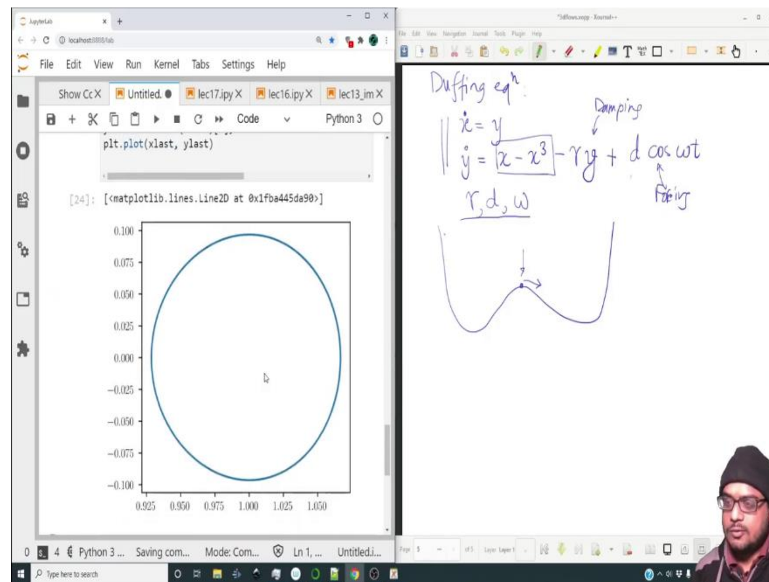

So, let me say in last equal to. So, let me remove this particular plot and let me only focus on, ok. So, it does go to a limit cycle; so for this particular parameter space, this parameter space.

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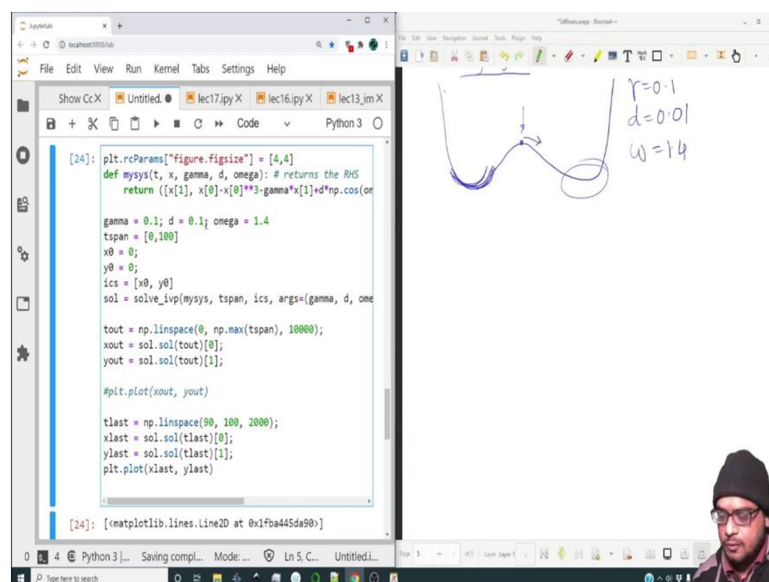

So, $\gamma = 0.1$, d is 0.01 and ω is 1.4, it homes onto a limit cycle; meaning ultimately it is oscillating something like this, it is the right limit cycle, so it has gone to this particular well, ok. So, now, what happens when this parameter is changed, ok?

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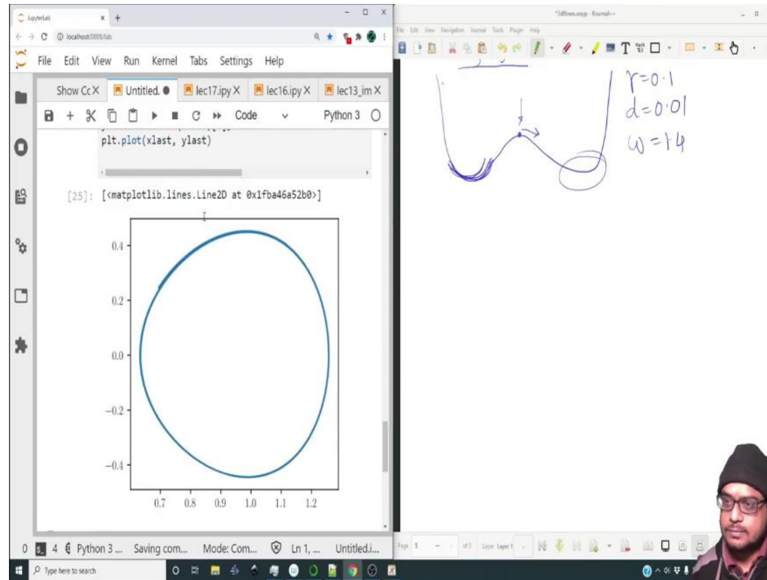

So, let me make d go back to say 0.1, let us plot this. So, in this case, so in this case as well, it does appear that we are achieving a limit cycle.

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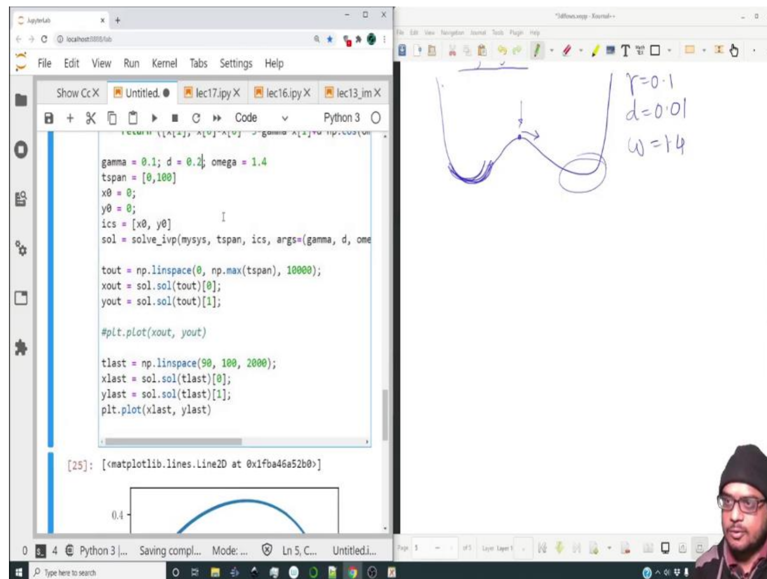

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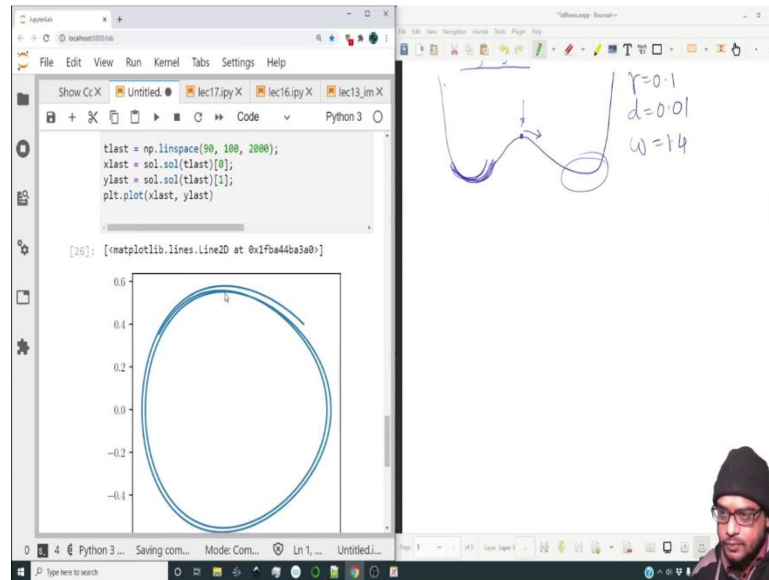

Let me increase the value of this d further, let me make it 0.2. Now, I think we need to run everything for a much larger time; so in fact, let me. So, let me tell you the motivation of what I am trying to do.

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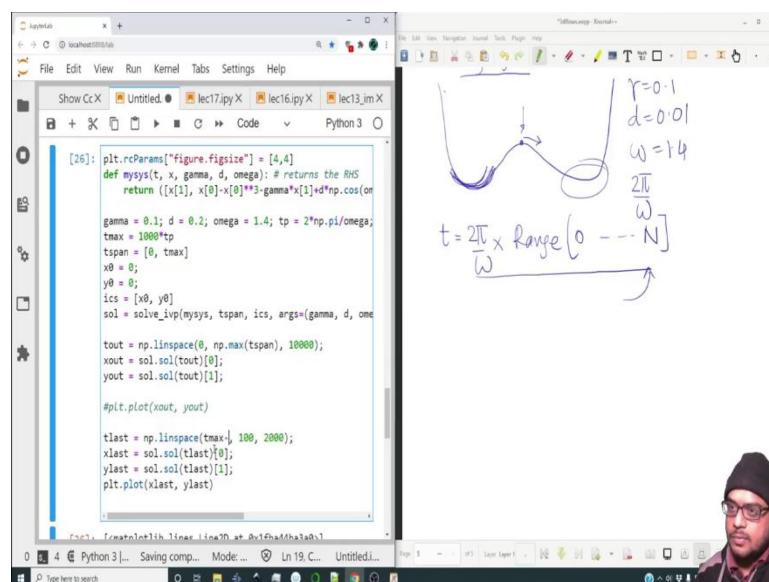

We will define the time in the number of time periods of the entire system. So, the time period of the system is going to be $2\pi/\omega$. So, we are going to have $2\pi/\omega$ times a range of integers, which will go from 0 to some number N.

So, by defining time like this, we will have N cycles in the entire system and during those N cycles, how the flow will evolve that is of the paramount question. So, instead of hard coding it as 100, let us do the following. So, because we have defined omega over here; so the time period tp will be 2 π/ ω and the.

So, let me say tspan = [0, tmax] and let me define tmax as some large integer 1000*tp. So, that it has 1000 cycles in built, ok. So, we can then soft code all of this. So, it instead of 90 to 100, we can say tmax - or we can plot the last 100 cycles. So, in, so we have 1000 cycles over here; let us say we want to plot the 1000, last 1000 cycles.

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So, we can write as Nmax, we can define Nmax as 1000 and then we can simply do Nmax – 100*tp and it will go all the way to Nmax or tmax, ok. So, this is how we can soft code everything and let us see when we run this, what happens?

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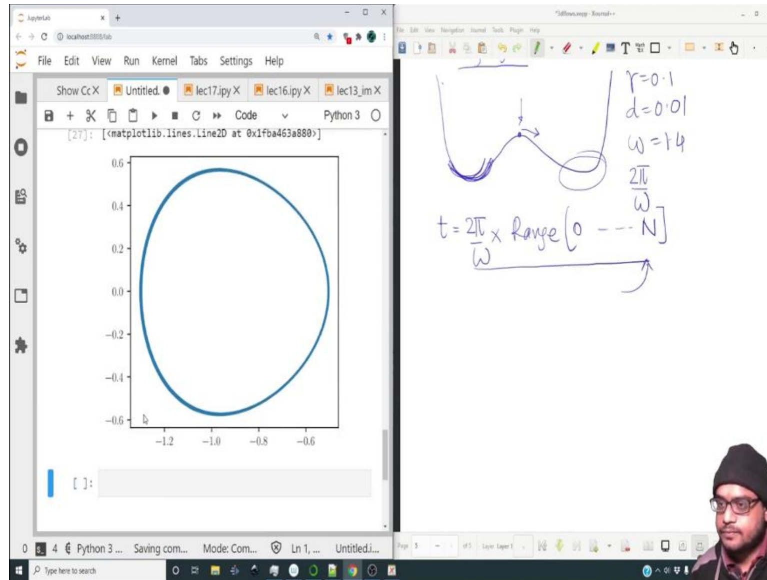

So, it looks something like this. Let me increase the accuracy of the solver, it is already increased to a very large extent. Let me increase the value of d further.

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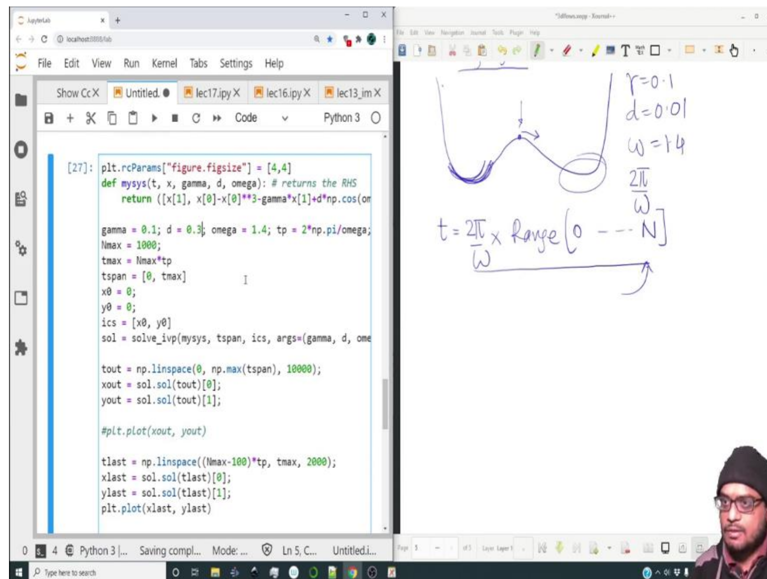

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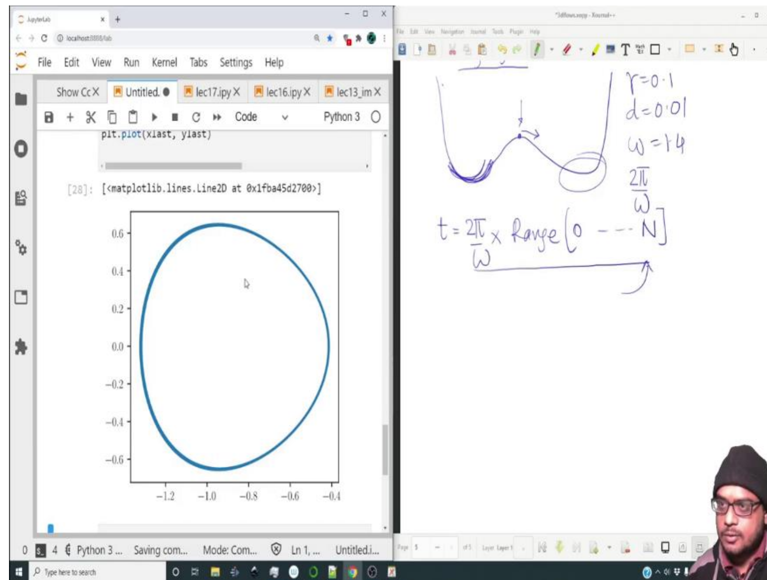

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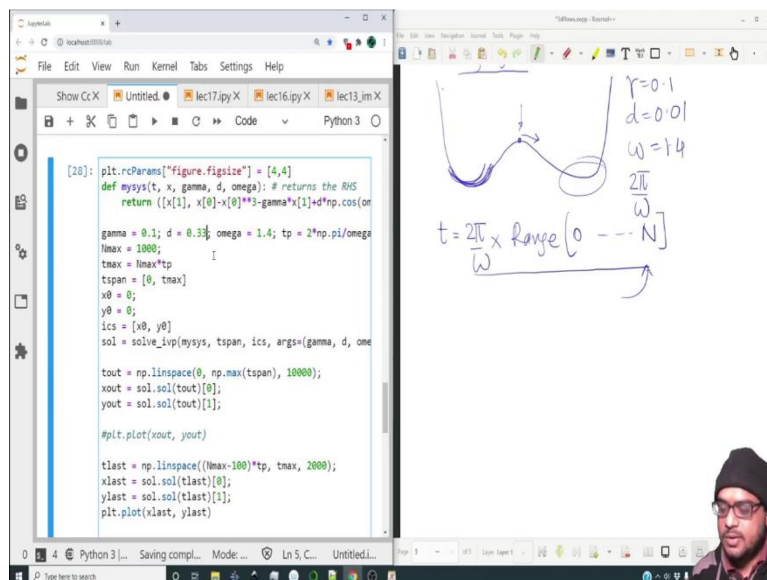

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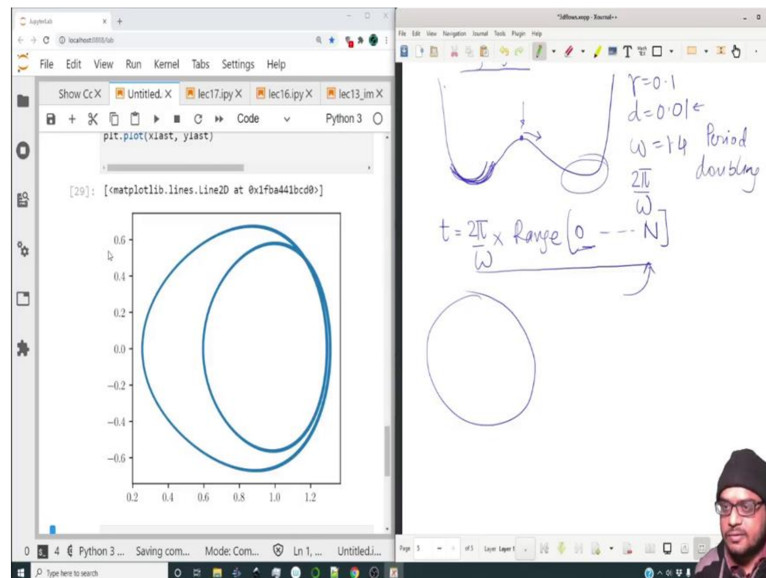

Let me make it 0.3, let us see what happens, ok. It still looks the same; let me make it 0.33, ok. So, what we observe over here is the fact that, from a single limit cycle, it has now evolved to a cycle which has two frequencies. So, it is starting over here, it is going like this, it is crossing one more time and going.

So, every two cycles it is sort of achieving the start of its entire cycle. So, we say that, we have doubled the period of oscillation, so ok. So, by increasing the parameter d, we have achieved period doubling. Let me increase it further, let us see what happens, ok.

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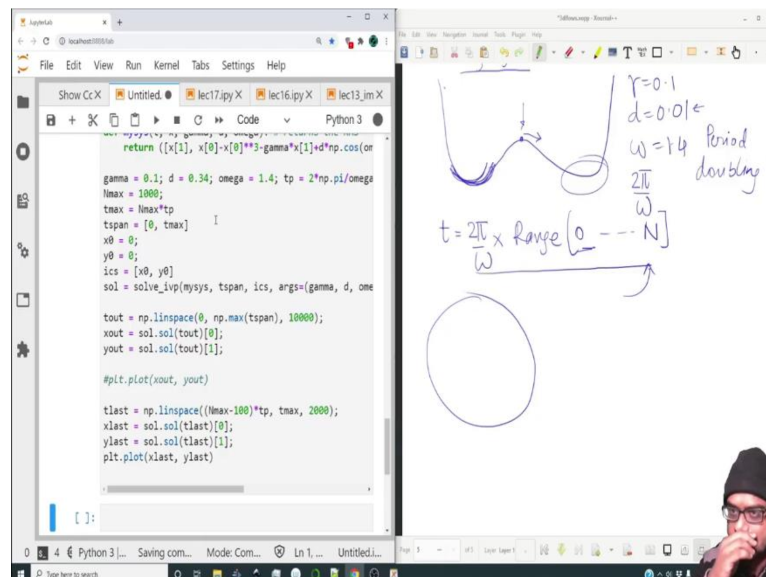

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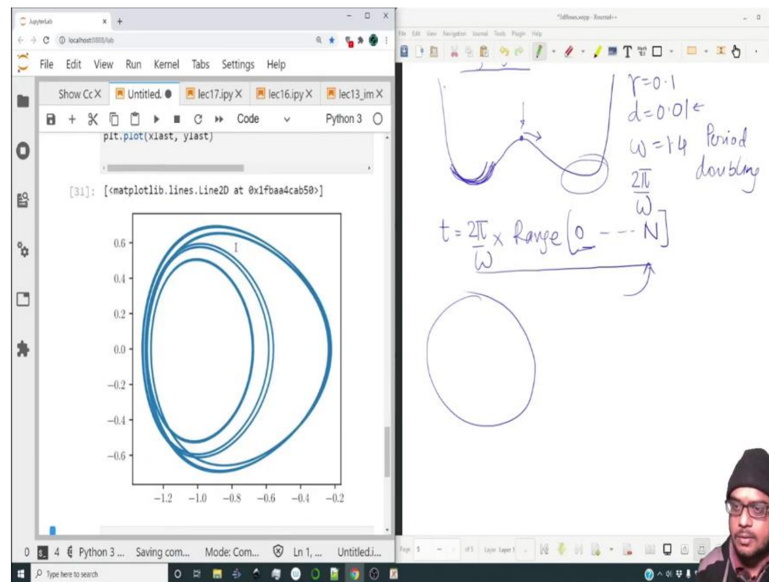

So, the period appears to have increased again, let us plot only the last 50 cycles, ok. So, it looks much much cleaner. So, we do see that, the number of periods before which it settles down into its into some frequency it is increasing. So, we say that the there is an increase in the multiplicity of the period, ok.

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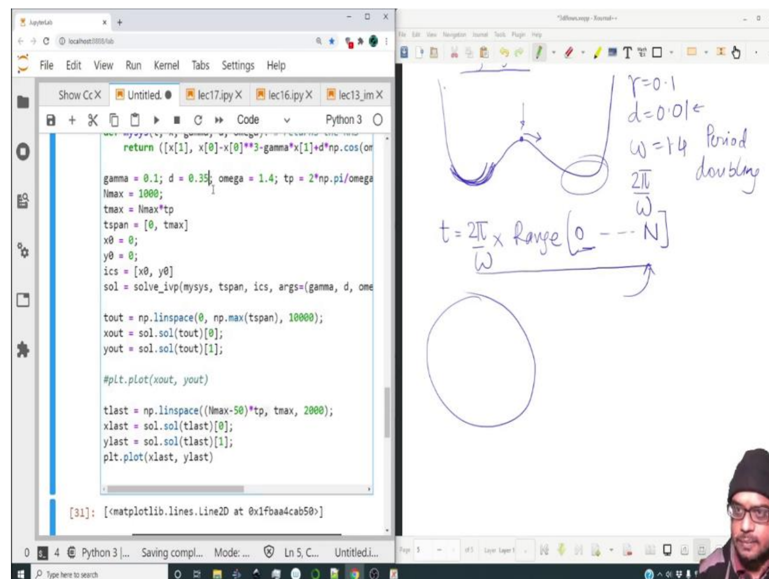

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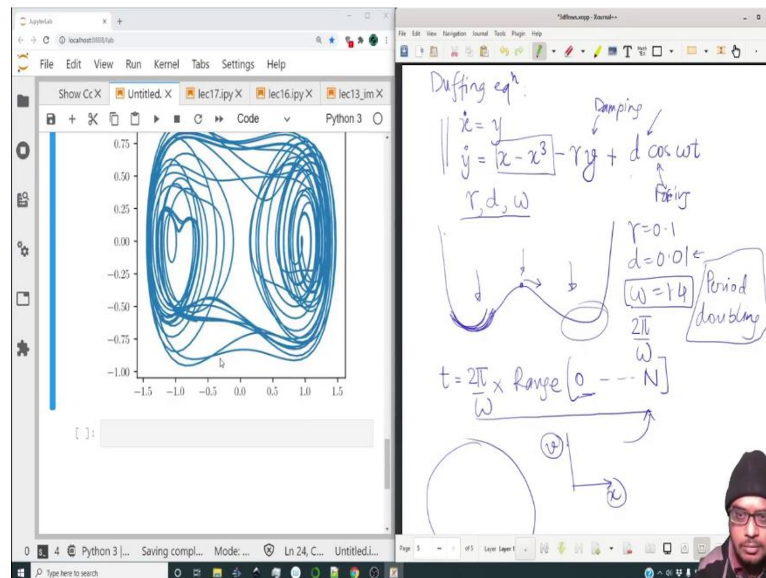

Let me make it 0.35 and boom; we no longer have a periodic thing going on and things tend to go wildly from one side to another side. But the forcing frequency regardless is still this; we have not, all we have changed is like the damping term in the governing equation, one of the driving term.

So, the sort of dissonance between the natural frequency of the oscillator and the driving term is causing, the system to not settle down to a limit cycle; but rather it is going from one well all the way to the other well, it is having a set of velocities which is quite different ok, it never seems to settle down.

So, we say that we have gone into a chaotic regime, through a period doubling cascade. And once you go deep into this particular topic, this is not a topic, not a set of lectures where we are going to discuss all this in details; but I just want to show you how you can do your own scientific experiments or mathematical experiments or numerical experiments by this means, ok.

So, now we are more concerned about this stroboscope; that is after each time period, how does the velocity and location of the particle look like? I am more interested in v x after each time period and that will give us a lot of insight into whether things are chaotic or random or what, ok. So, let us do that. So, I already have the N max with me.

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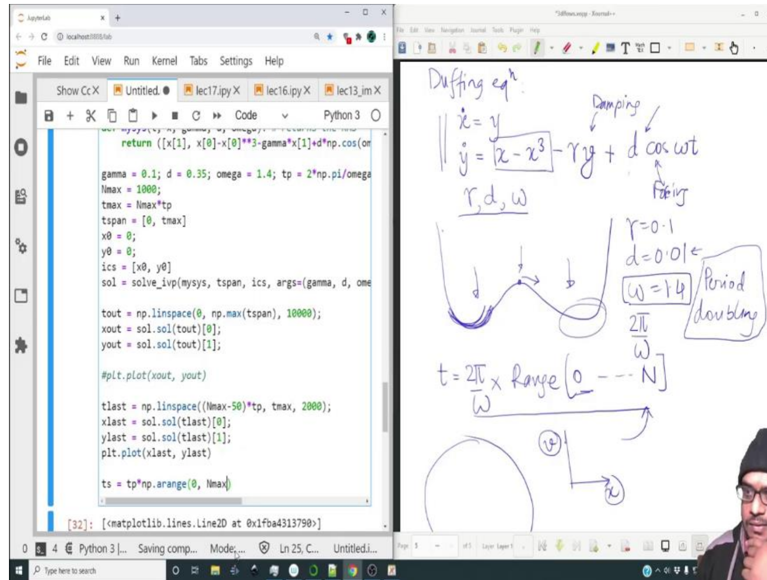

So, let me define. So, t strobe or let me call it ts = tp*np.arange(0,Nmax). In fact, I do not really bother with the initial transients; so I will just go from Nmax - 100 to Nmax. ts = tp*np.arange(Nmax - 100,Nmax).

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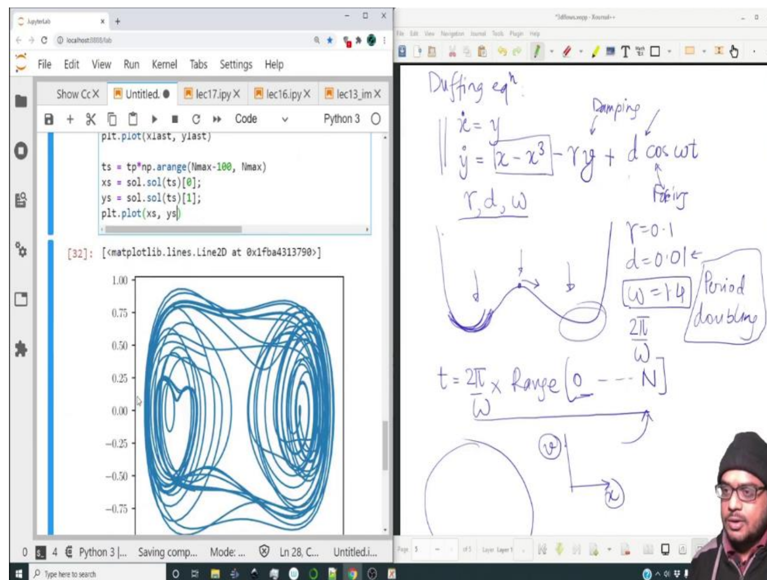

And the reason I have not chosen 0 is because, initially there will be a lot of transients; like we have seen in the previous figures and we do not really bother about the transients, because we let the system evolve to its limit cycle.

And once it is evolve towards limits to its limit cycle; then we want to analyze the system, whether it has a frequency = the forcing frequency or twice its frequency ok, that is what

we are more interested to know, ok. So, this is what the time strobing is. Then we simply extend the previous snippet; we can copy this and we can make this x strobe, y strobe, t strobe, oops t strobe, t strobe and we will plot x strobe and y strobe. Just for completeness, we will keep this figure as well.

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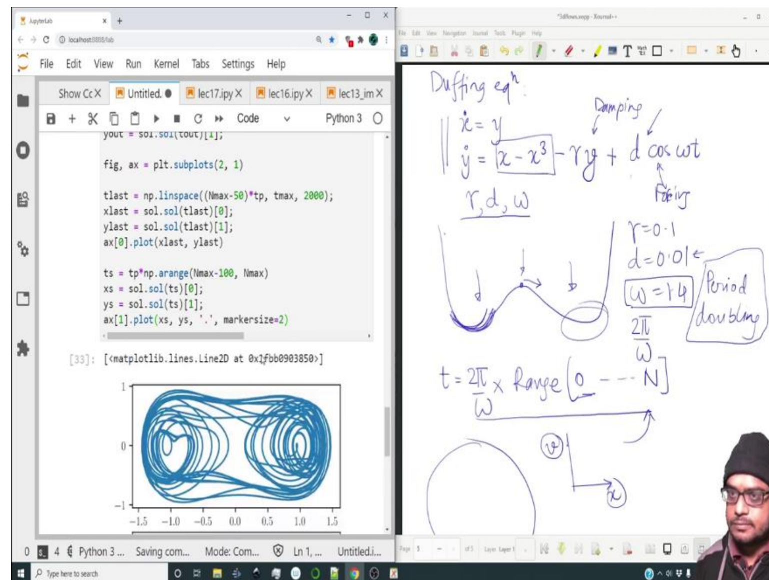

So, in order to keep that figure, we will define a set of subplots. So, fig, ax = plt.subplots (2,1) 2 rows and 1 column; then this will be ax[0].plot and this will be ax[1].plot, ok. Let us and we want to plot this as a strobe; so each time the strobe we will see only one point. So, let me mark this as a point and let me reduce the marker size to say 2.

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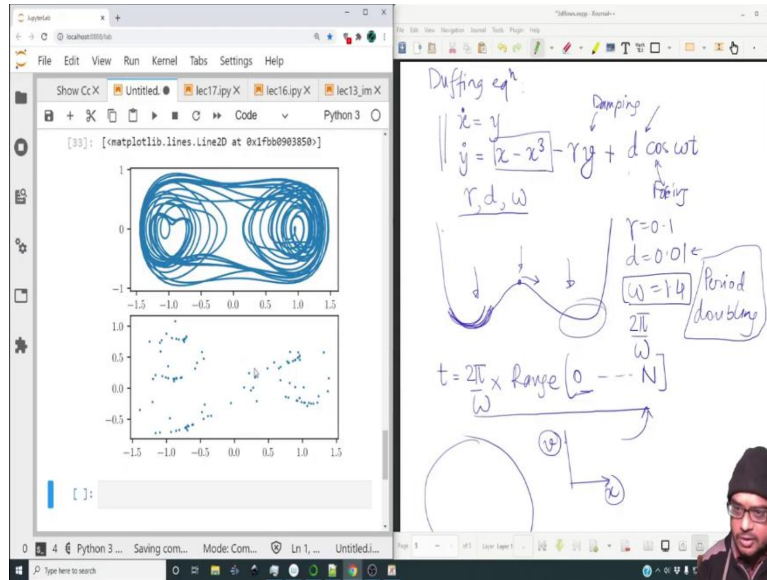

So, let me run this, ok. So, we it appears that there is some kind of some curve along with these points are lying; it is not scattered all over the place. If it were to be random, if the evolution of these orbits were to be random; then we should have had a case where after strobing after each time period, we should have the point on the phase space all over the place. But it does appear that there are certain points on which it is appearing.

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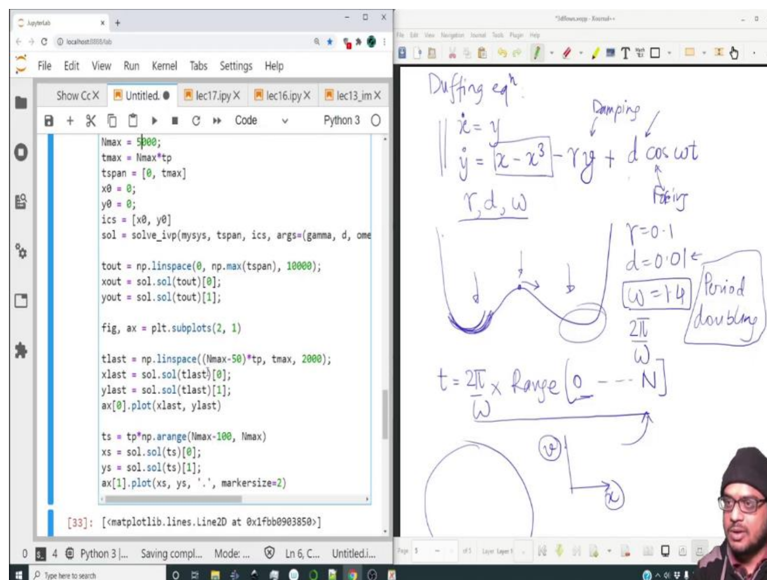

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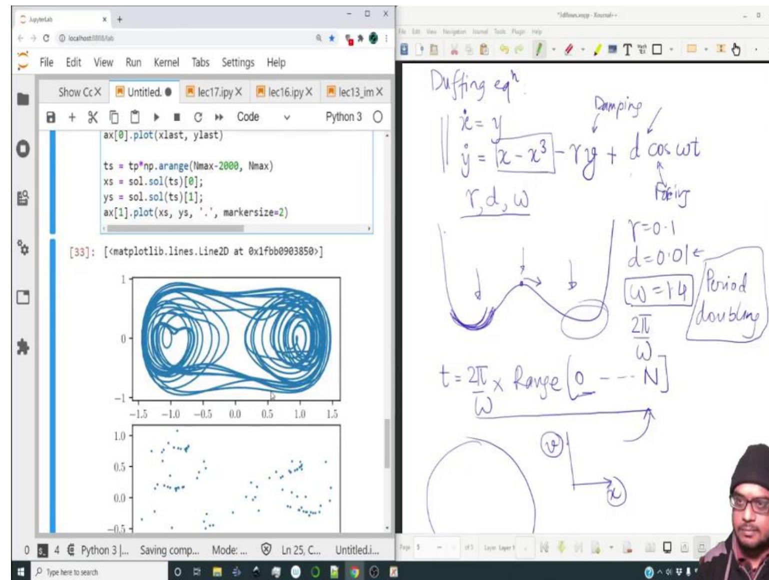

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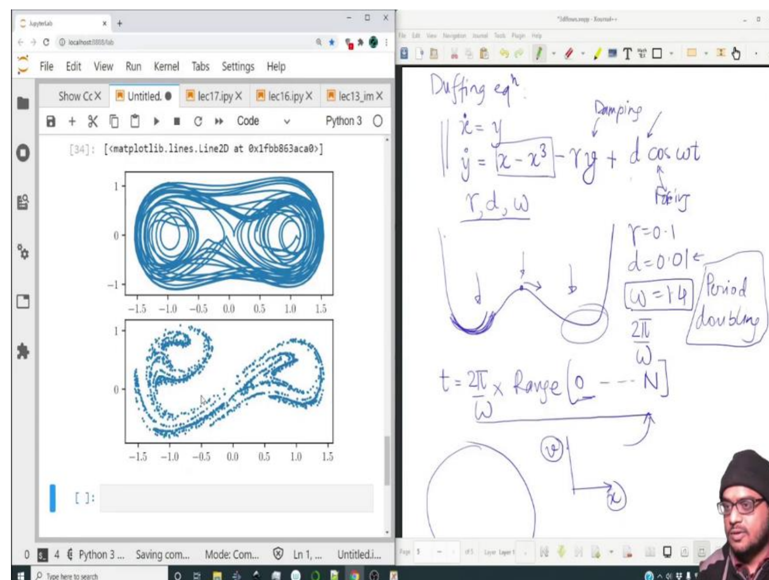

So, in fact let me increase Nmax to 5000 or let me make it 5000 and let me plot the last 2000 cycles; wow this looks much better, in fact let me increase the size of the figure.

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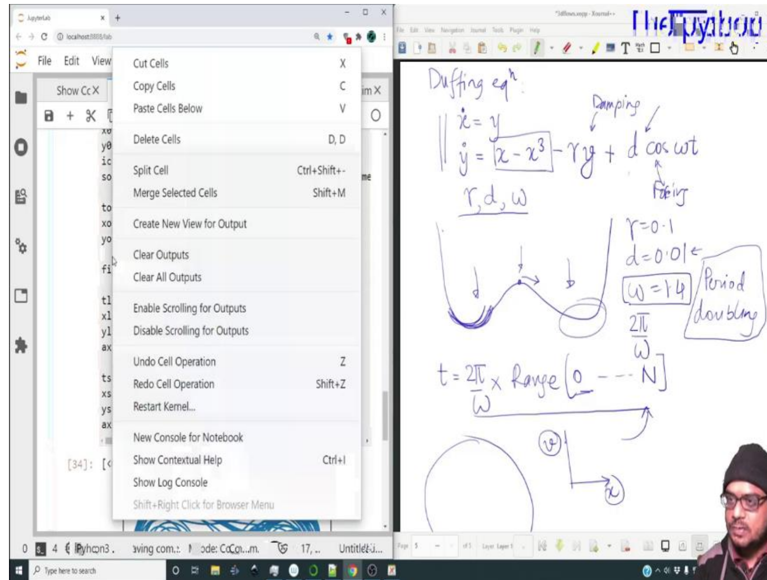

So, if I want to increase the size of the figure; let me simply split this over here, so that we have one computational cell and one plotting cell, we do not want to club both of these together. And let me increase the figure size; so the way to do it is to change the parameters over here.

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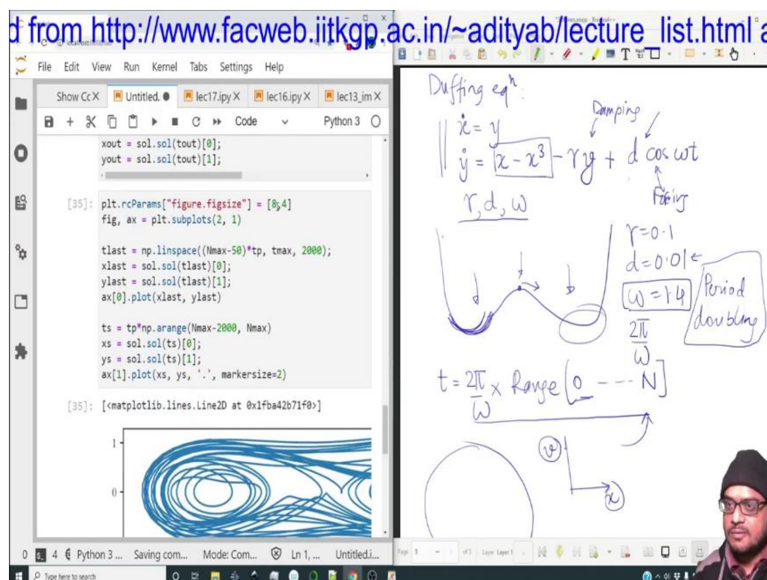

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So, let me make this 8, so, alright. Well, let us see whether that is the width or not, ok. So, this is the width of the figure in inches, this is the height of the figure in inches. So, let me make it 5 or let me make it 6; we can we can and let me make this 12, I will maximize this, ok.

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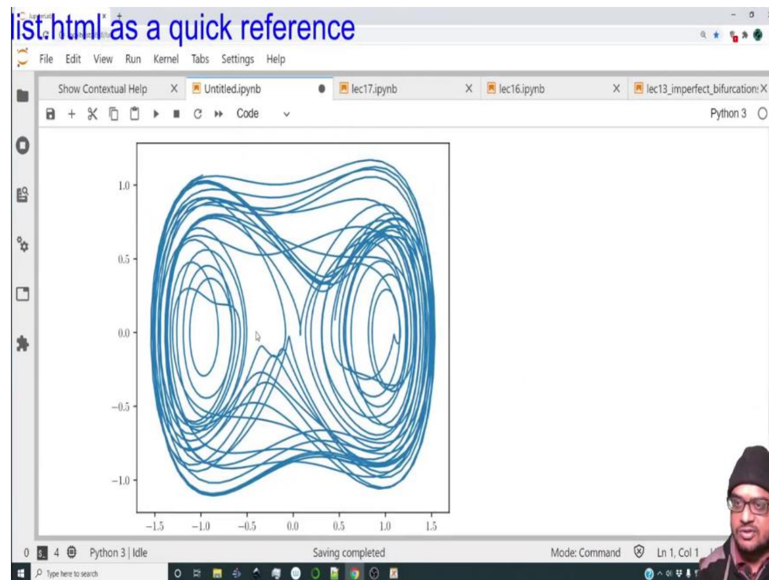

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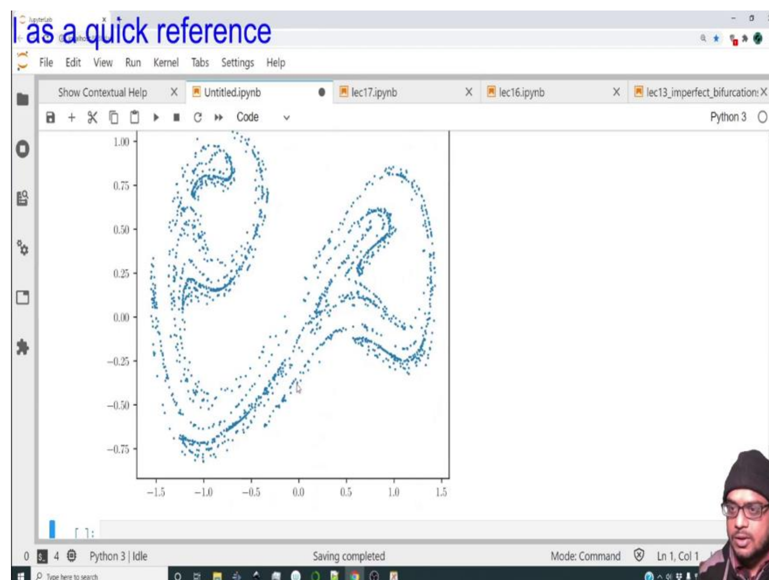

So, this is how the trajectories look like; but this is how the Poincare map looks like. So, it is clear that, along these points in the phase space is how the point is iterated once it starts from a certain point near the limit cycle; but it is not really approaching any cycle, it is going from one well to another well, it is not able to settle down, ok.

And it is not random, it is chaotic; meaning the points will not overlap, any small change will map on to some other point, there is no multiplicity in the mapping, meaning two points will not map to one, ok.

The trajectories are never repeating and it appears to stem from. So, you can see that there are certain curves that begin to appear; but they are not really solid curves, they are just a bunch of points which are close together. In fact, I am not going to do it on my computer; you can try it, it will take a bit of time, you can increase the number of, the number of cycles, you can make it 30000 and you can plot the last 10000 iterations, you will get a clearer picture of how the Poincare section for the Duffing oscillator looks like.

So, with the help of this, we are able to see how the Duffing equation, when it is forced by this kind of a cosine term; it undergoes a transition from a period one orbit to period two orbit and then eventually it is goes to chaos. And we are able to see that the Poincare map depicts the curves along which the phase space trajectories are lying on.

For example, it is not lying on this space over here or it is not lying in this space over here, and we have essentially reduced the dimensionality of the problem by one; because we have we have sort of removed time from the picture.

We are snapping the picture every time period, so we do not have to worry about the trajectories as the meander through space, ok. So, every time you do a Poincare map, you are reducing the dimensionality of the problem by one and this is a very neat way of assessing the system.

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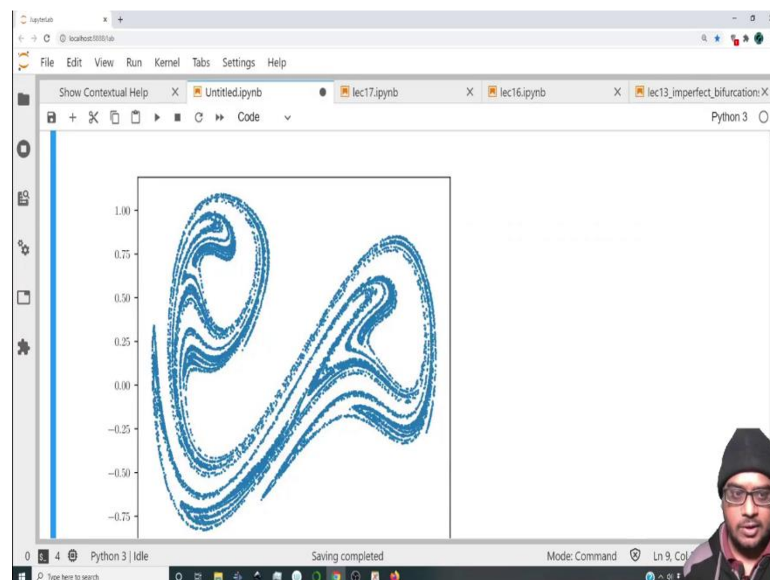

Well off camera I have done the simulation for 30000 time steps and plotted the last 20000 iterations; it looks quite nice, it is the very famous Poincare section of a Duffing oscillator.

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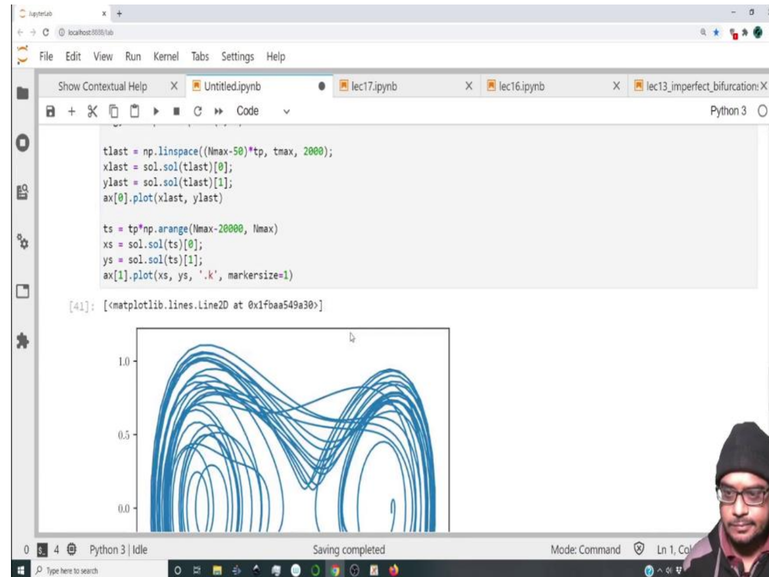

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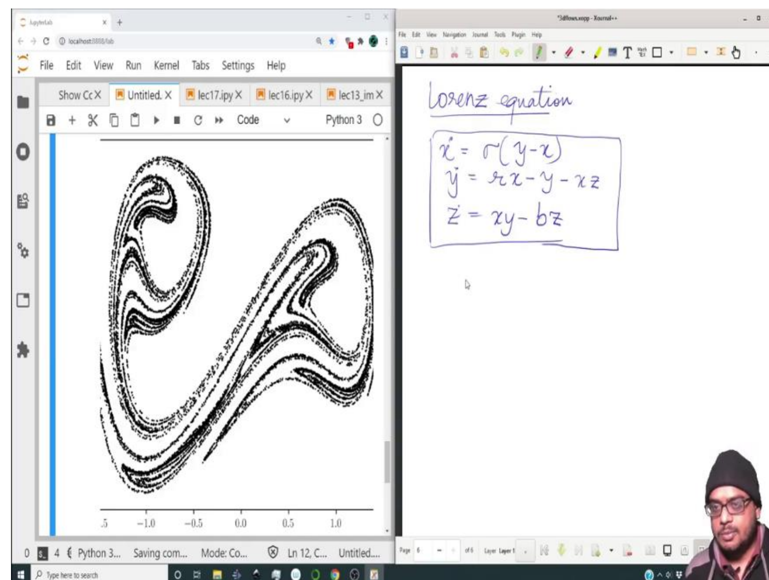

Let me just reduce the marker size to 1 and let me make the markers black, just for effect. This looks nice, this looks like the original plot in one of the papers, I will link it down. So, lastly to conclude this particular module, we studied the Lorentz equation; well studying not in the classical sense, because this is itself a very involved topic. I am just

going to scratch the surface and show how you can investigate this further as per your own interest.

So, Lorentz derived a set of non-linear equations which was used to model convection rules and at it is bare it can be written as following. So, $\dot{x} = \sigma(y - x)$, $\dot{y} = rx - y - xz$, and $\dot{z} = xy - bz$. So, these three equations are called as the Lorentz equations. And depending on the value of sigma r and b, we obtain different kinds of behaviors as seen from the evolution of trajectories in the phase space.

So, just a side note, σ represents the Prandtl number, r represents the Rayleigh number, and b represents the aspect ratio of the geometry. But anyway, let us focus more on what this particular set of equations helps or rather represents.

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So, let me go over to the notebook and try to code this in. And for this purpose, I will reuse some of the previous code; I will reuse this particular solver and that will help us in reducing our effort.

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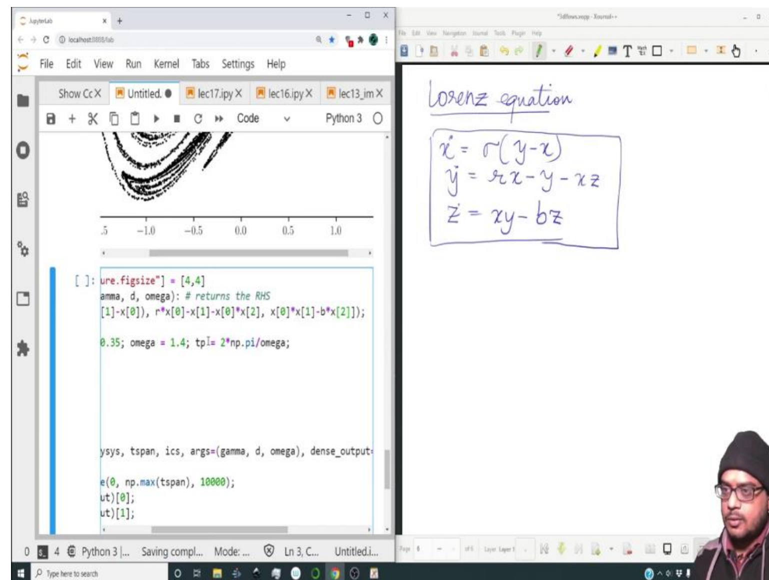

So, over here this will be. So, let me call σ by s. So, this will be x[1] – x[0]; this will be r *x[0] – x[1] - x[0]*x[2]. And lastly, we will have x[0]*x[1] – b*x[2], ok. So, x is a vector, it is a three column vector, time is t, and gamma has to be replaced by sigma s, d will be replaced by r, and omega will be replaced by b.

So, let me define certain values. And so, after experimenting a lot, numerically of course, with various values of the parameters; Lorentz was able to obtain very weird behavior for a particular set of values. So, we will set those values in and all these equations have a very fantastic history; in particular when Lorentz used a computer to solve all this, obviously there were tapes and all these things to store data.

So, what would happen is, he would run these equations on his computer, he would integrate this numerically; but after a while he would run out of tape, so he would use the previous solution as the initial condition for this equation. And what he did obtain was; even though there was, there were more, there was a larger accuracy in the representation of numbers, the tapes would have only a finite amount of digits.

And so, the loss in precision over there would translate to a small perturbation resolution. And even though he would obtain a similar looking number, a very small difference in the number which was not printed on the sheet; but which was stored in the computer would lead to a markedly different trajectory. So, this was later known to be as sensitiveness to initial conditions, ok. So, read this up, you will you will find this very interesting.

In case you have not read all this already; but I will put some links in the description, you can have a look. For now, we will focus on trying to solve this equation numerically and see what we can make out of it.

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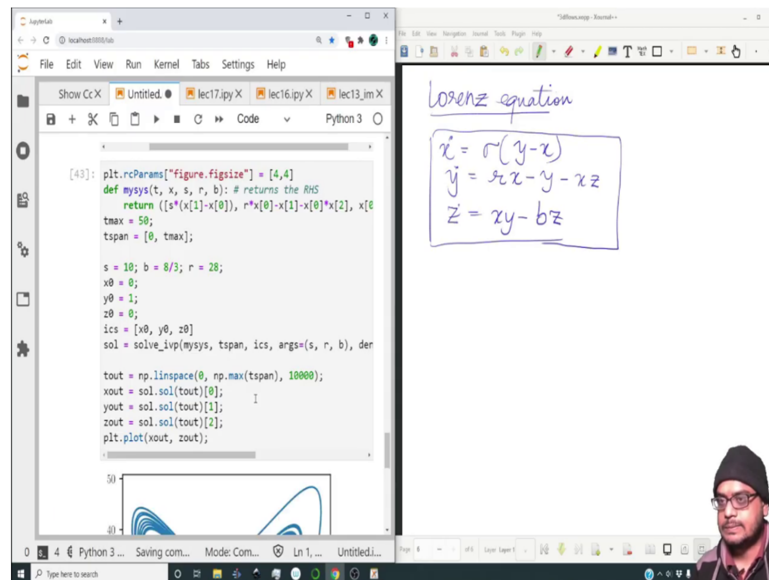

So, t max let me say it goes from. So, the t span goes from 0 to 50, initial condition we need to put in z0 as well. So, let me start the initial condition from 0, 1, 0; let me add z0 to the series, the arguments, right.

So, the arguments would be s = 10, b=8/3. So, these are the classic values which we found upon extensive experimentation and we have to pass these arguments into the solver, so srb, alright. So, simply we will have t out that is pretty much it; let me run this and see if there is some error, there is no error. So, let me just write zout = tout[2]. So, now let me plot the projection of x and z for example. So, plt.plot(xout, zout); let us see how this plot looks like, alright.

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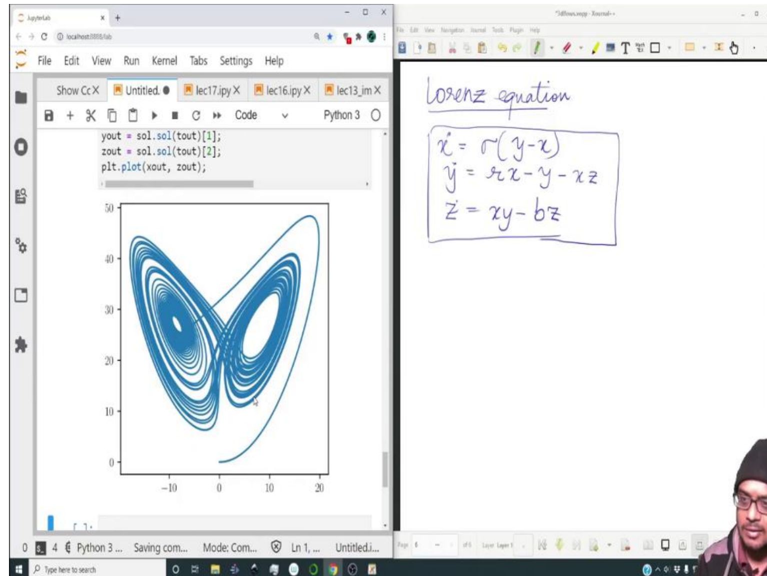

So, the plot looks something like this and in popular science this is called as. So, whatever I just spoke about that sensitiveness to initial condition that is called as a butterfly effect. And it has something to do with the fact that this trajectory projected on the x z plane does look like two wings of a butterfly or that the wings of a butterfly.

And the key observation is the trajectory is really winding itself on one side going on to the other side; it is not really stabilizing on a certain point or a certain attractor, it is sort of oscillating between these two wings.

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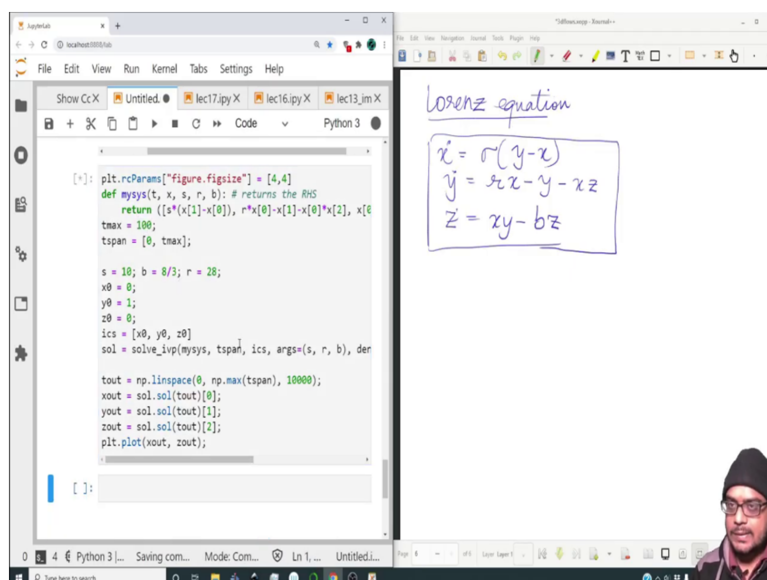

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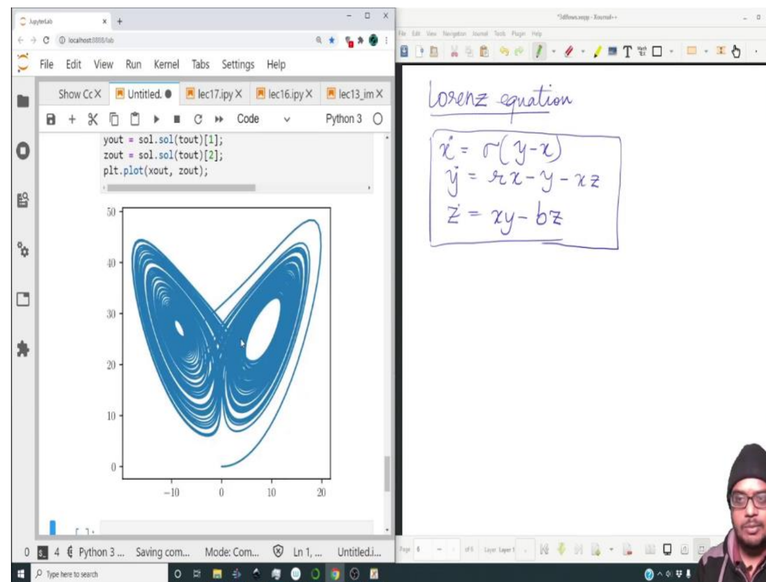

And let me just increase the time limit; I mean it is not just , it will never approach a certain value, it will keep on doing this, it will never intersect, it will not cross the same path again.

It may appear that these are intersecting in the projected plane; but in reality as we will see very soon, they are not really intersecting, they are just viewing around the various trajectories without intersecting.

And so, let us have a look at how this the trajectory looks in the Cartesian phase plane; that is xyz, we would like to represent the 3 dimensional trajectories for this particular initial condition, alright. So, for that purpose we would require Mayavi; I have already imported Mayavi at the very beginning of the particular notebook.

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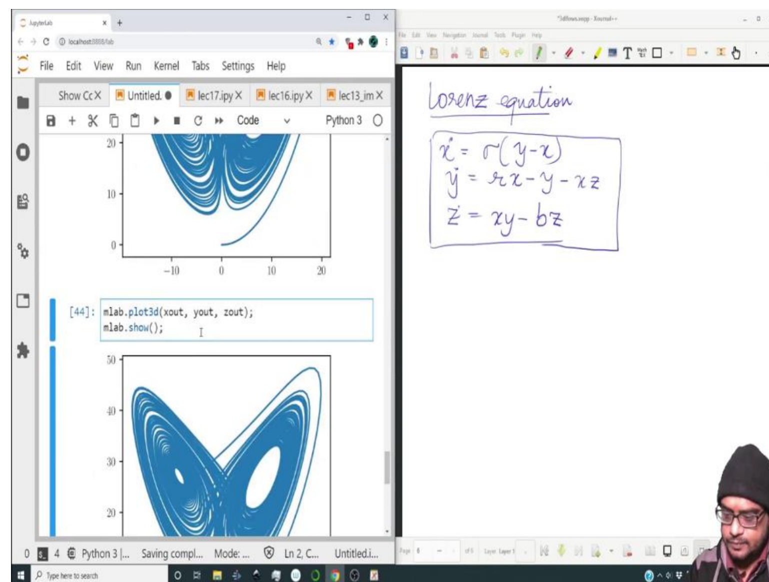

So, for that we will simply do mlab. So, I have imported mayavi as mlab. So, mlab.plot3D that is the function. So, I will write xout, yout, zout as simple as that; then we will write mlab.show, alright. So, let me create a new cell for the 3 D plot; because I want to keep it different from.

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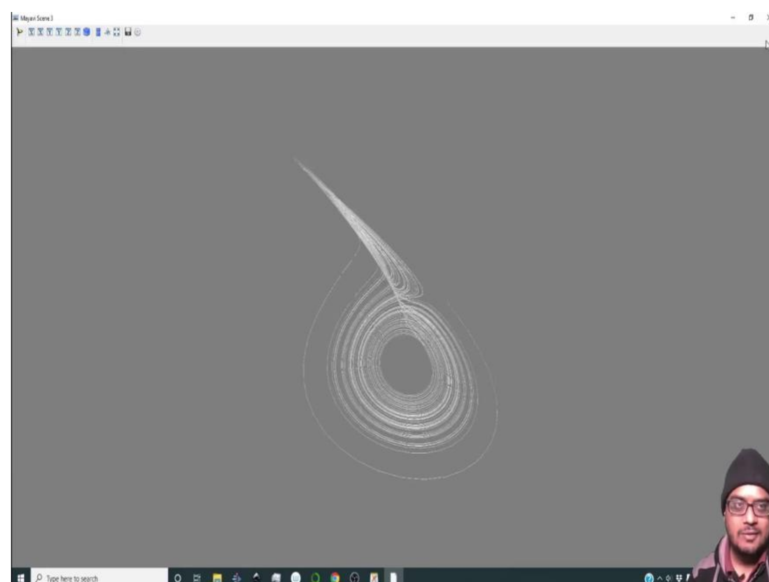

So, let me just confirm this is still, ok. So, let me run this and we have a Mayavi screen and very nice; it does look like a very weird looking structure with nothing seems to overlap. Let me in fact; increase the size of the tubes. So, typically mlab uses stream tubes or it uses tubes to represent all of this. So, let us just modify this glyph. So, it is typically called as a glyph, it is used to represent all of this.

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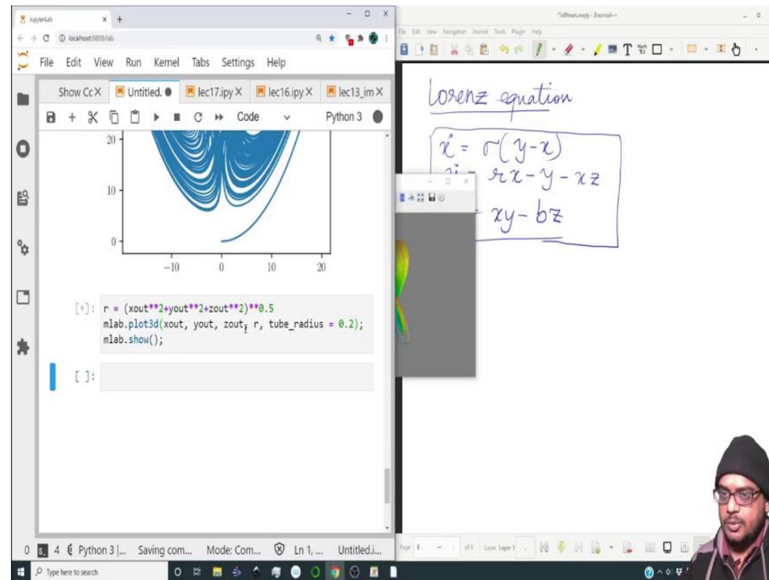

So, one thing we can do is create a variable r, which is like the distance from the origin. So, let me define it as $xout^2 + yout^2 + zout^2$ and then take a square root of all this. Let me pass it to this for coloring the curve. So, the curve will now be colored depending on the scaling of r ok; it is a very convenient way of showing which trajectories are nearer to the origin and all that. So, after this, let me also increase the thickness of the tube. So, tube radius = 0.2, alright.

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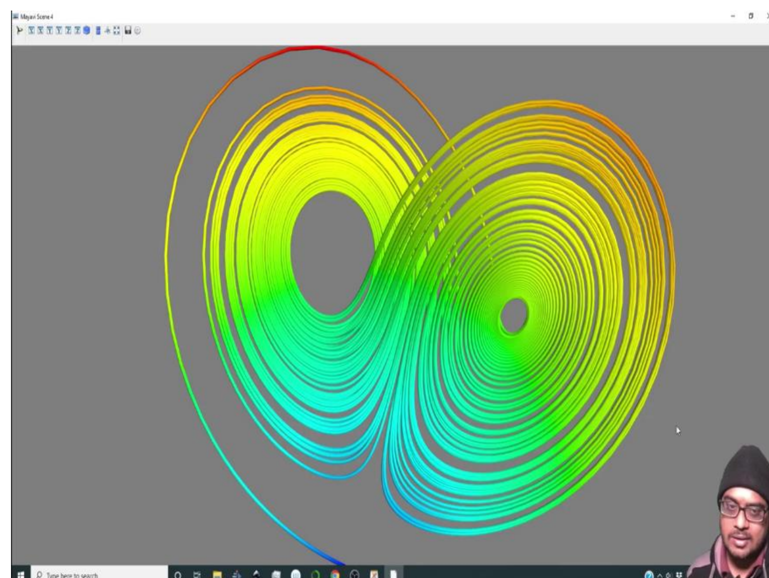

Let me run this, alright. So, this looks good, ok. So, this is the Lorentz equation trajectory; this is the initial point and these are the two wings of the butterfly. And the trajectory home seen onto one of these wings; it goes to the other wing, oscillates over there, goes to the other the other end, it is like an infinity loop you know.

It is like looping over this; but it is never really settling down on any one trajectory. And it is amazing; I mean it would seem like there would be some kind of stable orbit, because of the concentric nature over here, but really none of the lines intersect. They are always separate, they will never overlap and the entire phase space is split between these two sort of flat shapes; I mean these are not really flat as you can see.

But these are sort of two shapes onto which the trajectories will home on to and this set of, this entire set this shape it is called as a strange attractor, ok. So, let us let us try to draw another trajectory, let us try to draw another trajectory.

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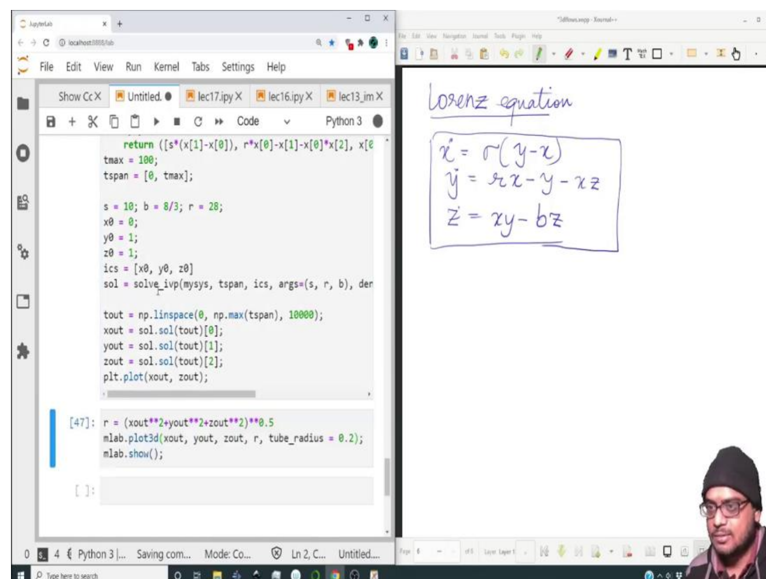

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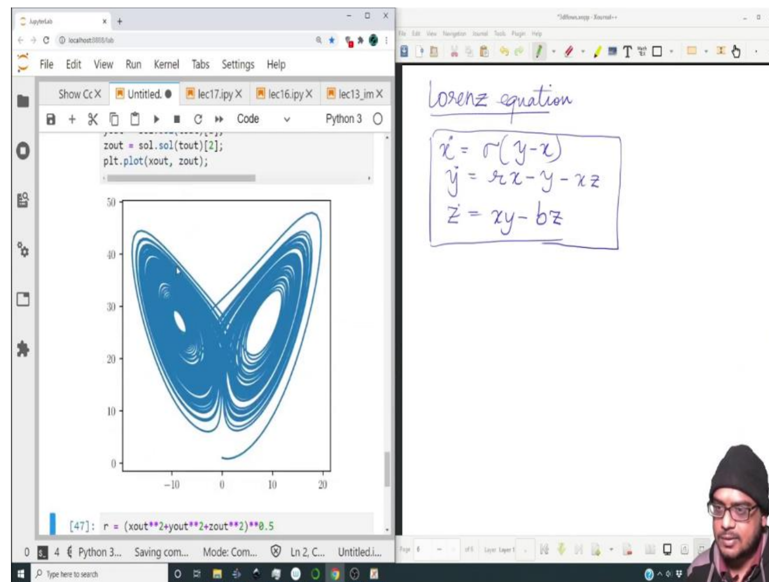

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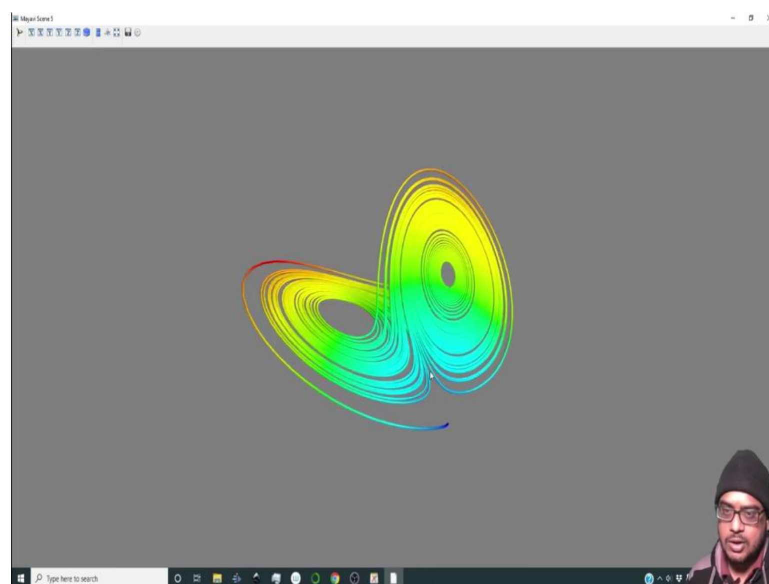

So, let me make this also 1. So, let me run this, ok. So, let me run this and you can change the value; but you will see that it always tries to get back to this weird looking surface and that surface is this strange attractor. So, now let us also see the time series of the z x, the z point. So, what Lorentz did was; once he obtained the solution, let me reduce the time, it is perhaps a bit too large.

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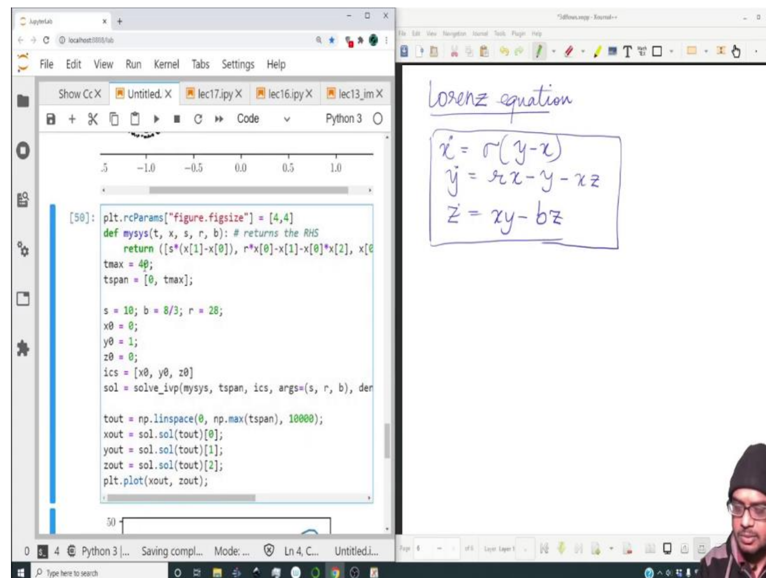

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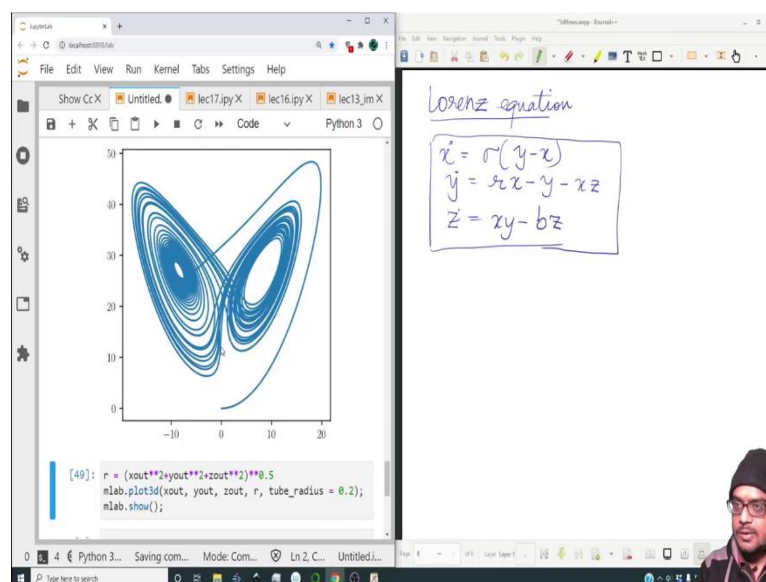

So, let me write it 40, ok. So, what Lorentz observed was, it was hovering around on one end; then for a certain value it would go to the other end, then it would stay there for a while, then it would go to this end. So, it would oscillate between the two ends and he plotted this particular time series.

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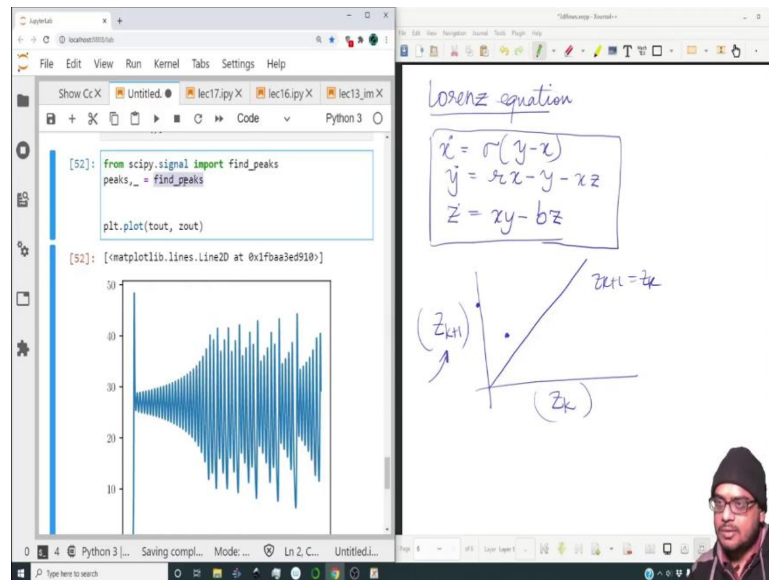

So, let me do a plot plt.plot. So, I want to plot the time and the z. So, I want to plot tout,zout. Let us see how the plot looks like. So, it starts from 0, it then shoots up, then oscillates for a while; then it seems to be locked in this a periodic orbit. So, now, the basic idea that he had was; if I plot successive maximum values of z.

Let me. So, he thought let me plot successive maximum values of z. So, basically he wanted to plot z[k + 1], again z[k] and these are the local maxima. So, whether a given maximum value of z would help us in predicting another, the next consecutive maxima that is going to appear. So, if it would be the same, we would have this 45 degree line; if z[k + 1] = z[k] that is for a locked oscillator, where you oscillate over the same value, you know that you are going to get the same amplitude each time.

But, what about this case? It seems to fall, it seems to rise. So, it is it is not really repeating itself; but what about it ok? So, in fact let us pick out the maximum values of z and try to derive this particular plot, ok. So, for this we will need to pick out the maxima, the local maxima of this entire time series. So, let us do that.

So, for this what we will require is, find_peaks function of scipy. So, from scipy.signal import find_peaks alright, then. So, find_peaks is a function which resides inside the signal sub module of spicy, alright. After this we are going to find out the peaks. So, peaks sorry, peaks, _. So, this is a way of assigning the output of find peaks to nothing.

So, find peaks returns the index, where the peaks are found and it returns a dictionary which contains additional information. So, we are not really worried about the dictionary.

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So, if I double click on this, I should be able to get the entire the function call and all that how to do that.

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So, in this contextual help, we can see that the output it returns peaks and it returns properties, where it is a dictionary containing all these different things. We are not bothered about that, we just. So, I can assign it to underscore, which is nothing; I do not assign it to a variable.

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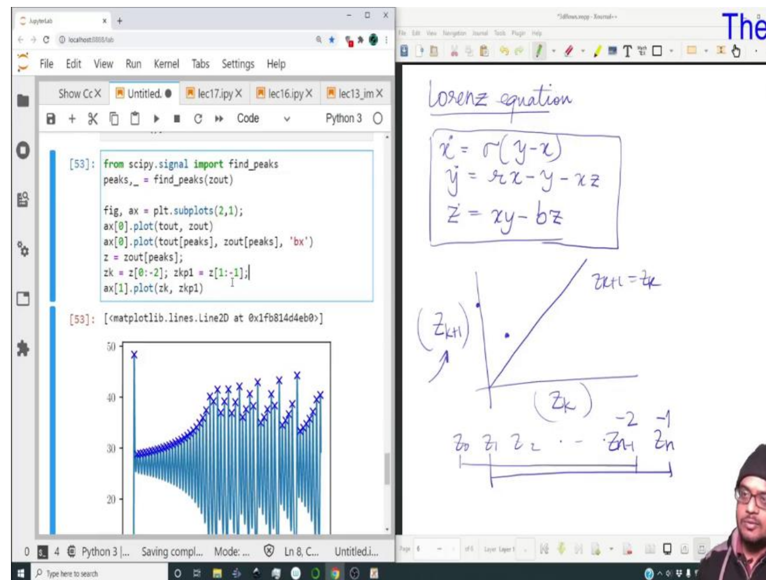

So, to this function, I will simply pass the value of zout, alright. Once I do that, I can then superpose on this particular plot. So, what I will do is plt.plot and I will plot on top of this curve all the index corresponding to the peak. So, tout, the index will be peaks and zout, the index will be peaks and let me mark them by a black cross.

So, let me run this and see, oh sorry blue cross. So, excellent , it shows us all the local maxima appearing on this curve; it is a useful thing to know how to do this, because in signal processing and all that, you may end up using such kinds of functions, and scipy.signal has a lot of such, a lot of these functions, it is quite useful. So, once we have done this, we can now plot what Lorentz plot to see whether a certain value of z is going to help us to predict the next value of the local maxima, alright.

So, we will go over here. In fact, let me assign all of this to a subplot. So, I will do fig, ax = plt.subplots(2,1); then ax[0] , ax[0] and ax[1] we are going to plot z[k] and z[k + 1]. But for that, we have to define what z[k] and what z[k + 1] are going to be. So, we have already found out the sequence. So, the z sequence is going to be zout peaks, alright.

So, now let us assign z[k] as z[1:- 2]. So, basically if this is the sequence of z's we have z 0, z1, z2, zn- 1, zn. So, I am picking up this entire sequence and dumping it into z k. So, this is z 1 to - 2. So, this is index - 1, this is index - 2. So, this is a way of splicing the array together. And similarly, can you tell me what z[k + 1] is going to be? So, it is going to be basically this array, alright.

So, it is going to be obviously, z, this has to be 0; because the index origin is from 0, it is unlike MATLAB, it is like C, alright. So, this is going to be from 1 to - 1. Now, simply you have to plot them.

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So, let me run this and if you need to only plot. So, we are trying to plot the iterates; we are not trying to plot a curve, we are simply trying to plot an iterate. So, dot k marker size = 2 and we end up with this kind of curve.

Let me just change the aspect ratio of this curve and for good measure, we will also plot the y = x line on top of this. So, let us see it goes from 30 to 45. So, x dummy =, so this will be xd , xd, let us draw a magenta line, ok.

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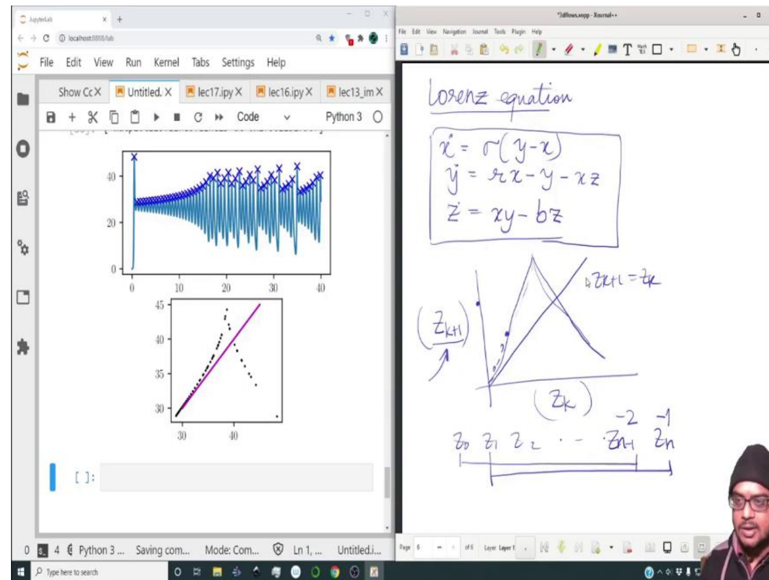

So, the lower curve, you know it is quite interesting to see what the lower curve really means; it means that, despite the seaming a periodicity and all these kinds of things. So, if we just plot, whether the k + 1th maxima; I mean how k + 1th maxima and kth maxima are related, we obtain a discrete curve, set of curves like this. And actually it is not a curve, it is a set of points which has a very small thickness; but it is not a curve, we cannot fit an equation through it ok, it is something which has a very small thickness.

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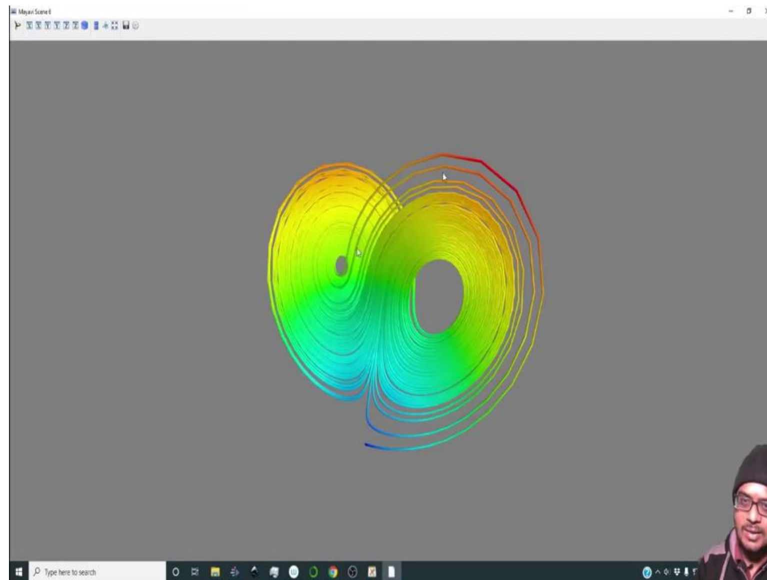

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```
tmax = 200;
tspan = [0, tmax];

s = 10; b = 8/3; r = 28;
x0 = 0;
y0 = 1;
z0 = 0;
ics = [x0, y0, z0]
sol = solve_ivp(mysys, tspan, ics, args=(s, r, b), den

tout = np.linspace(0, np.max(tspan), 20000);
xout = sol.sol(tout)[0];
yout = sol.sol(tout)[1];
zout = sol.sol(tout)[2];
plt.plot(xout, zout);
```

```
[58]: r = (xout**2+yout**2+zout**2)**0.5
mlab.plot3d(xout, yout, zout, r, tube_radius = 0.2);
mlab.show();
```

```
[56]: from scipy.signal import find_peaks
peaks,_ = find_peaks(zout)

fig, ax = plt.subplots(2,1);
ax[0].plot(tout, zout)
```

$$\dot{x} = \sigma(y-x)$$
$$\dot{y} = rx - y - xz$$
$$\dot{z} = xy - bz$$

$z_{k+1} = z_k$

(z_{k+1}) (z_k)

$z_0 \; z_1 \; z_2 \; \cdots \; z_{n-1} \; z_n$

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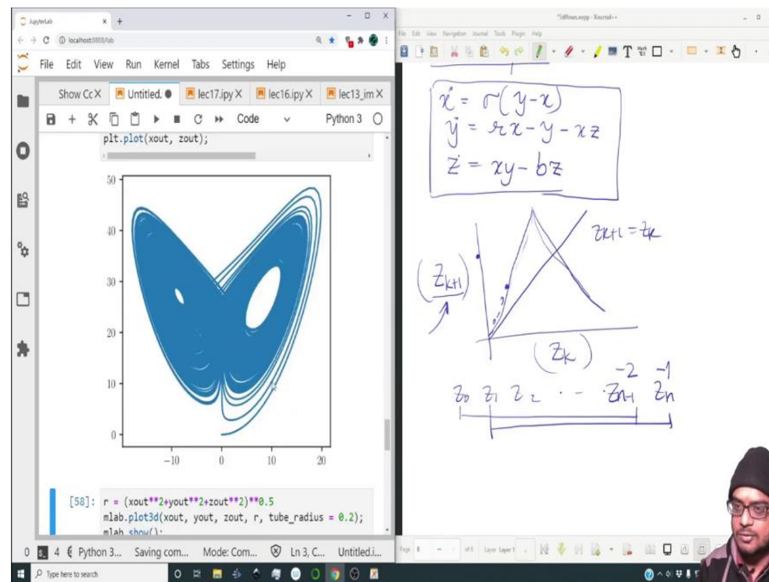

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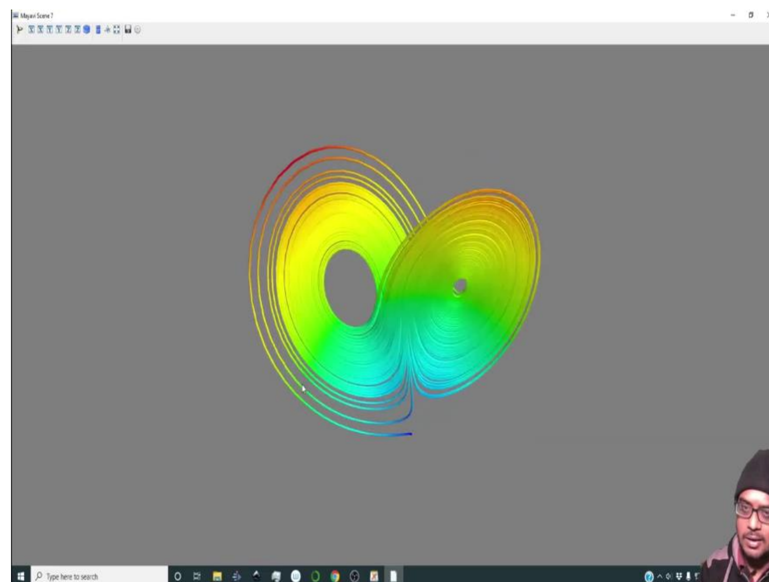

In fact, if I increase the number of; if I increase the time span to say 200 , we will see better what it is, ok. So, this is the projection onto the exit plane , this is how the 3 D butterfly looks like and it is a bit jagged, because of not resolving the time well enough. In fact, we need to take maybe 20000 points; why do not we do that, we have the computing power, it is not a big deal.

Be careful when you are doing this on a underpowered computer, it may take a while. I am running a fairly recent computer, ok. So, this is how the trajectories look like in phase space or the trajectory looks like in phase space.

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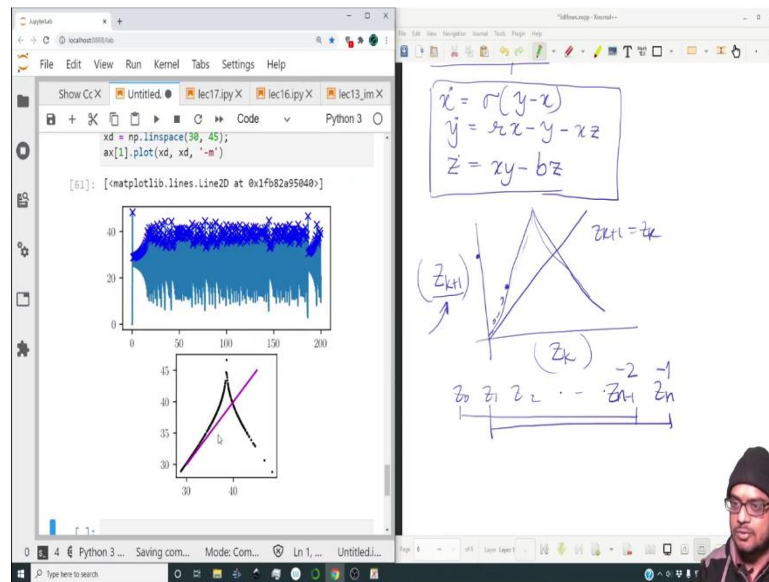

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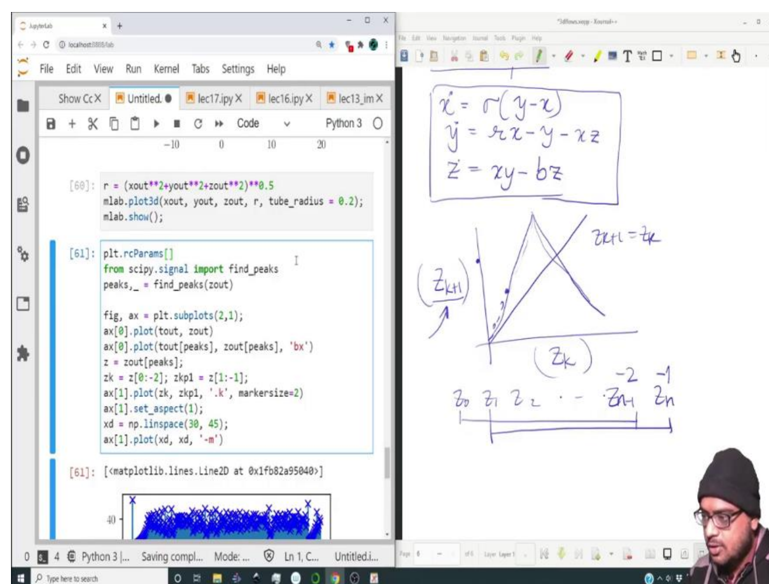

And now let us see this particular signal. So, now, we do have a much smoother looking curve; let me just increase the figure size, because it is rather small.

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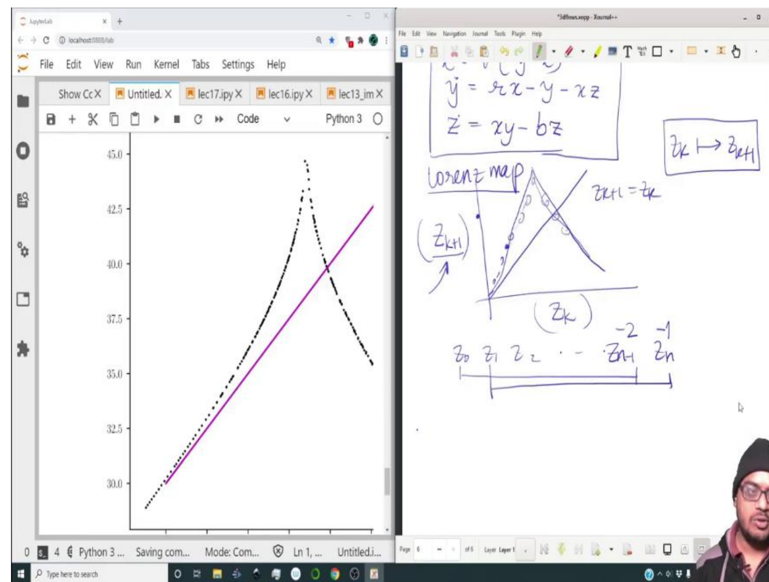

So, this curve it does appear to be a curve; but it is really a locus of points with a very small thickness. And this represents the fact that the iterates are indeed bounded to this particular set of points ok, it is not going to deviate.

So, it is not going to happen that one point is over here, the other point is going to be over here; you are always going to land up somewhere on this curve through iteration. So, if it were to be a closed orbit, you would know that the next point and this point would land on the magenta line; because it has always a fixed amplitude, but not in this case, it lies on this curve and it is a very famous curve.

This particular curve is called as the Lorentz map; because it maps a value of z[k] on to z[k + 1] ok. And that sort of, well it is not a functional relationship that graphically is this set of black curve, black points ok; that is the map which helps us achieve how z[k] maps onto z[k + 1].

So, before concluding, I would like to discuss a bit on the Poincare section of the Lorentz equation. So, in the previous, the previous set of equations we had seen that, the stroboscopy or when you strobe an equation at a definite time period; you can reduce the dimensionality of the equation and sort of have a window into how the evolution of phase points looks and we saw that it led to weird looking curves, stranger thing happened for 3 D.

But in this case the Lorentz map was really a reduction to from three dimensions to one dimension; you are just looking at one coordinate and assessing how the evolution sort of distributed around this, this particular set of points.

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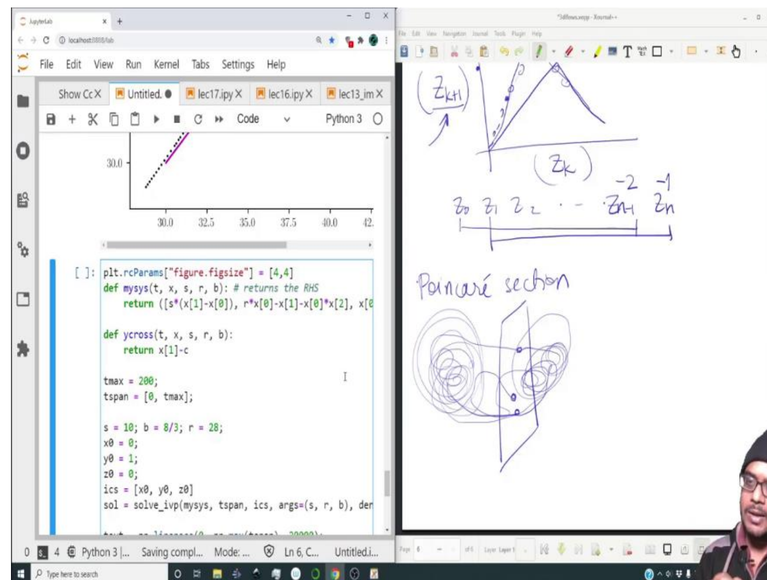

But we can also alternately find out the Poincare section of the Lorentz equation. So, we have a set of trajectories like this and the set of trajectories like this so we can. So, we do not really have a forcing over here; so we do not have really a time period with which we can measure at what frequency you have to strobe.

But instead of that we can sit at a particular plane; we can sit at a particular plane in this phase space and see how the trajectory crosses it, goes over here, when it crosses it over here, and how it crosses. So, we can sort of look at the evolution of the recurrence of these points on that particular plane, ok.

Whenever the points cross a certain plane, we will plot it on that particular plane; it is like instead of strobing, you are sitting at a place and looking where the particle is going through that plane, ok. So, let us try to reuse this particular code and this will show a very useful function or rather useful extension of the solve ivp function and that is to define events, ok. So, we will define so everything else remains the same, we define an event like this.

So, let us say we will define an event as to when it crosses the line y = 0, ok. When this particular trajectory crosses y = 0 that is an event which we want to note. So, we will write y cross. And this will have the same inputs as r; b and we will return simply x 1. We will simply return x 1; because if we return x 1 - something like a constant, then it will give us x 1 = 0.

So, whenever x 1 = c or very close to that; because we are doing a numerical integration, it will never be equal, it will always be in an in a neighborhood of c, then it will trigger the event, ok. So, once it triggers the event, we can sort of get an information of when those events have been triggered, ok.

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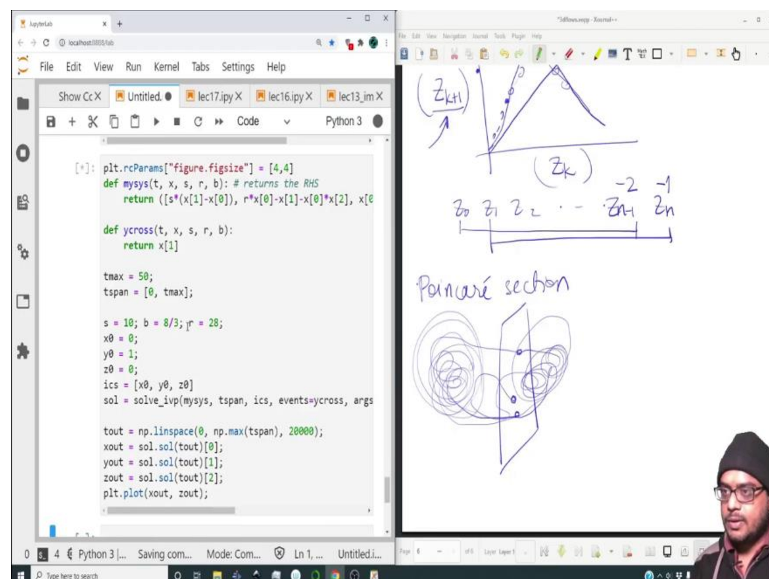

So, in this we will write events = y cross and that is it. So, let me run this; maybe I need to reduce this a bit, let me let me make it 40 or 50.

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So, we have this curve. Now, let us probe or let us look into what sol is. So, in sol we will look at these particular things.

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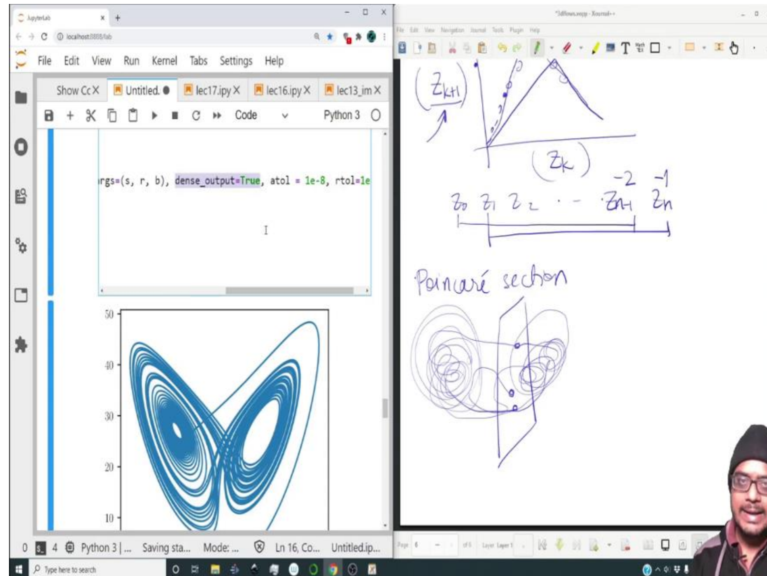

And we have so much information inside sol solely because, we have made dense output = true; if you do not declare dense or to dense output to be true, we will not get all this information.

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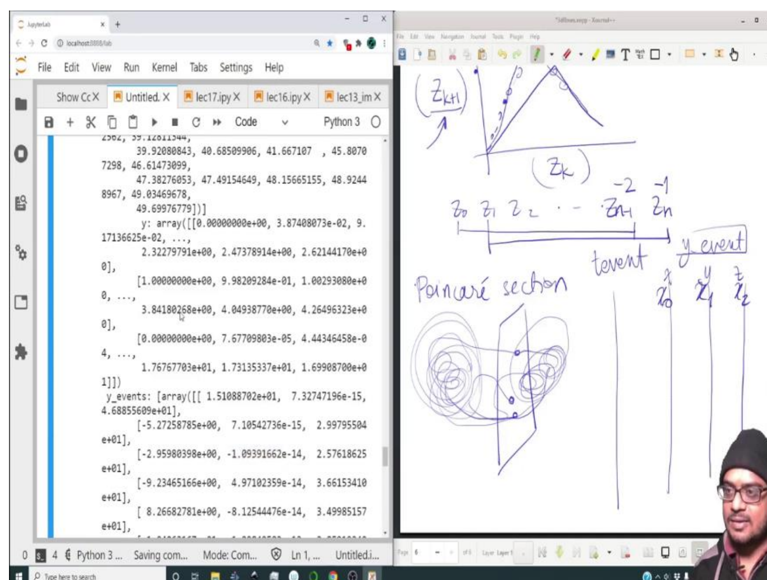

So, events = y cross. So, t events is given and y events contains all the x, y and z corresponding to those t events where y is close to 0. So, now look at y events. So, this is one set of x, y and z; look y is very close to 0, again y is very close to 0, ok. So, we have t events, which is a series of times; then we have y events, whose first column is the x values, second column is the y values, third column is the z values.

So, do not be confused between y event and x y. Solve ivp uses the default variable as t and y and y can contain x, y, z, w, v all these things; it depends on your declaration, essentially this is x 0, x 1 and x 2 from the function definitions that we had before.

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Now, let us see out how we can extract this. So, we can set t event = sol dot t events; but let me show what this contains, let me print this. So, it is actually an array of an array. So, we have to simply plot the 0th element of this. And how do I know it is an area of an array? Because look there is a right brace over here.

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So, now once we do this, we will have an array. So, t events should contain 0 and the reason why they have this array of an array is, because you can have multiple event functions inside this. And so, t event 0 is all the events for one particular function; t events 1 would be all the events where some other function is satisfied and so on ok.

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So, we are more interested in 0 and the corresponding x e. So, let me print sol dot y events; a similar thing will happen here as well. So, when I plot the 0, it is this and all the rows.

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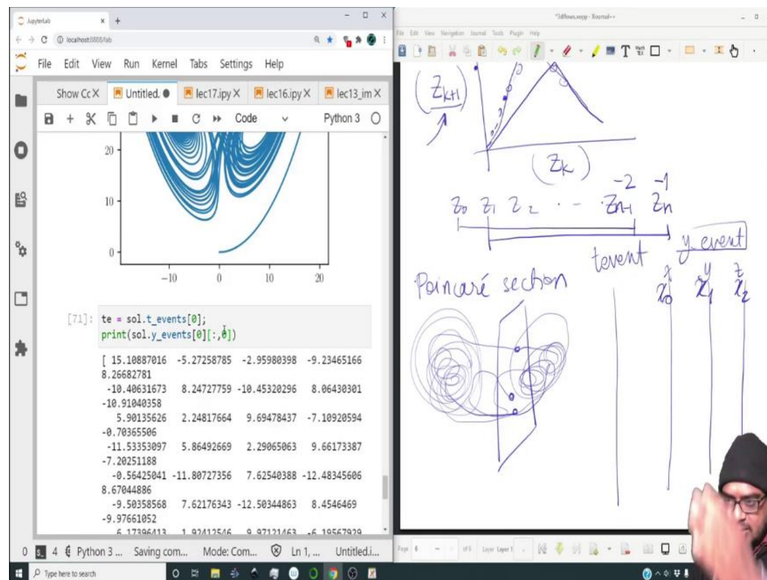

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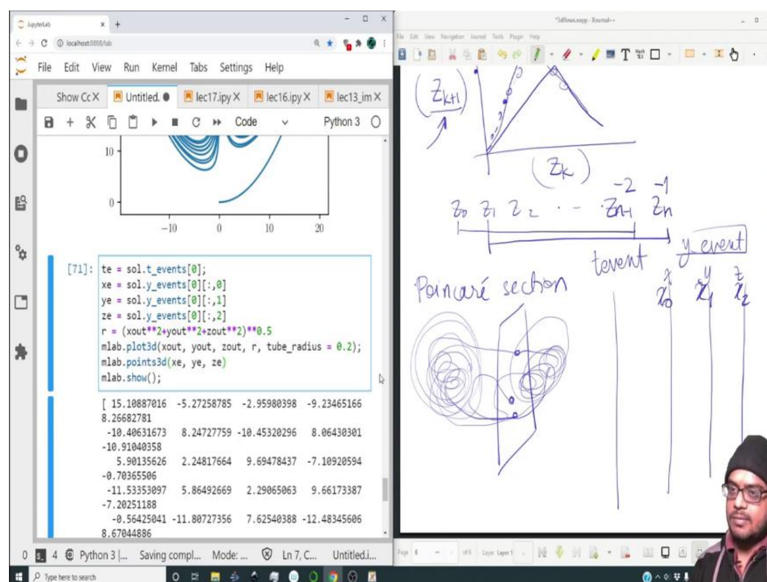

So, all the rows of the first column and these are all the sorry of the 0th column. So, these will all be the x values; then though there will be y values and there will be z values. So, let me declare them two separate variables. So, this will be x c, let me copy this; this will be y e, then this will be z e.

So, this will be 2, this will be 1. And look if we have more events, all these 0 will be 1; if we have another event, it will be 2 and so on. So, with this we can now plot the corresponding points. So, for that let me in fact take the 3 d curve that we had and then on top of the 3 d curve, we will also plot these points. So, mlab.points3d(xe, ye, ze) alright.

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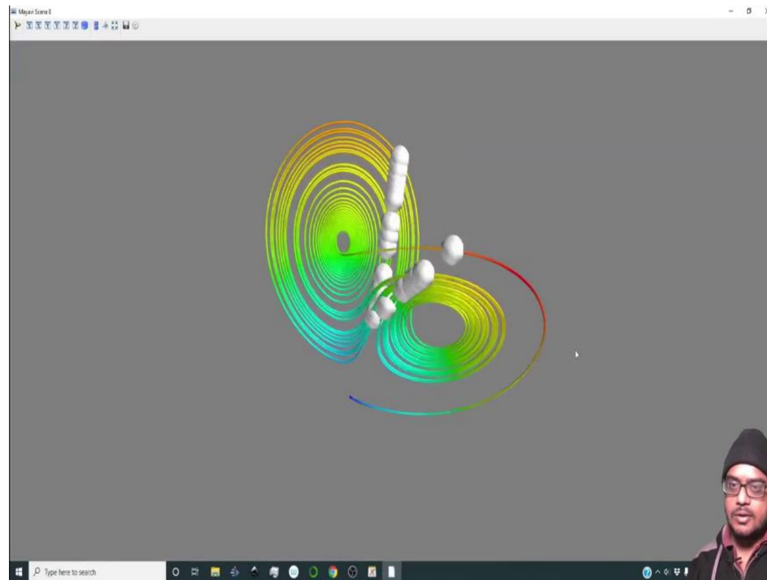

So, let me plot this, excellent. So, look at this these; so obviously these are very big glyphs, we can reduce the size of this by going over here.

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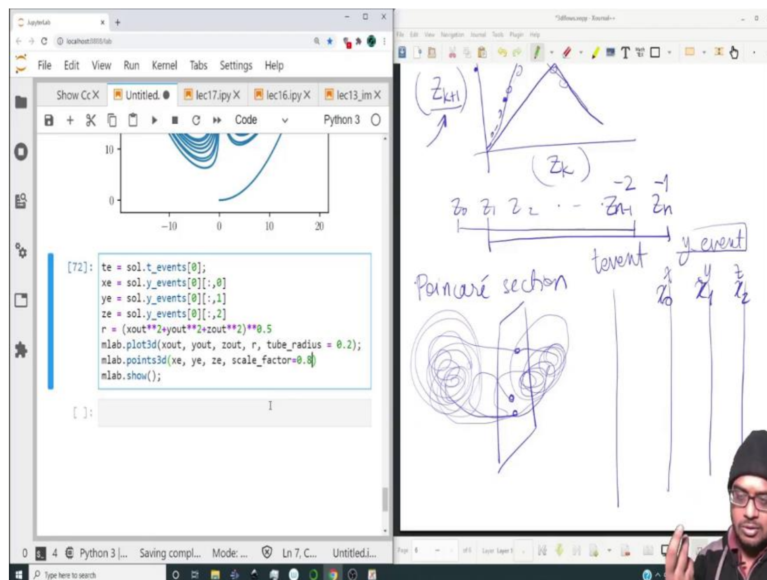

And we can say scale factor = 0.8 and you can look at the contextual help for this function in order to understand how to scale those balls or points as they may be ok, so, alright.

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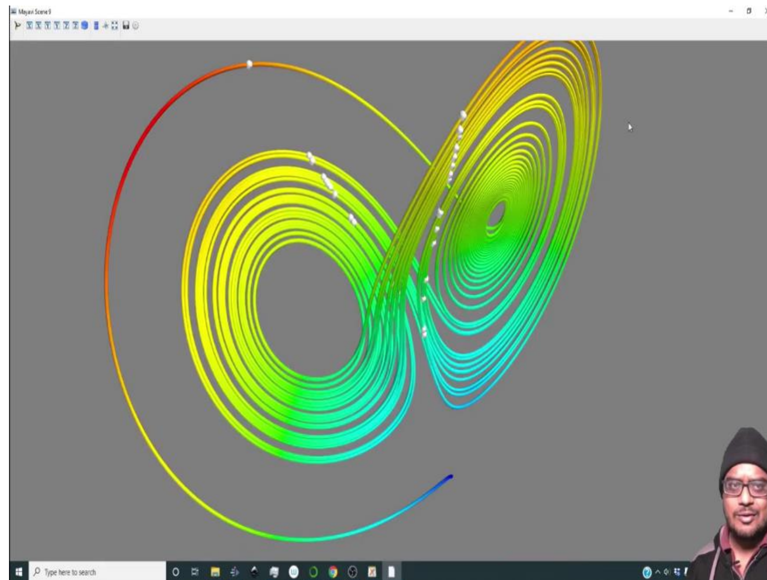

So, these are the points where the y axis is being crossed. And how do I know that? I mean it all looks very colorful, there is no axis and this is just a small drawback of using mlab; the axis is not there by default and there are various ways of doing it, you can go look at it, but I am not going to do it over here. In fact, I am going to plot y out over here and what I am going to color it? So, the fourth variable inside plot 3 d will be to color the set of curves.

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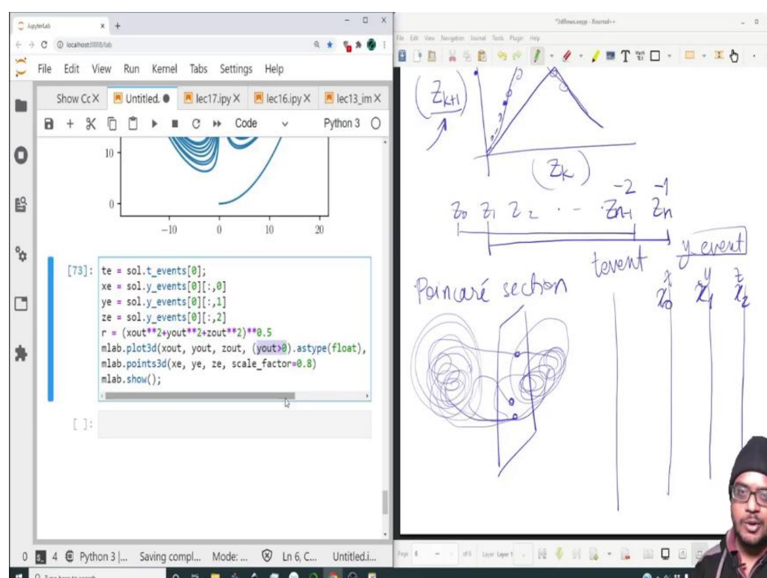

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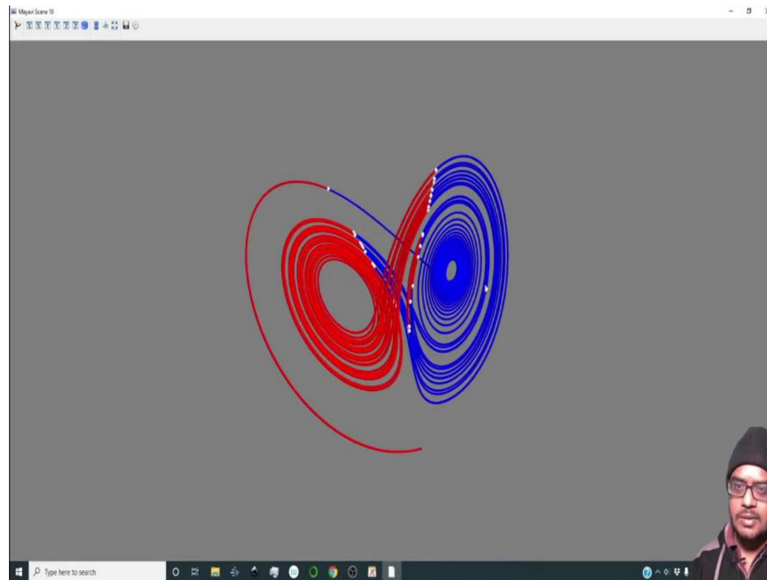

So, it will be y out greater than 0 and I will cast it as a float. See because y out greater than 0 will be a Boolean, it will be true or false; then dot s type float will convert that Boolean into a floating point number. So, true which is one Boolean will be converted to 1.0 float. So, with this, we should be now able to see, great. So, this is how we can distinguish the crossing of the two curve when y changes and these are the points.

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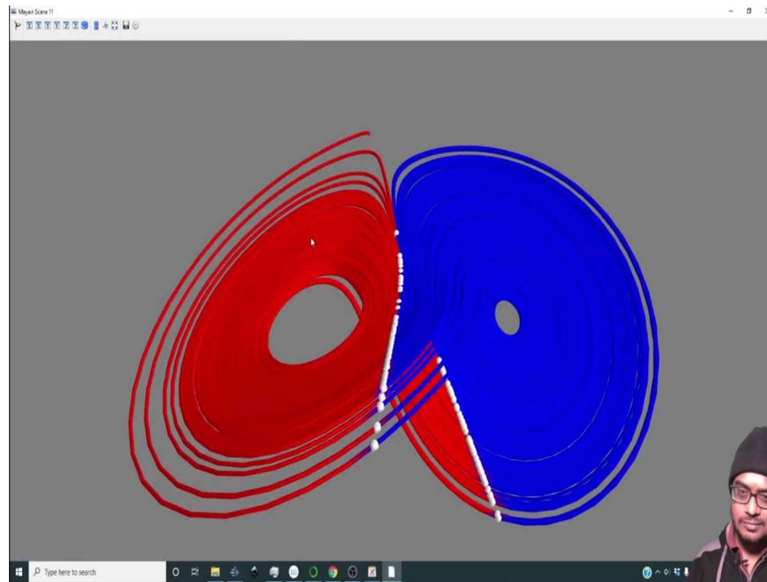

So, now let me increase the time, so that we can have a better Poincare section. The Poincare section does not really give a good insight for this particular problem; because the we cannot really, well, I mean it depends on how you look at it. The interesting thing is it does lie on that on a particular locus of curves with a seemingly zero thickness, but it actually has thickness, ok.

So, the theory of this is quite complicated, it requires a lot of insight into the entire mathematics of this, which is much beyond the purview of this course. I will try to link some other NPTEL lectures or some lectures by Strogatz and you can have a look.

But this is how, once you do have a look at those; you can really visualize those things using python or octave and that will give you really confidence to tackle these problems. I mean once you go into more difficult problems, it is always good to visualize something, seeing is believing right.

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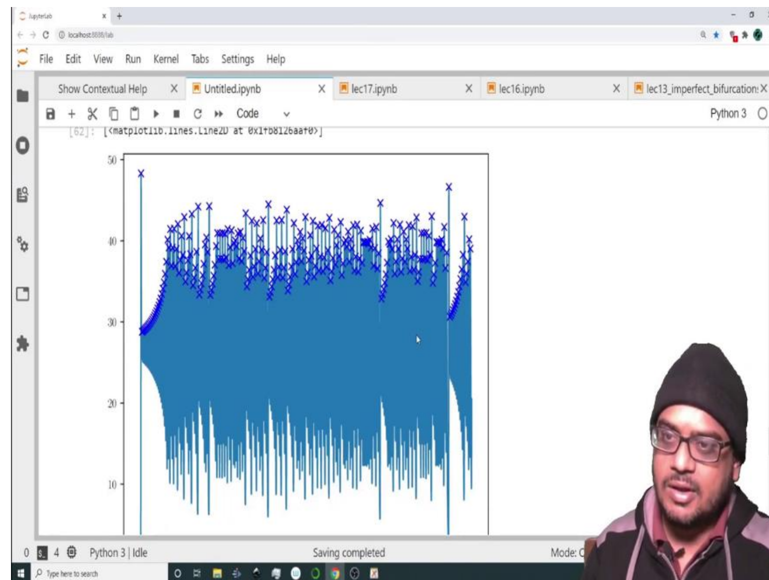

So, with this I end this particular week, you can see how a periodic it is. So, we have looked at a lot of things and we have looked at various disparate problems and we have seen how we can address them by means of python. A very similar code set will be used for octave as well, all these things will be available from the website; you can have a look, you can run those codes on your computer.

In fact, I encourage you to do that; it will give you a lot of confidence when you can do these problems on your own. So, with this I conclude week 3; next week we will be back with some random numbers, until then it is goodbye, bye.