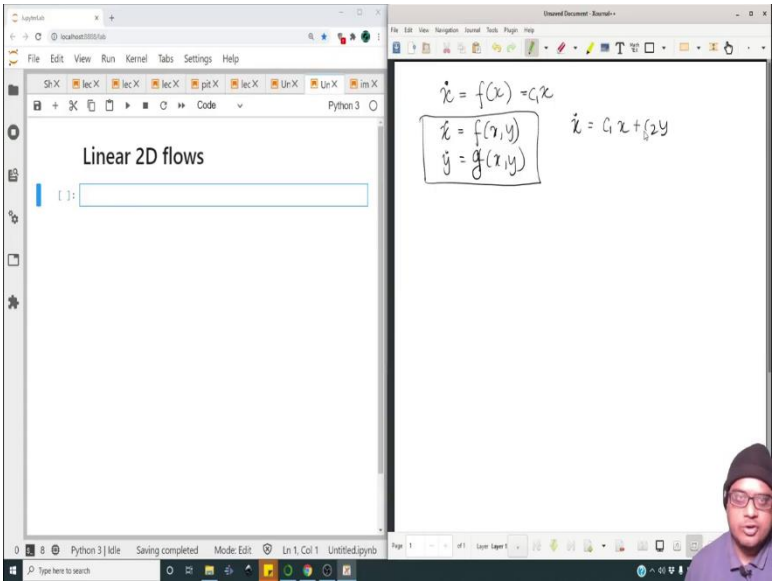
Tools in Scientific Computing
Prof. Aditya Bandopadhyay
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 14
2D flows – linear systems

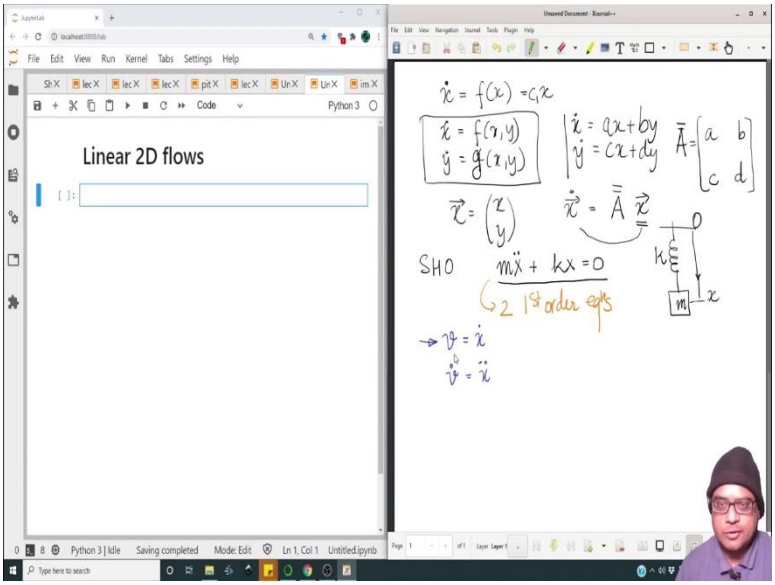
Hello everyone, in this lecture, we going to start with 2 Dimensional flows, in particular we will start with linear 2-dimensional flows. So, what do we win we mean by linear flows?

(Refer Slide Time: 00:42)



So, in 1 dimensional flows, we had something of this form. So, $\dot{x}$ is some non-linear function of *x*; I mean this is an autonomous equation and when $f(x)$ is linear that is $f(x)$ is of the form *x* or $c_1 x$, then this equation is a linear equation. What do we mean by linear 2 D flow? So, 2 dimensional flows first of all we must have $\dot{x}$ equal to some function of *x* and *y* and $\dot{y}$ is equal to some other function of *x* and *y*. So, this essentially encompasses 2 dimensional flows.

(Refer Slide Time: 01:45)



Now, when $f(x, y)$ and $g(x, y)$ are linear functions of *x* and *y*; meaning $\dot{x}$ is some $c_1x + c_2y$ or more generally we can say $\dot{x}$ is $ax + by$ and $\dot{y}$ is $cx + dy$. Then such kinds of flows are called as linear flows. And the reason why we study linear flows is that, the behavior of linear flows will give us more insight into the behavior of non-linear flows; it holds the key towards understanding what the behavior near various fix points will be.

And I reiterate the fact, this is not a theoretical course on non-linear dynamics; but regardless of that by pure geometry, we will try to make use of Python and the equivalent script in octave will be also on my website. So, how we can use these tools to visualize and understand the basic idea behind interpreting such kinds of equations?

Once we have cast the equation in this linear form, we can then write this equation down in a matrix form. So, $\vec{x}$ is the vector x, y and so $\dot{\vec{x}}$ is some matrix times $\vec{x}$. In this particular case, the *A* matrix will be this *a, b, c, d*.

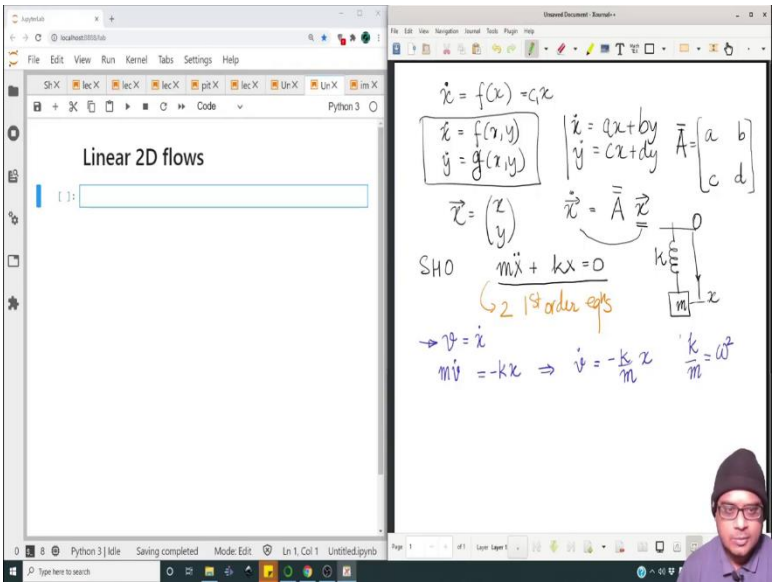
And wherever there are matrices involved, we can bet that the eigenvalues and the eigenvectors have something to do with how this system of equations, this dual system of equations fundamentally behaves; because the eigenvalues and eigenvectors hold the key towards unraveling the structure of a set of equations, which is sort of bounded together or mapped by a matrix. So, essentially, we are mapping this state with the rate of change through this kind of a matrix, alright.

So, let us consider the very simple linear equation and the linear equation in this case will be a simple harmonic oscillator. So, what is the governing equation for that? So, $m\ddot{x} +$

$\omega^2 x$ or $\omega_0^2 x$ this will be kx is equal to 0, ok. So, this particular equation is what governs the motion of. So, if this distance is x from this, the spring constant is k, the mass of the block is m; this equation governs how x changes in time.

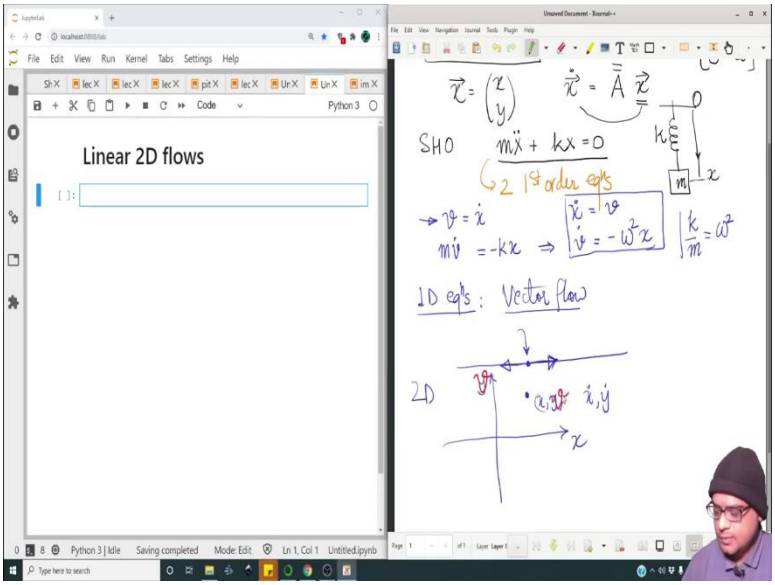
Now, this is a second order differential equation; we can cast it in the form of 2 1st order differential equations, ok. This the first task will be to split this equation into 2 1st order equations. So, let v be equal to $\dot{x}$ and consequently $\dot{v}$, which is essentially $\ddot{x}$, ok. So, this is what basic substitution will be.

(Refer Slide Time: 05:33)



So, if $v = \dot{x}$, we can write $m\dot{v} = -kx$ or essentially $\dot{v} = -\frac{k}{m}x$. Where $\frac{k}{m}$ we can sub, we can I mean if you remember your high school physics, $\frac{k}{m}$ is like the ω^2 parameter of an oscillator, ok. We are not going to discuss about how this comes into being, but it is related to the frequency of oscillations.

(Refer Slide Time: 06:07)

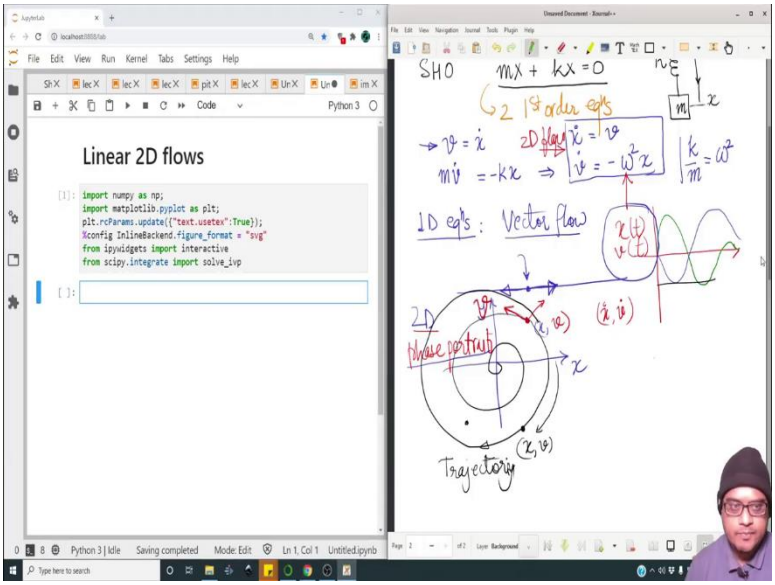


So, $\dot{v}$ becomes $\omega^2 x$. So, this we have two equations. So, $\dot{x} = v$ and $\dot{v}$ becomes $\dot{v} = -\omega^2 x$. In the case of 1 dimensional equations, we had an idea of a vector flow.

And the vector flow simply told us on a 1 dimensional line, how does a point move and the motion of the point was dictated by the velocity or $\dot{x}$ corresponding to this particular location. So, if $\dot{x}$ at this location is positive, in this particle will tend to move towards the right; if $\dot{x}$ is negative, it will move towards the left.

So, the vector flow in 1D showed us, how points would move on a line and from that we were able to see whether various points on the line would be attracted towards a certain point on the number line or it would be repelled away from certain point. But in 2D, we have now 2 dimensions to play with. So, we have both x and y and at a given collection of x and y, there will be two velocities $\dot{x}$ and $\dot{y}$. In this particular case, it will be x on the x axis and v on the other axis.

(Refer Slide Time: 07:54)



So, $\dot{x}$ and $\dot{v}$ are like telling us how this point in this plane would move given this kind of flow. So, this is called as a 2D flow and depending on this right-hand side, it would tell us how this point would move. If at a certain x and y, $\dot{x}$ is positive and $\dot{v}$ is also positive and obviously it will, this point will tend to go towards a higher x and a higher v.

If at this point $\dot{x}$ is negative, but $\dot{v}$ is positive; then the motion would be something like this, it would go towards a lower value of x and a higher value of v. So, such kinds of diagrams are called as phase portraits. And they tell us how various points in the 2-dimensional abstract plane, in this case x and v would move in response to this right-hand side. Now, while this information is very useful the reality is; you will solve this to obtain x as a function of time and v as a function of time.

And then you will plot, you will have a time series of x how it will vary and v how it will vary. But here we are not so much bothered about the time it takes, but we are more bothered about the overall behavior of the system and it will be clear through examples. So, essentially, we are parameterizing these time series with the help of this 2-dimensional plot; here the time no longer comes into picture.

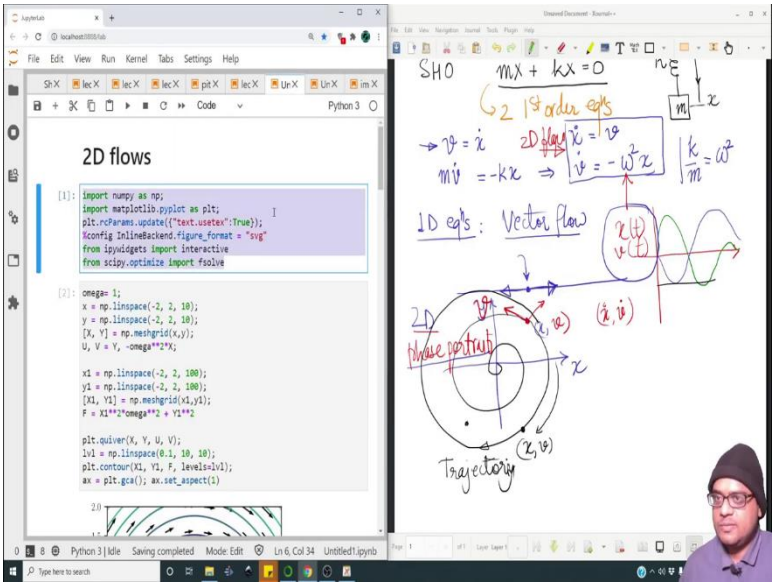
It is just the trajectory in the phase space; it could look something like this or it would look something like this ok, depending on what this right-hand side function is. But we do not know how much time it might have taken for it to reach this phase point.

So, phase point in this 2D plane is a combination of x and v, which the particle might have reached or the particle might have originated as well. So, for all the points on the 2-

dimensional plot, we can sort of find a trajectory. So, these kinds of curves are also called as trajectories.

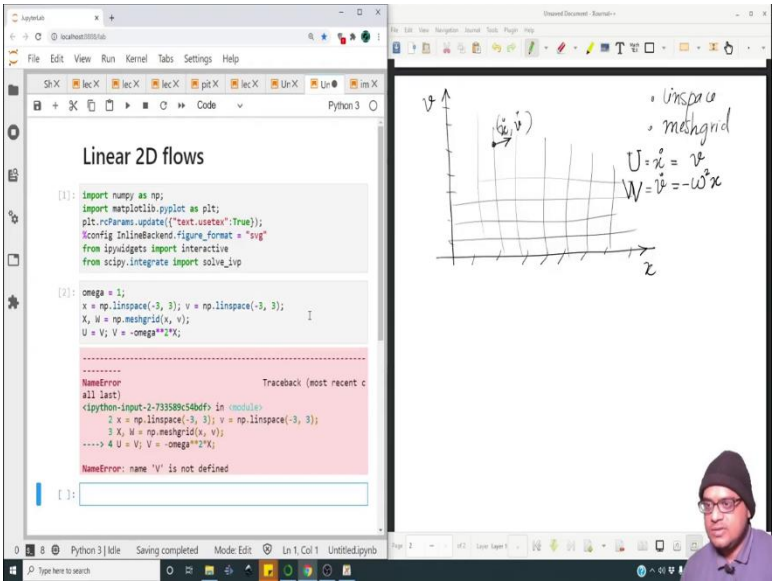
And this trajectory it tells us what phase points the curve passes through in order to reach at some equilibrium or whether it goes on forever; there can be various conditions and we are more interested in how those conditions are rather than time series. Of course, the time series is very important to understand the time taken for a certain motion, ok. So, let us try to first have a qualitative understanding of how this flow would look like.

(Refer Slide Time: 11:23)



In order to do that, we go to our console; let me just copy this, we will require this in our lecture. We do not need solve, so we will get rid of this; but we will eventually need solve ivp. So, from scipy dot integrate import solve underscore ivp. So, let me execute this, alright. So, in order to find out the flow, let us do the following; let us define first the 2-dimensional phase space. So, let me show what I am trying to do over here.

(Refer Slide Time: 12:02)



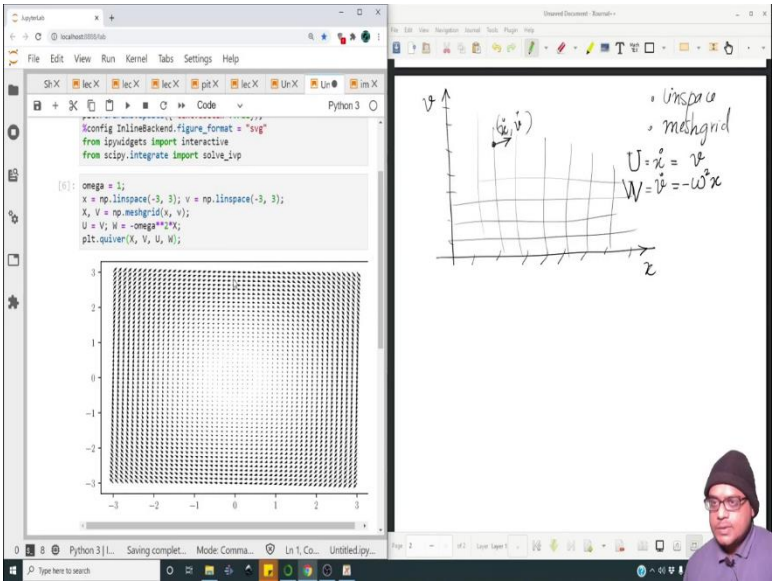
So, essentially, we will choose all the values of *x* and all the values of *v*; we will make a linspace, then we will convert it into a mesh grid, ok. The first point is to make a linspace, then to make a mesh grid; once we have that, for all the points in the *x y x v* plane, we will see how the $\dot{x}$ and $\dot{v}$ look like. So, the $\dot{x}$ and $\dot{v}$ are like the *x* velocity and the *y* velocity for this particular 2D plot. If you recall $\dot{x}$ will be equal to *v* and $\dot{v}$ will be $-\omega^2 x$.

We will we would like to see how the vector points at each point in the domain. So, let us do that. So, let me go over here and say x=np.linspace(-3, 3); v=np.linspace(-3,3); x, w=np.meshgrid(x, v). Now, once we have this, let us define.

So, let me call this as u and v; I mean that is very convenient, let me call this as capital U, let me call this as capital V. So, it is like the velocities in the *x* and *y* directions respectively. So, we will define the function. So, U is simply equal to V, yes U is equal; we have already used V, ok. Let me call this as W, rather this be V; let me call this as W.

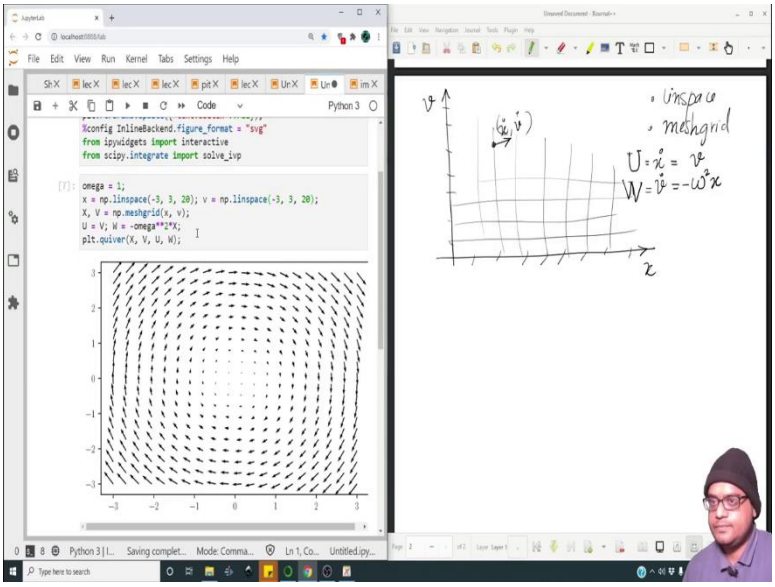
So, I am calling it as U and W; because we have already used the capital V variable. And V will be equal to $-\omega^2 x$, ok. So, these are the definitions, we have to define ω. So, let omega V equal to 1; no problem, alright. Let us execute this and see if there is some error ok; V is undefined, obviously this has to be W, alright.

(Refer Slide Time: 14:39)



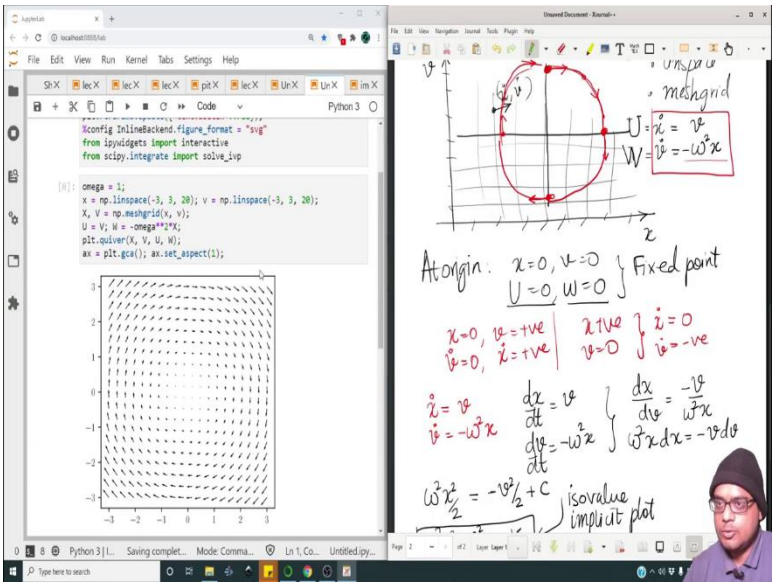
So, now we are in a position to make a plot of how the vectors look like. So, we will say plt.quiver(X, W, U, V). I have messed it all up; as per the convention over here, this should be V, this should be W, ok. So, let us not confuse this anymore; I am using the same convention that I have written in the notepad.

(Refer Slide Time: 15:22)



So, there is too many arrows for us to see what is going on; let me reduce the density of arrows. Let me just take 20 points over here, ok. So, we have plotted in the *x*-*v* plane, how the velocities look like. So, let us look carefully what is going on. So, we have the origin over here, where the arrow lengths are quite small and it means at the origin, the velocity is very close to 0. Let me just set the aspect ratio properly.

(Refer Slide Time: 15:49)



So, ax=plt.gca(). I get current access, ax.set_aspect(1), alright. So, at the origin, what do we have? So, at the origin v; so, x will be 0, v will be 0 and correspondingly U will be 0 and w will be 0. The x velocity is also 0, so the w velocity is also 0 and this is obviously a fixed point. So, the arrow lengths are going to 0 over here. Now, let us look at a positive value of v. So, let this be the origin. So, positive value of v and x equal to 0. So, over here x is 0 and v is positive.

So, when x is 0 and v is positive; the $\dot{v}$ is 0, so x 0 v is positive. So, $\dot{v}$ is equal to 0 and $\dot{u}$ or rather $\dot{x}$ is some positive number. So, the velocity is like this over here; by the same logic when we have v negative, the velocity will be something like this over at this point. So, at this point, x will be positive and v will be 0. So, when I substitute this, this results in $\dot{x}$ equal to 0 and $\dot{v}$ equal to negative, ok.

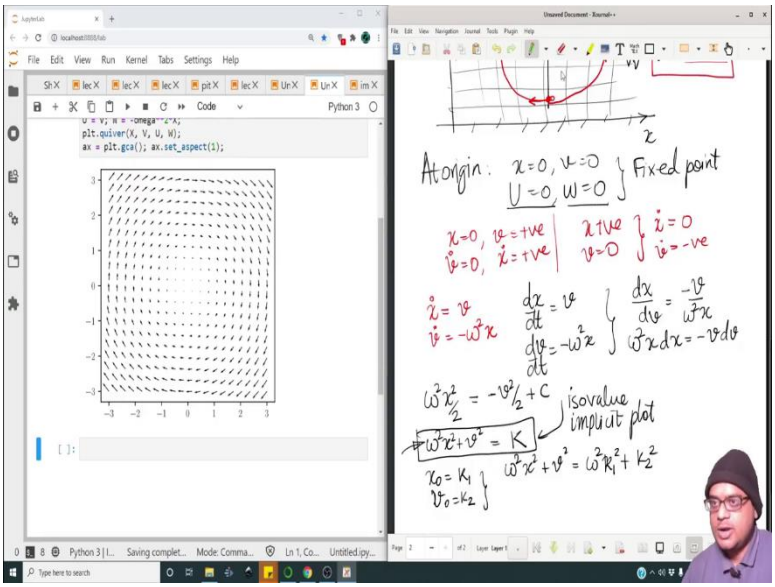
It is $-\omega^2 x$. So, the velocity will look something like this and the same logic something like this. So, we see that, if we had an initial condition over here; the particle would be pushed like this all the way to here, the particle would be pushed like this, it would be pushed like this. So, this is how the particle trajectory is expected to look like.

Now, yeah, I mean you can sort of see that how everything is trying to go in circles, alright. But can we find out that particular curve, ok? How do you find out what this curve is going

to be? Let us see, let us start with this equation. So, we have $\dot{x}$ is equal to v and $\dot{v}$ is equal to $-\omega^2 x$ and this implies, $\frac{dx}{dt}$ is equal to v and $\frac{dv}{dt}$ is equal to $-\omega^2 x$.

Let us divide this. So, we have $\frac{dx}{dv} = -\frac{v}{-\omega^2 x}$ and thus $-\omega^2 x dx = -v dv$. Let us integrate this. So, we have $\frac{-\omega^2 x^2}{2} = -\frac{v^2}{2} + C$.

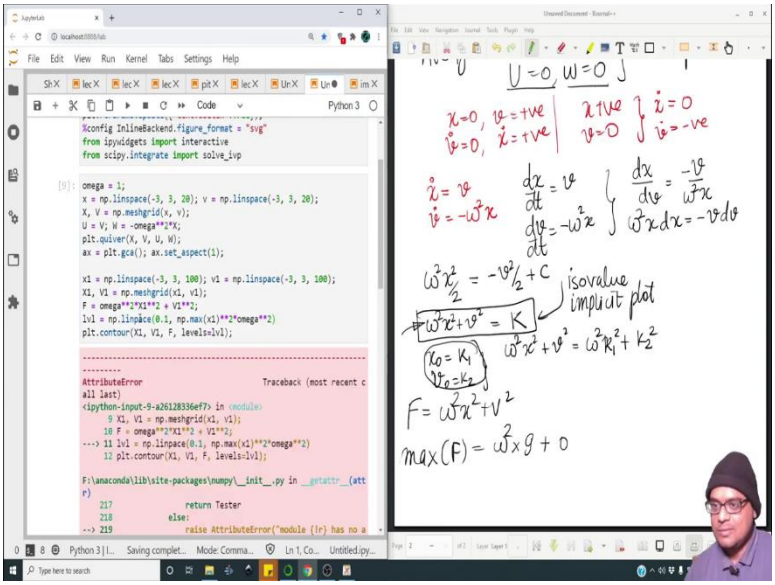
(Refer Slide Time: 19:11)



And actually, we can write this as $\omega^2 x^2 + v^2 = K$. Now, what is the value of the constant? It depends on the initial condition. If we choose $x_0 = k_1$ & $v_0 = k_2$; then we have $\omega^2 x^2 + v^2 = \omega^2 k_1^2 + \omega^2 k_2^2$.

So, depending on what the initial conditions are, we are essentially setting this functions iso value. So, now, it all boils down to finding out the iso values of this particular implicit function, essentially, we have landed up with a situation where we need to do an implicit plot. So, let us go ahead and do that; let me go to this curve over here, this code over here.

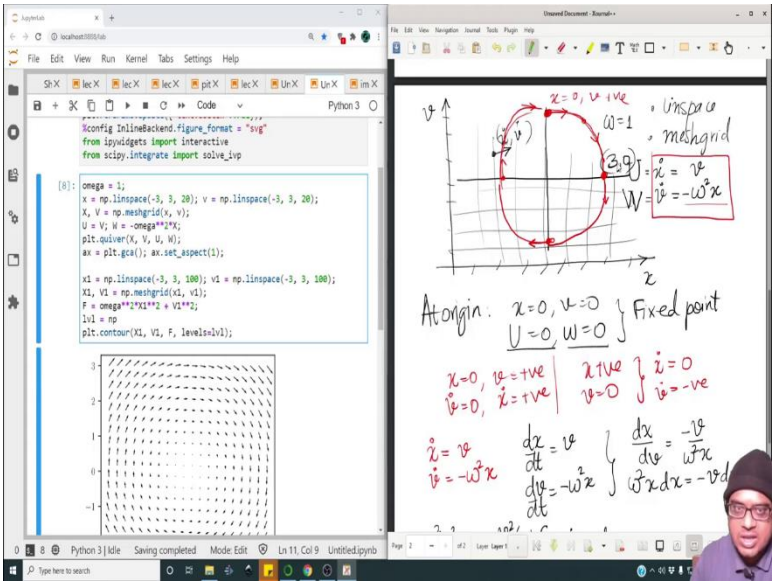
(Refer Slide Time: 20:29)



Let me define x1=np.linspace(-3, 3). Let me take more points, v1=np.linspace(-3, 3, 100); now we will have capital X1, V1=np.meshgrid(x1, v1). Then, we will define a function capital F=omega**2*X1**2+V1**2 and we need to plot. So, essentially our, the function on the left-hand side this is F.

So, F=omega**2*x**2+v**2 you know depending on which iso value of F we plot, we should obtain various contours corresponding each to a different initial condition, alright. So, let me go ahead and plot this. So, plt.contour, we will make a contour in the X1, V1 plane of the function F and levels equal to lvl. Let me define what lvl is. So, it is say we define a bunch of levels starting from 0.1 all the way to. So, the domain is extending from -3 to 3, alright.

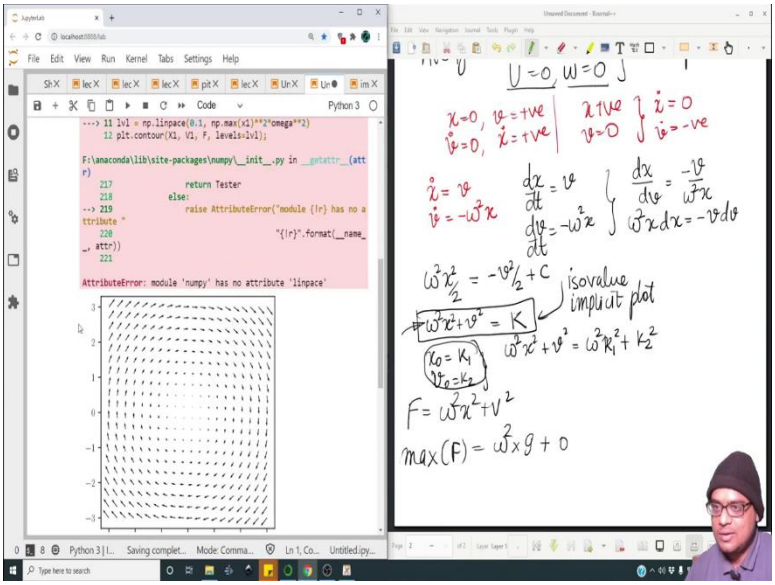
(Refer Slide Time: 22:09)



So, the domain actually extends from -3 to 3. So, this point is going to be (3, 0) and $\omega = 1$ or rather let us see what the maximum value will be. So, it will be $3\omega^2 + 0$, ok. So, the maximum value of this function in the domain that we are plotting in, it will be $9\omega^2 + 0$.

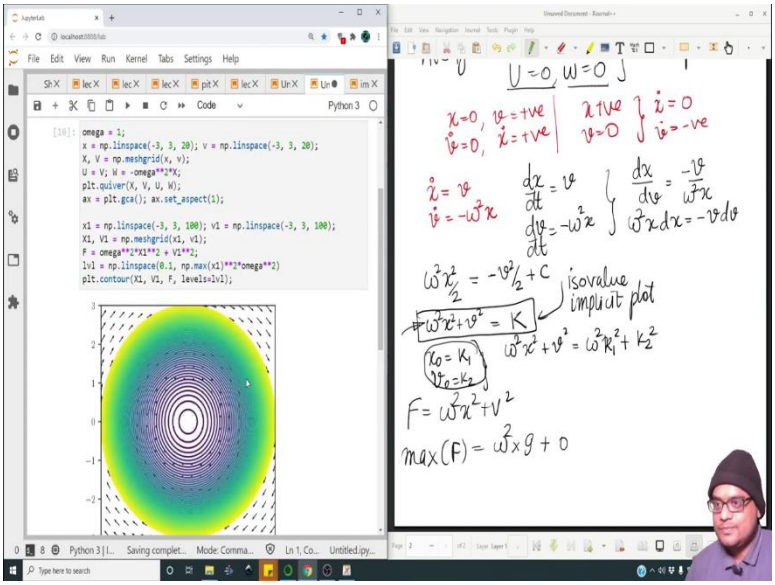
So, let us take levels; so lvl varying from 0.1 to 9; let me not hard code that 9 as well. So, let me just say np.max of x1 squared times omega squared, ok. So, I have not hardcoded anything. Because if I change the domain, I should still be able to plot the good iso levels, ok. So, better not to hardcode anything or rather to minimize hard coding, alright ok.

(Refer Slide Time: 23:20)



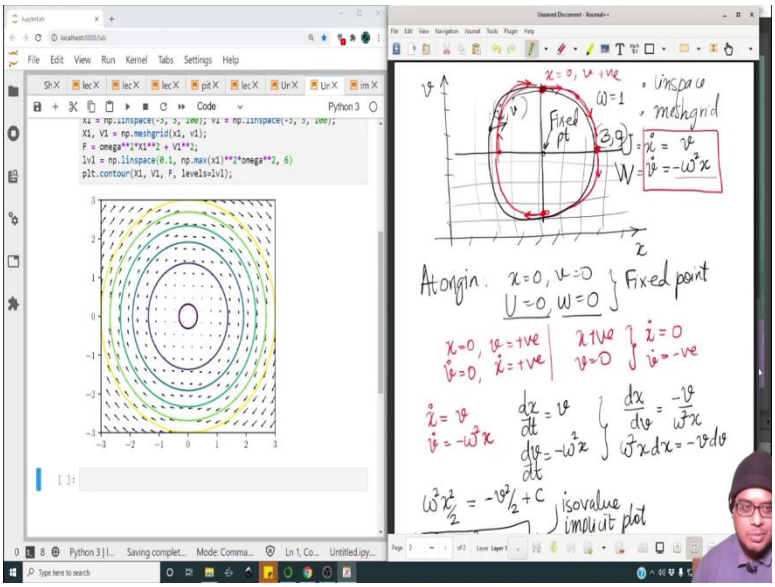
So, let us run this ok, there is an error np has no attribute linspace; where is it? Which line is it? We have written linspace.

(Refer Slide Time: 23:36)



So, it should be linspace ok. So, that is the too many iso values over here.

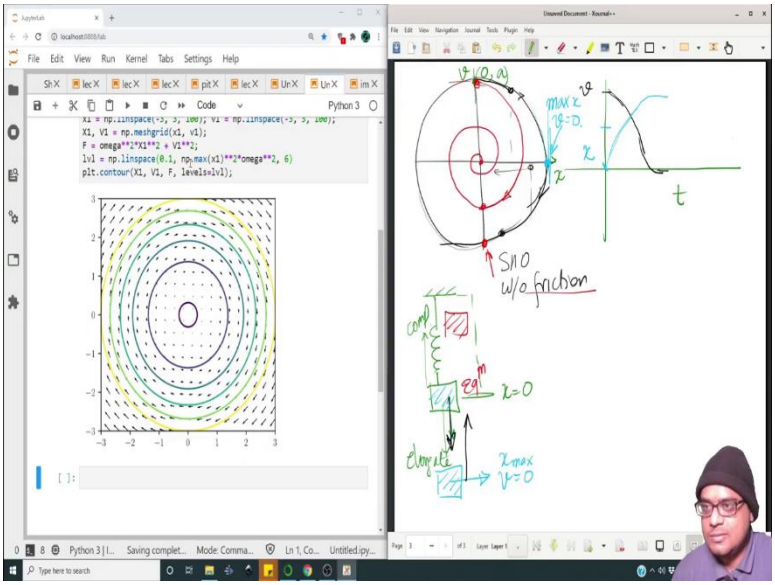
(Refer Slide Time: 23:46)



Let me reduce the number of iso values. So, we will take only say 6 iso values, great. So, these are the iso values, alright. And the arrows actually indicate how the flow is looking like and the closed orbits, ok. So, this particular curve that we had predicted, it is going to

be a closed orbit and this is the fixed point, ok. So, this is the simple harmonic oscillator in the phase space. Anyway, can we interpret this? Can we interpret this plot?

(Refer Slide Time: 24:27)



So, what is going on? We have a closed contour like this and the arrows seem to go like this, ok. So, let me try to interpret it. So, if we start off at the point over here; so, what does this mean? So, this is if I reiterate, this is x, v. So, x value is 0 over here; but v has some positive value. Say that it has a value of a; it means that if I take the spring mass system, it is the equilibrium configuration, ok.

So, x is 0 over here and x can go negative; it will, it means that, the spring will compress and if x becomes positive, it means that the spring will elongate. So, x equal to 0 is like a situation where the spring is not tensed or not compressed. And now a positive value of v means, I am imparting a certain velocity in the positive x direction; I am imparting a velocity in this direction.

So, what will happen? When you impart the impart a velocity, the x value will increase. So, we are moving towards this; so, the x value has increased. So, if we plot the velocity, so we see that the velocity has reduced ok, it is reducing; but if we simultaneously plot the x ok, so its increasing and it will go all the way to this point. At this point what do we have? We have some positive value of x and the velocity becomes 0.

So, this particular point it corresponds to the case, where the block has gone all the way to the end, where the spring is now at its maximum elongation. The mass decelerates and reaches the maximum farthest point at which point the velocity becomes 0; but x is maximum. So, over here we have the maximum value of x and the velocity is 0. So, something like this and that corresponds to 0 velocity, something like this.

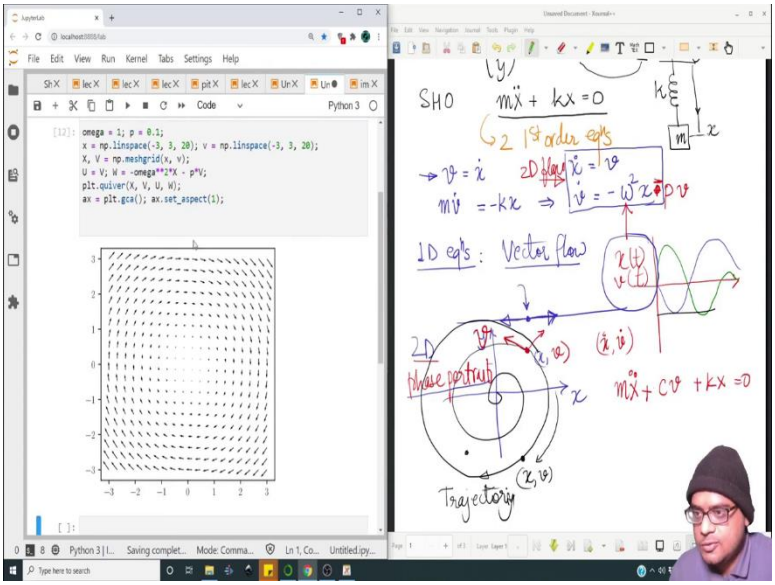
And then we see that x begins to reduce, ok. So, the x is now reducing as we go and the velocity is becoming negative; meaning the block is not traveling in the negative x direction and the velocity is increasing. So, this is how you can justify the presence of such closed trajectories; it goes all the way till it reaches the initial point and this is a simple harmonic oscillator without any friction.

Now, can you tell me what will happen if this particular system were to have friction losses? If we were to have friction losses; what would happen? Now, when the particle would go like this or the when the block would travel like this; at each point, the block would be losing energy. So, once it reaches the other symmetric point. So, what is this point going to be? It is the point where we have maximum compression.

But will we have the same velocity a that we had started with and the answer is no; because due to friction, you are losing energy. So, the velocity that the block will reach at the end at the other point, that at the point of maximum compression it would reduce. So, we would have something like this and it would go like this and eventually you would reach a condition, where the block has no velocity and it is at equilibrium.

So, this is the motion with friction, ok. So, I how, I hope you can appreciate how we can visualize this and it is let us see whether we can make a simple change to account for a friction. How will friction appear in this equation?

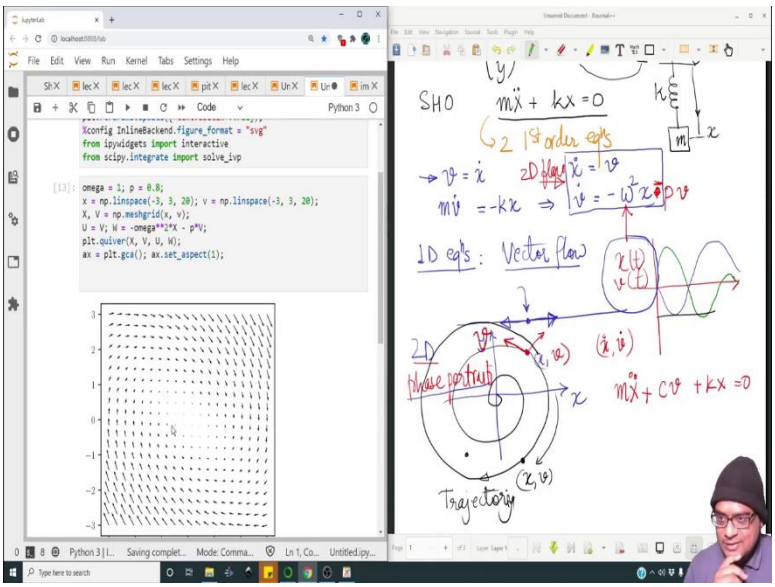
(Refer Slide Time: 28:52)



So, in this equation that we have over here; there will be $m\ddot{X} + cv + kX$ ok, this will be equal to 0. So, $\dot{X}$ will be still equal to v and $\dot{v}$ will be $-\omega^2 x$ plus some new constants say, C; let us call the new constant also, let me call it p to avoid any conflict plus, or rather it would be $-pv$, ok.

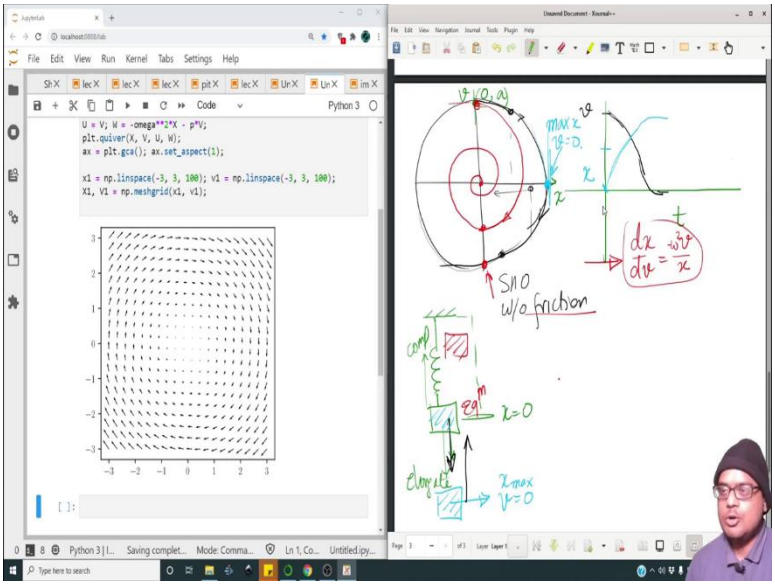
So, let us make this particular modification, ok. So, W will be $-pV$; let me define $p=0.1$ and let me remove this contour plot, because that integration will not be exact. So, let me remove it ok, let me remove it. So, let me run this and let us see how the arrows look. Let me increase the value of p something like 0.8.

(Refer Slide Time: 30:16)



Now, look how it starts to look more like a spiral ok; it looks more like a spiral and it is slowly losing energy and it will go to the origin.

(Refer Slide Time: 30:32)



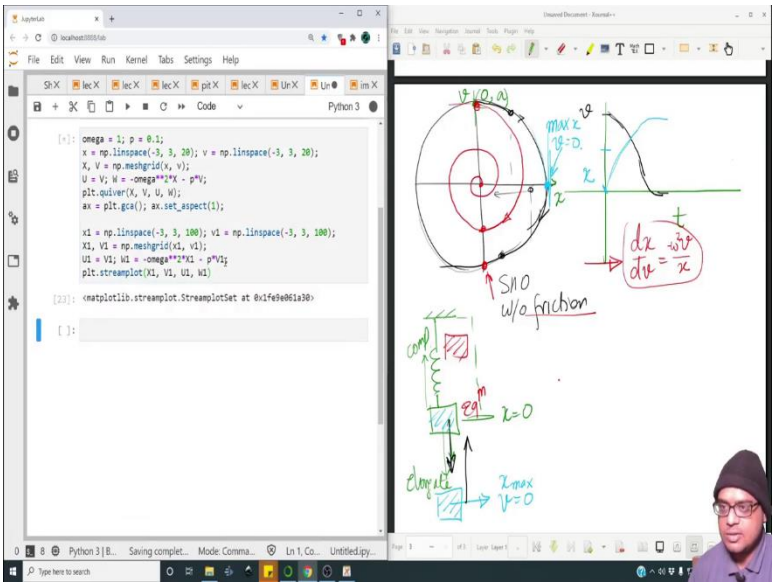
While we have obtained the analytical expression for the closed contour in the previous case by integrating, by dividing the two equations and integrating by parameterizing time rather; we can also make use of the streamline plot in order to plot this particular trajectory for the dissipative case, ok.

When there is friction, we do not have an analytical solution per say; but even if we do not have an analytical solution, we can use this particular vector field to integrate numerically and find out how the trajectory will look like. So, what is happening?

When we have done with this particular thing; so $\frac{dv}{dx} = -\frac{-\omega^2 v}{x}$, this is tantamount to having an initial point somewhere and then integrating it and seeing how it goes, ok. This is what we are essentially doing and that we found out analytically; but instead of doing it analytically, we can also do it numerically.

And Numpy I mean even gnu plot MATLAB whatever; when we draw streamline, plotting a streamline is trend amount to using that velocity field and finding out the trajectory in that velocity field. Even if the velocity field is time dependent; you freeze the velocity field and find out the trajectory on that frozen velocity field.

(Refer Slide Time: 32:03)

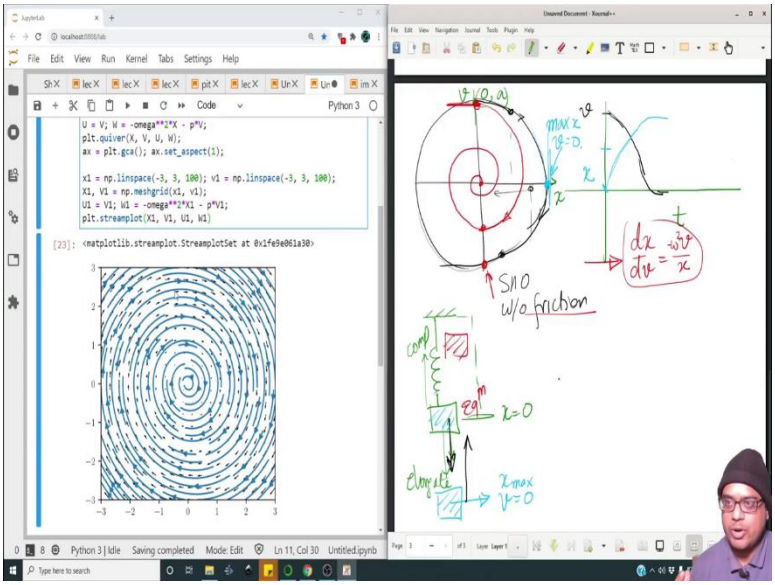


So, now I brought back x1, v1; let me copy the expression with friction over here. And let me substitute this by V1; this will be X1 and this will be V1. So, the reason why I am having two different grids is, I want to have the vector field with few points plotted; if I plot, if I do not plot it with 20 points, if I plot it with 100 points, there is too many vector fields all over the place and we cannot discern anything useful.

So, I have kept the number of points on which the vector field is to be plotted quite low. On the other hand, things like the iso contour or the iso value or the contour or this particular streamline that we were going to plot; I want it to be on a refined mesh, so that the curves appear to be smooth that is the only reason why I am using two different sets of meshes, ok.

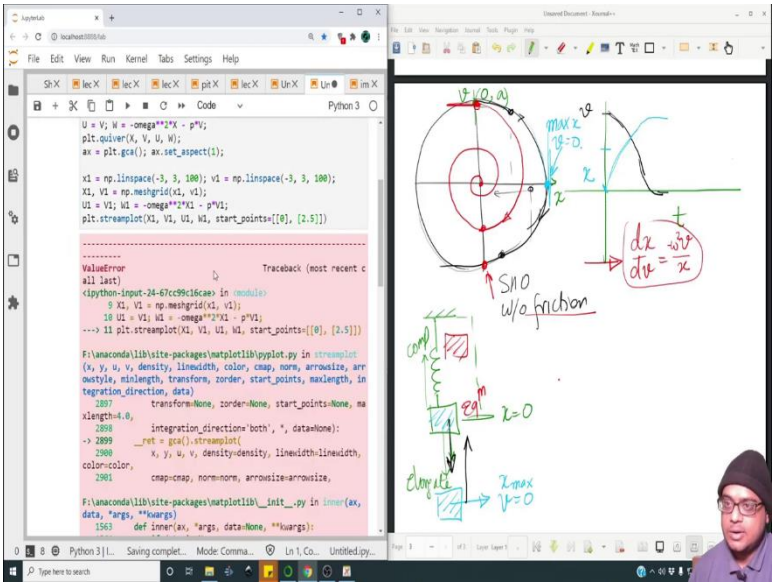
There is no other reason, I could have very well used a same mesh; it would have made, it would have made things very nasty, nothing else, ok. So, when I define U1 as V1 and W1 as this; I can now call plt.streamplot(X1, V1, U1, W1). So, I will pass the velocities to it.

(Refer Slide Time: 33:23)



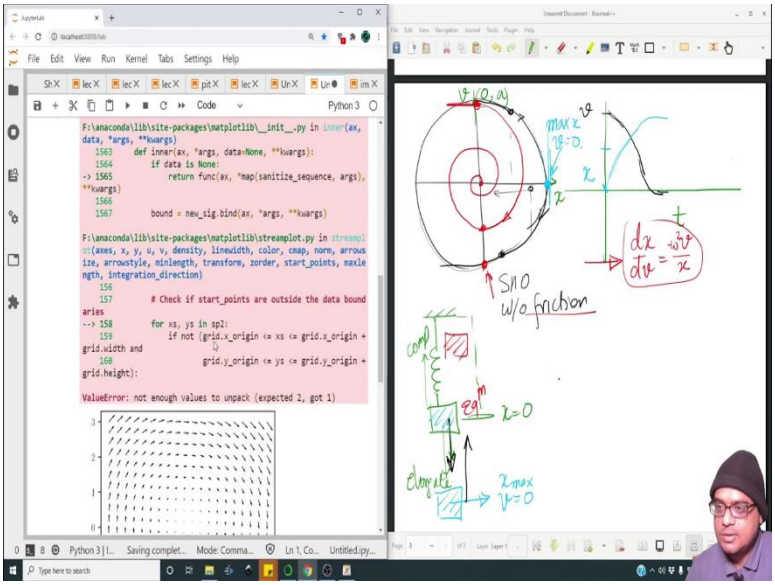
In fact, let us just run this much and see what happens ok. So, it is giving us a bunch of streamlines, very broken stream lines; because it is used it is plotting it using a uniform density. But we do not want that; we want the streamline or the trajectory to start from this particular specified point.

(Refer Slide Time: 33:41)



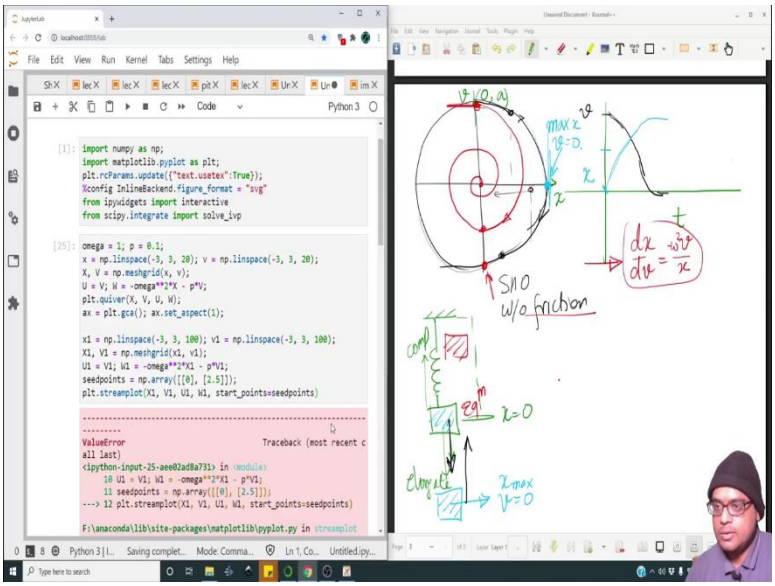
So, we will specify start points equal to. So, this will be two things. So, we will start with x as 0 and y at say 2.5.

(Refer Slide Time: 33:59)



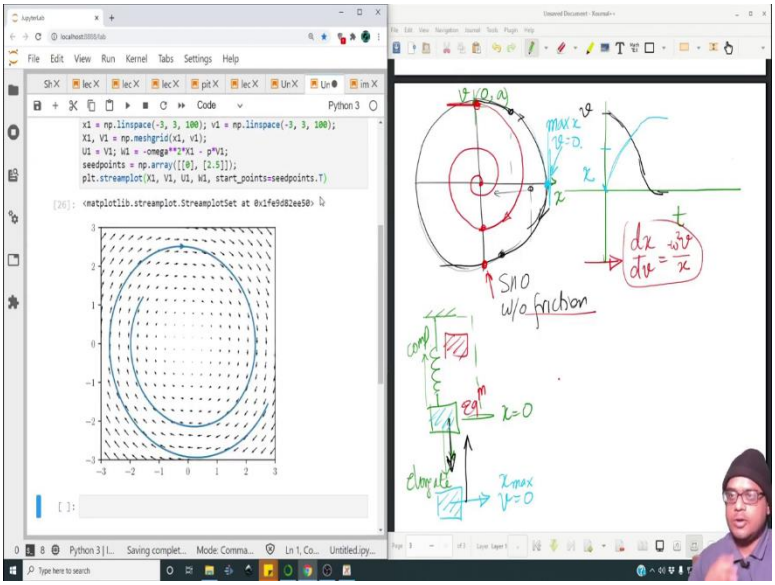
Let us run this, three is an error, gotten ok. So, I must declare this as a Numpy array.

(Refer Slide Time: 34:08)



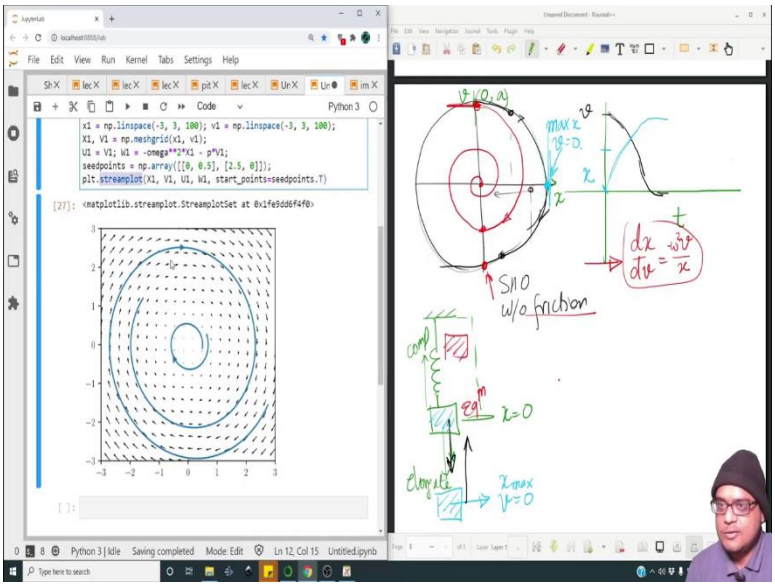
So, seed points equal to np dot array. So, there has to be two sets of points; one will be 0 and 1 will be 2.5. Let me then pass seed points to it; there is still an error, ok.

(Refer Slide Time: 34:32)



So, we must pass it as; I mean these are some idiosyncrasies of Python again.

(Refer Slide Time: 34:41)

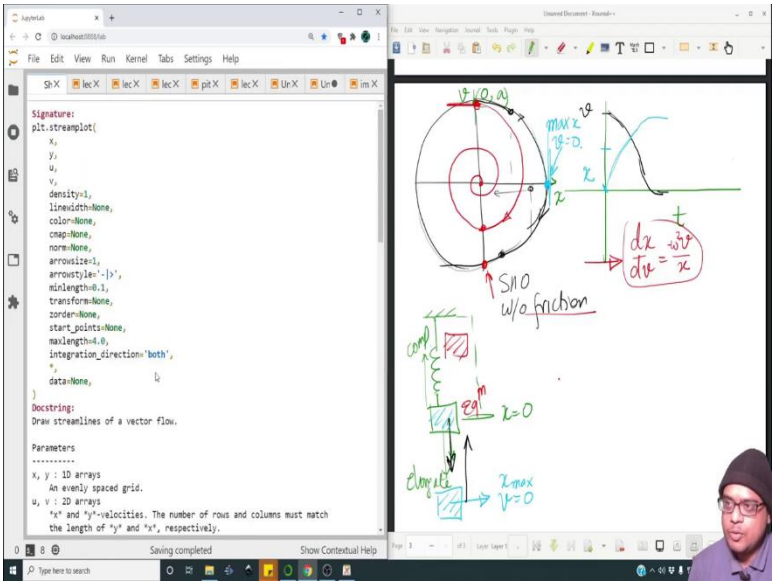


Even if we create different initial points; so, suppose there is one initial point at 0.5 and the velocity is 0, you must pass the transpose of that array. These are just small things that Python requires you to do. Obviously, the syntax will be different for the case of gnu Python, gnu octave, ok.

So, you have to pass the transpose of it; otherwise, it is not able to figure out what you have passed ok, it is quite simple. So, but now look what is going on; we have given this

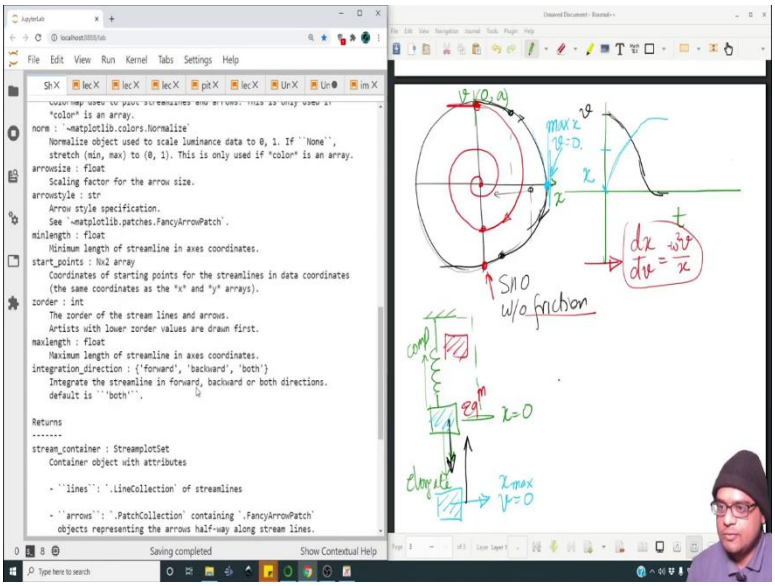
as the initial point, but it is giving us the forward contour and the backward contour and the reason is obvious.

(Refer Slide Time: 35:14)



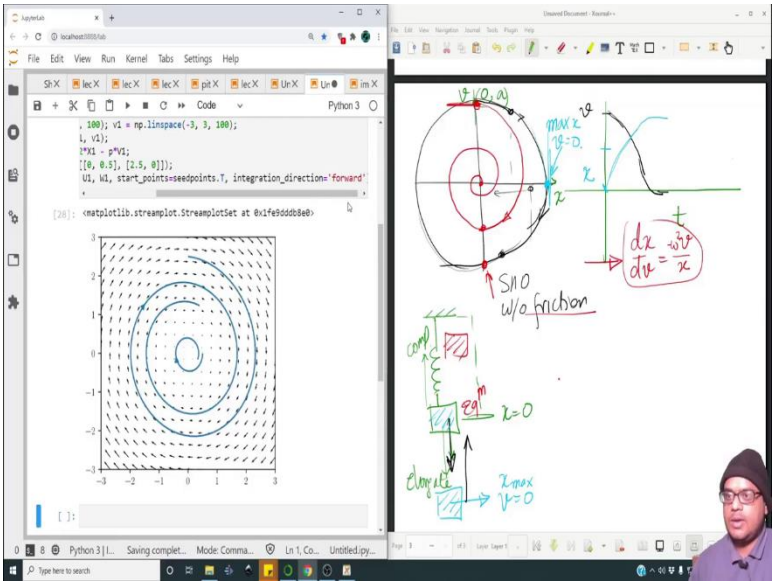
Let us go to the contextual help or stream plot; there is something called as integration direction, ok.

(Refer Slide Time: 35:20)



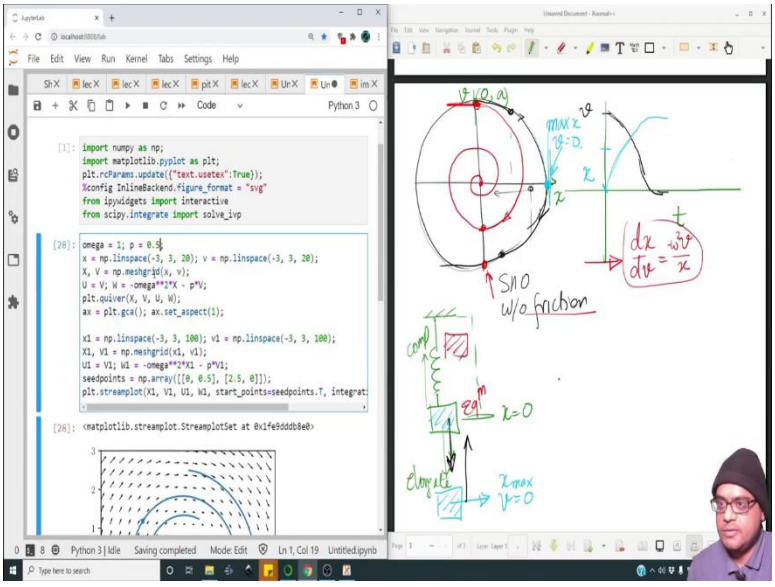
It says both; but integration direction we want only in the forward direction, ok.

(Refer Slide Time: 35:27)

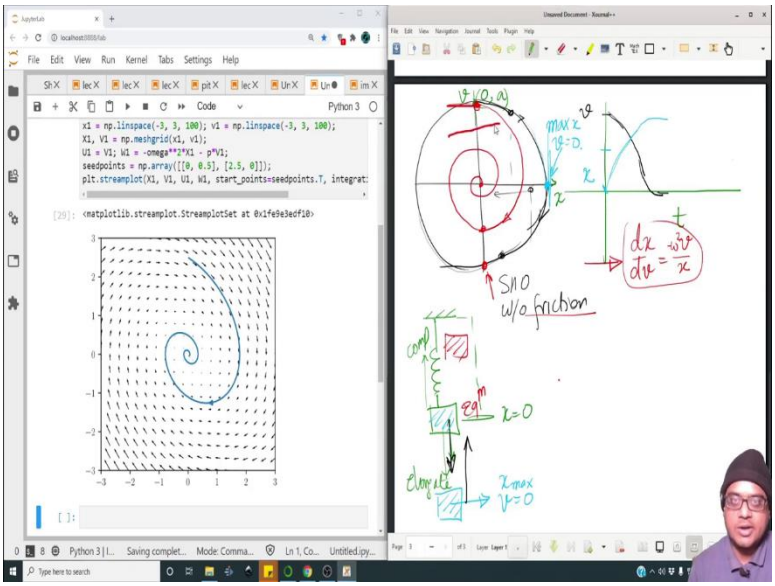


So, let us go over here and change the integration direction to only forward and this gives us the spiral, ok.

(Refer Slide Time: 35:44)

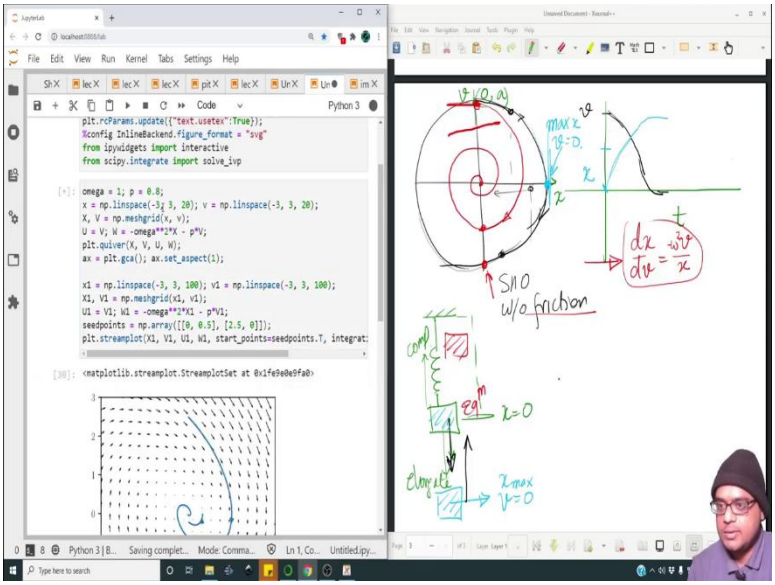


(Refer Slide Time: 35:45)

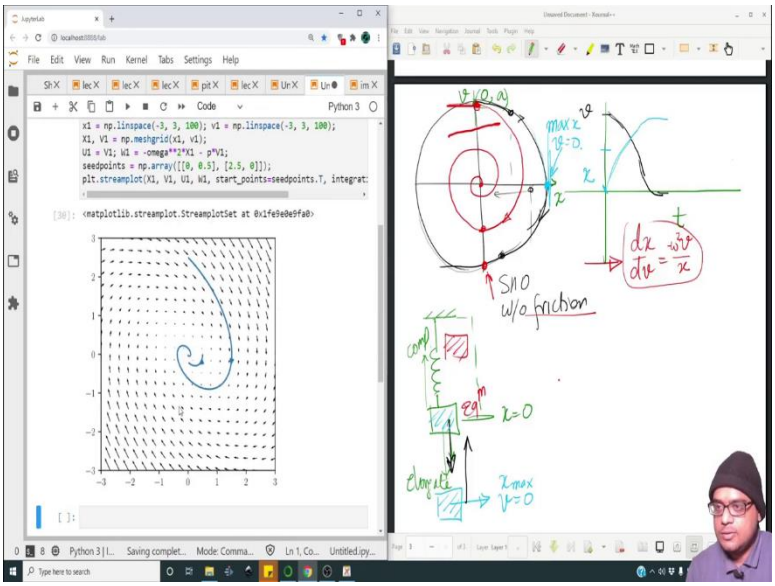


Let me now increase the parameter which controls dissipation to 0.5, let us see, ok. So, the thing quickly fizzles down to 0 as we had predicted, ok.

(Refer Slide Time: 35:56)

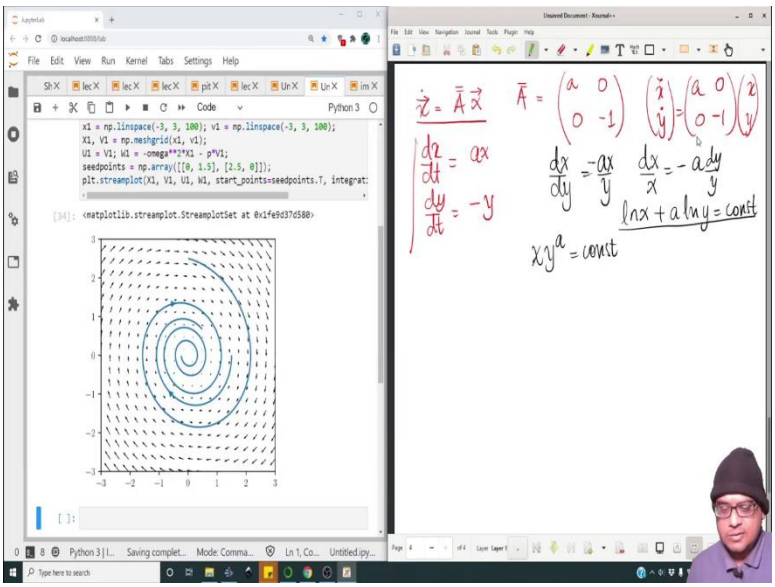


(Refer Slide Time: 35:57)



Let me make it 0.8, so that two contours go to 0, ok. So, this is how you can plot a streamline and how you can visualize the vector flow for simple cases.

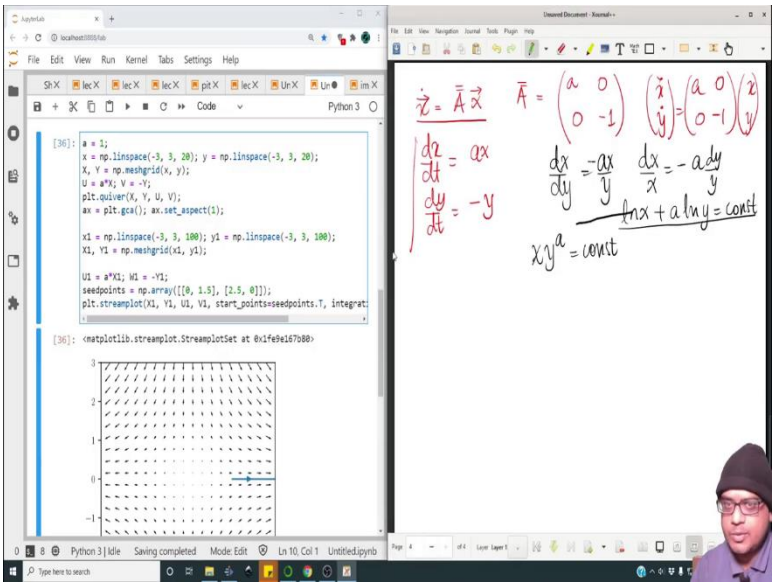
(Refer Slide Time: 36:10)



So, now, let us move ahead to find out the behavior of a prototypical linear system. So, $\dot{x} = Ax$, where A is given as a, (0, 0, -1), ok. So, let us look at how in this particular system would look like over a matrix, which looks something like this. So, when we have this, we can write $\frac{dx}{dt} = ax$ and $\frac{dy}{dt} = -y$; this is because $\dot{x}$, $\dot{y}$ will be equal to (a, 0, 0), (-1, x, y).

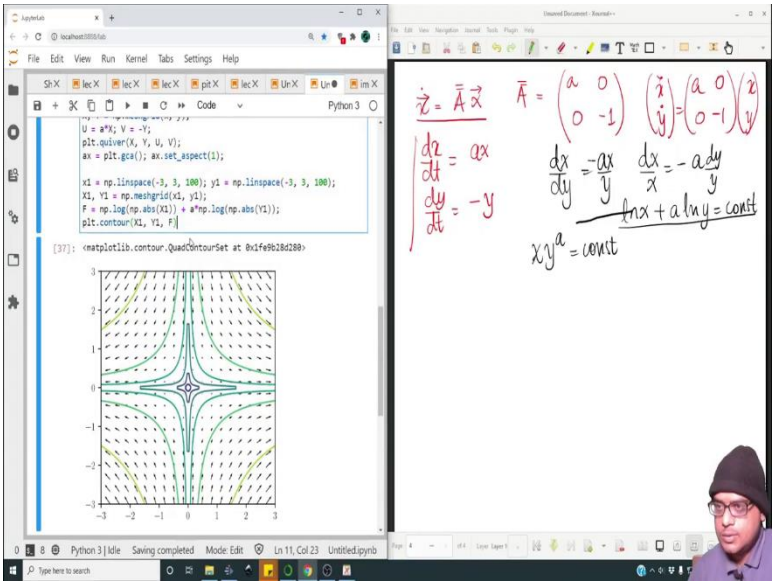
So, once you do the multiplication, you will obtain this. So, with this; we can divide these things and we can write $\frac{dx}{dy} = ax$ or rather $-\frac{ax}{y}$. And so, this becomes $\frac{dx}{x} = -a\frac{dy}{y} and\ lnx + alny = const$; which means $xy^a = constant$ or alternately we can simply do a plot of this as well, do an implicit plus of this.

(Refer Slide Time: 37:55)



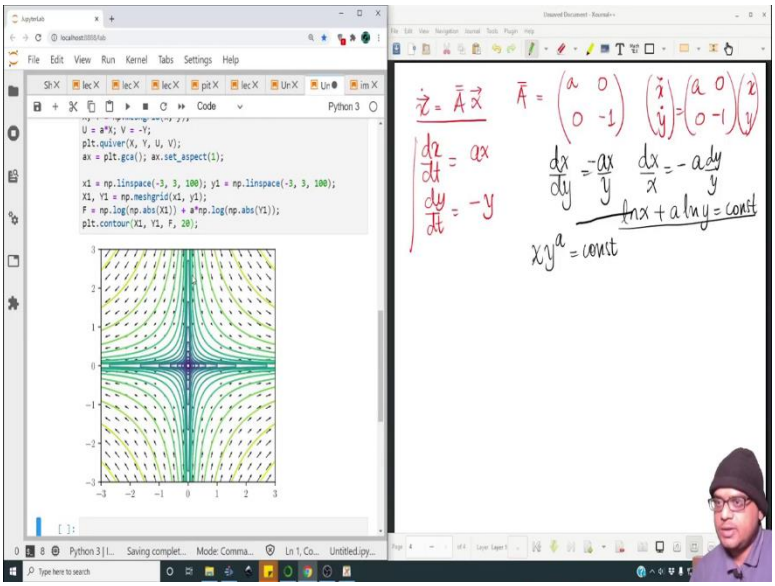
So, let me copy this particular code over here, let me go over here and write down. So, we do not need this, we will simply require a; we can put y over here, ok. So, U equal to aX and V will be equal to -Y X, Y, U, V and this will be also X1, Y1, this will be y1. But instead of having the stream plot, we can make use of the iso function that we have obtained over here.

(Refer Slide Time: 38:47)



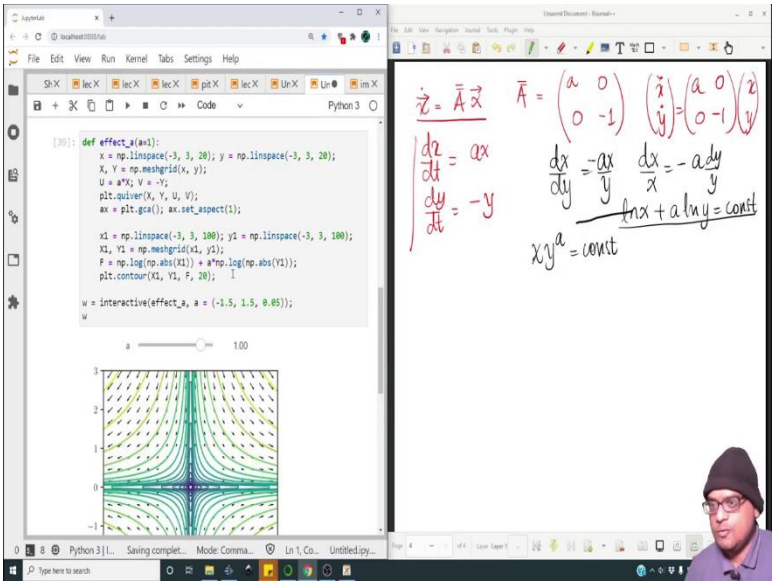
So, you can define F=np.log(np.abs(X1)) and the reason why I use absolute is, because we are taking a log+a np.log of np.abs of Y, ok. We do not need this; we do not need c points. So, we will do simply plt.contour(X1, Y1, F, 20). So, this has to be X1.

(Refer Slide Time: 39:27)



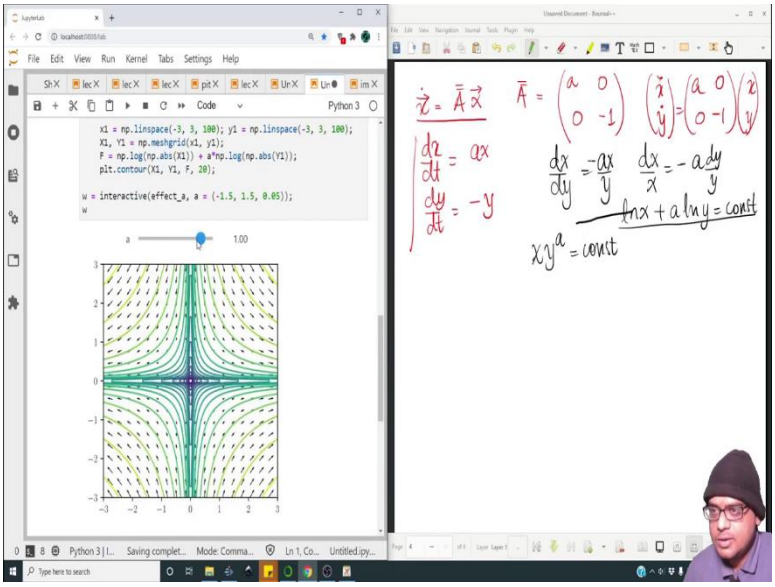
Let me change the number of contours to 20. So, when we want to study this equation, this particular equation for different values of a, let us keep a as a variable.

(Refer Slide Time: 39:44)

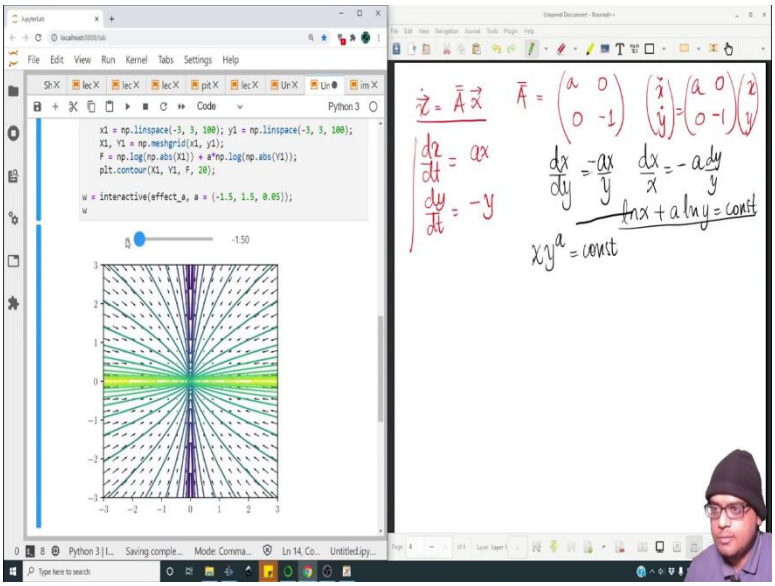


So, let us define a wrapping function, because we want to make it interactive. So, define effect of a and the input will be a; let the default value be equal to 1, ok. We have to put everything over here inside the function, w equal to interactive effect of a. A will go from minus 1.5 to 1.5 in steps of 0.05; then we have to display the widget as well, ok.

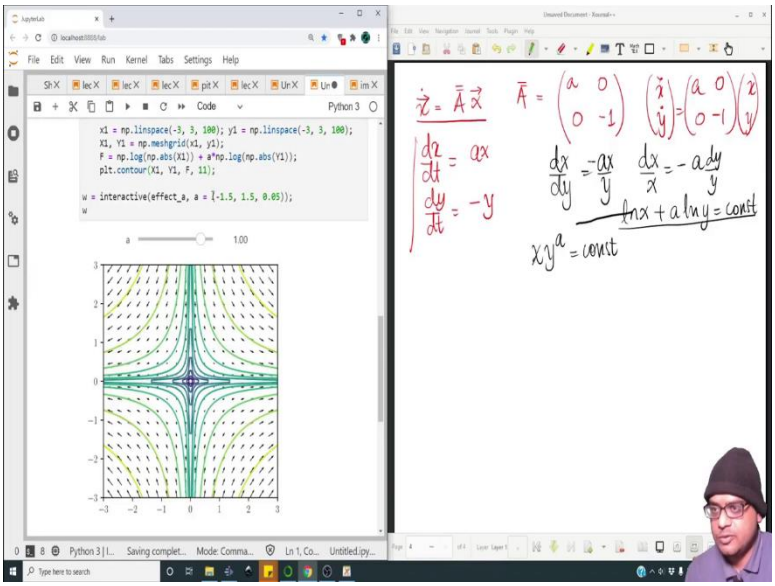
(Refer Slide Time: 40:23)



(Refer Slide Time: 40:28)

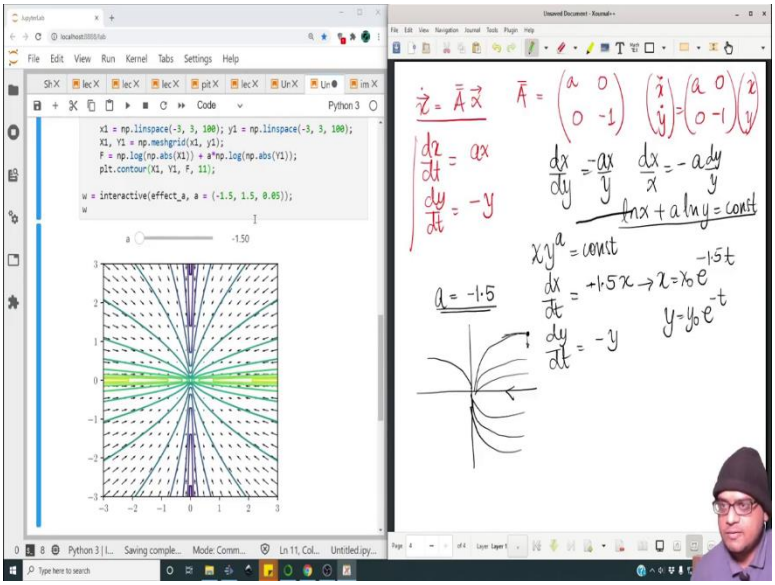


(Refer Slide Time: 40:33)



So, let me go to a value of minus 1.5; infinitesimal decrease the number of contours.

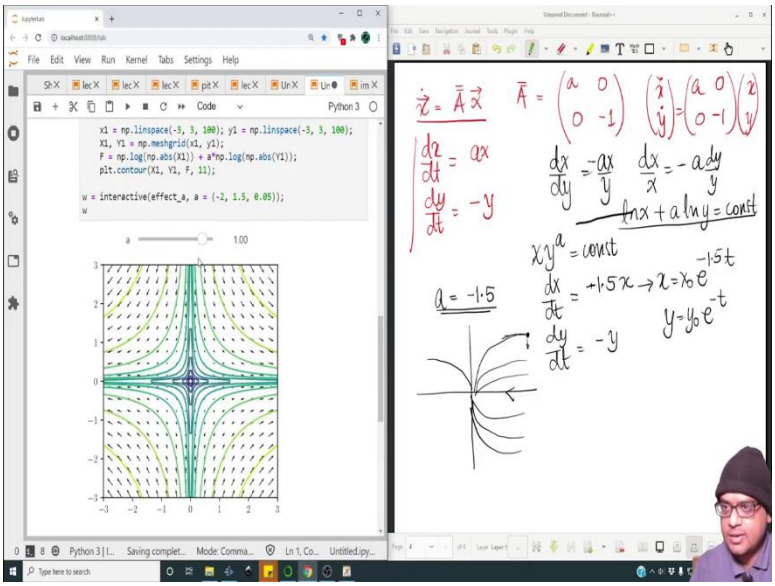
(Refer Slide Time: 40:35)



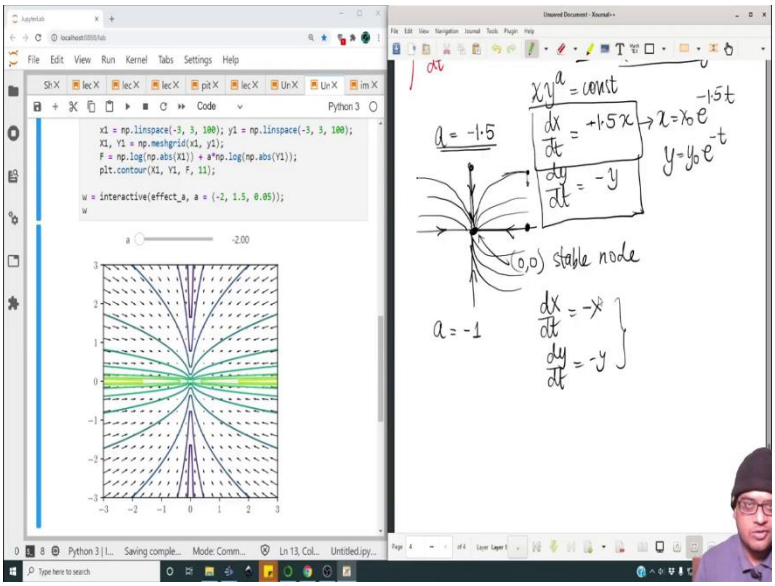
So, let us see what happens when a equal to minus 1.5, ok. So, a is minus 1.5. Let us see what the plot looks like. Let us look at this equation. So, $\frac{dx}{dt} = -1.5x$ and $\frac{dy}{dt}$ is going to be -y. So, this means that, $x = x_0 e^{-1.5t}$ this implies $y = y_0 e^{-t}$, so x decays faster than y, ok.

So, x decaying faster than y; means if I have an initial point over here, the decay in the x-direction will be faster than the decay in the y-direction, ok. So, it will look something like this, ok. So, that is why we have the curves which look like this.

(Refer Slide Time: 41:35)



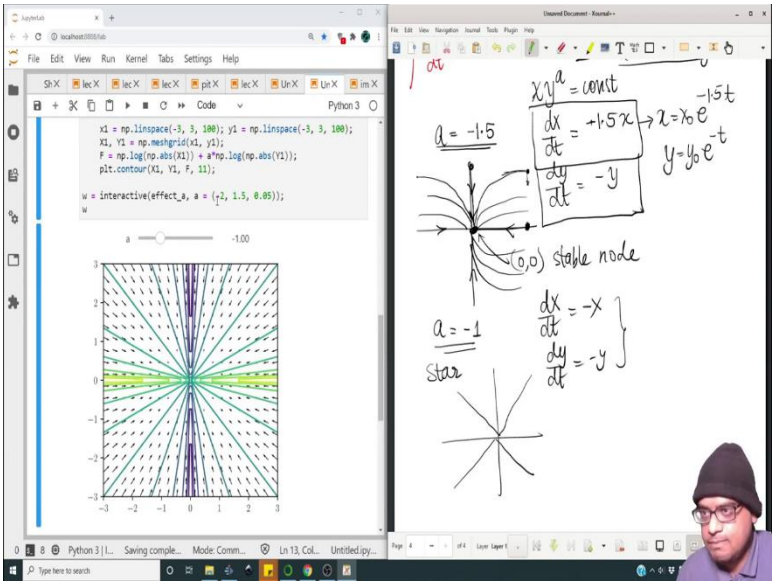
(Refer Slide Time: 41:38)



In fact, let me make this -2 to make it more emphatic ok; this makes it more emphatic; it is something like this, ok. So, the y-axis also attracts and the x-axis also attracts. So, if we choose a point on the x axis that is y equal to 0; the equation will be simply this and the point will go towards the origin.

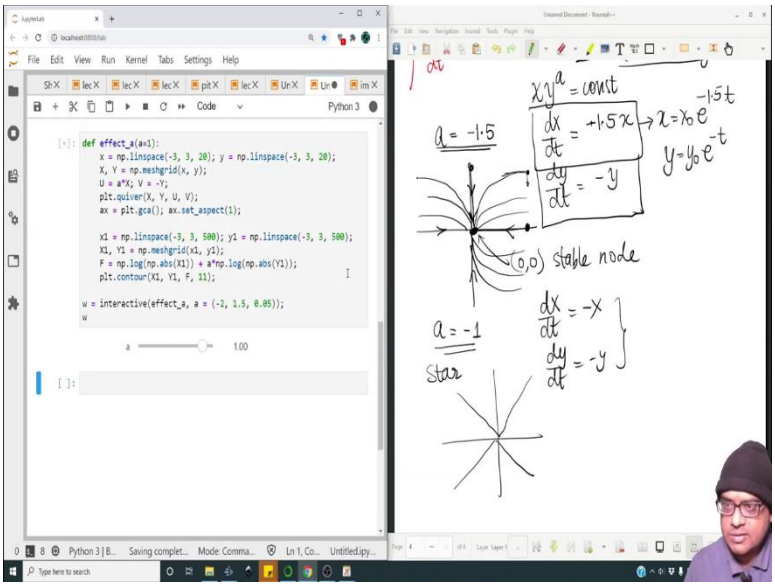
If we choose simply the y axis that is x equal to 0 and the point will also be attracted towards the origin; therefore, the fixed point in this system that is origin is called as a stable node. So, over a less than -1, it will be a stable node. Let us in fact now go to the case, where a is -1. So, what do we expect? $\frac{dx}{dt} = -x$ and $\frac{dy}{dt} = -y$. So, both the x and y are sort of attracted towards the origin and the same rate, ok.

(Refer Slide Time: 42:45)

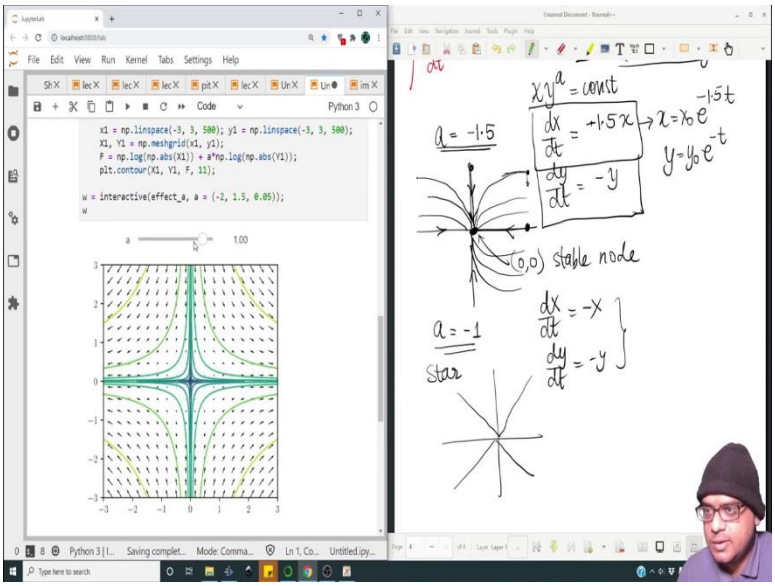


Start -1, so all the lines are coming towards the origin and this particular case is called as a star; it means that, all the points are coming towards the origin. The reason why you have these artifacts is because of the discretization of the mesh, nothing else, ok.

(Refer Slide Time: 43:14)

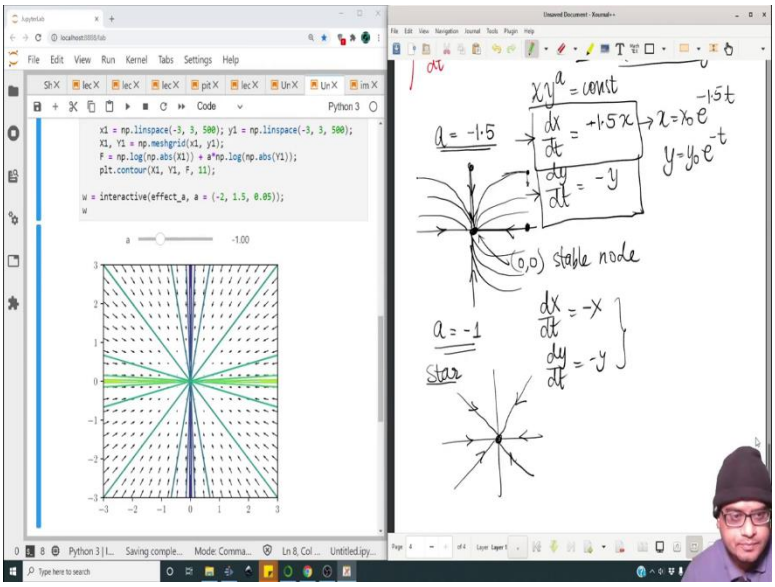


(Refer Slide Time: 43:17)



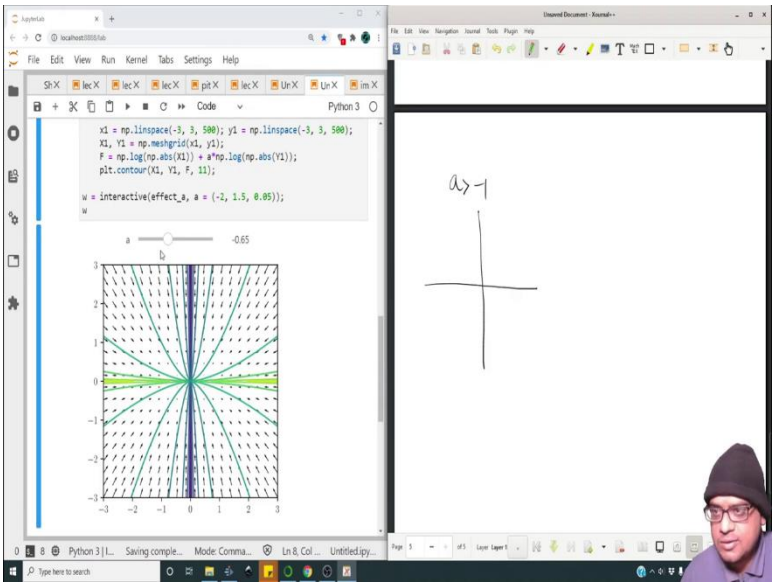
So, let me increase this to 500; let us see if that makes things is better. So, that is.

(Refer Slide Time: 43:26)



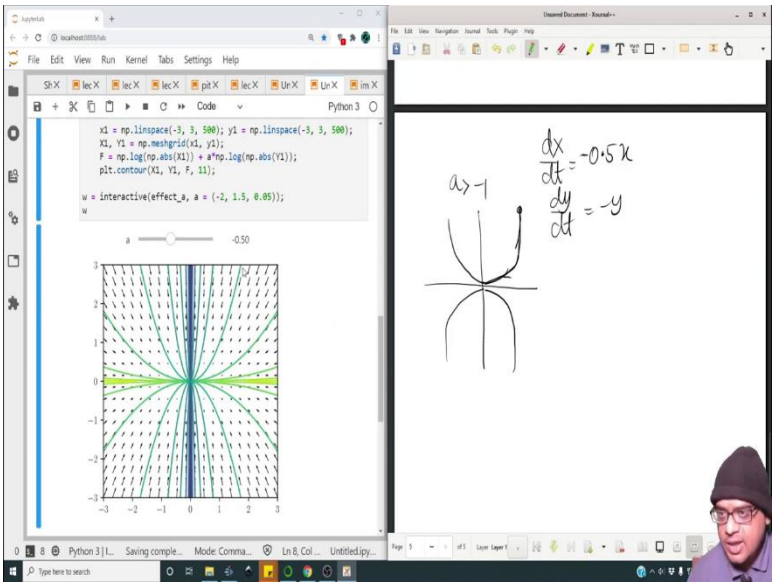
That makes things slightly more aesthetically good; I mean you can change all this by selecting certain levels of contours, but I am not going to do that right now, you can do it when you have time. So, this becomes a star; this means all the points in space are attracted towards the origin with an equal magnitude. So, *x* is also reducing, *y* is also reducing; over here *x* was reducing faster than *y*, but here it is all attracting. So, that the fixed point is called as a star; it is also a stable fixed point.

(Refer Slide Time: 44:06)



Now, what about a greater than -1, ok? See how the curves are bending.

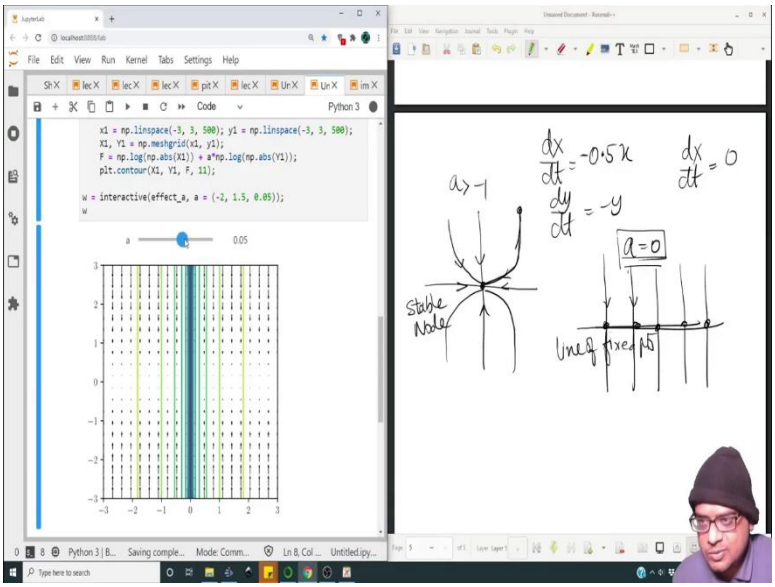
(Refer Slide Time: 44:21)



So, when a is greater than 0.1, say a is 0.5; then $\frac{dx}{dt} = 0.5x$, well $\frac{dy}{dt} = -y$. So, x is attracting slower than y. So, the trajectories are faster in the y-direction and then slower at the x-direction. So, the trajectories travel towards the x-axis faster than in the y-axis.

So, hence the trajectories look something like this ok; it is moving faster along the y-direction, it is just a swap of the initial case where a was less than -1, ok. So, the arrows also indicate the direction of the motion. Now, let us see what happens at a equal to 0.

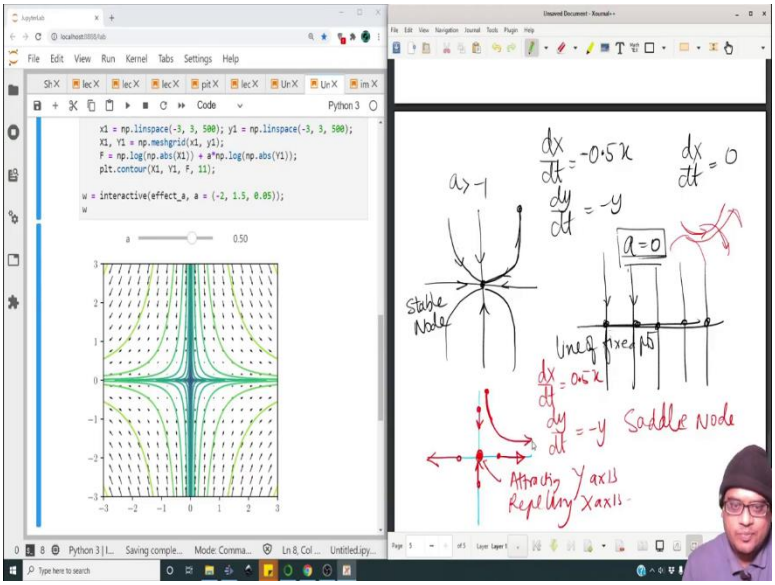
(Refer Slide Time: 45:16)



So, at a equal to 0, all the vectors are pointing towards the x-axis. So, a equal to 0, ok. So, before that, this is also a stable point ok; all the points are being attracted towards the origin. So, this is also a stable node or an attracting node whatever you want to call it. At a equal to 0, this thing becomes 0 and so $\frac{dx}{dt} = 0$, hence x is a constant. But divided is still -1; so, y things the points along this axis they converge towards x and so the x axis essentially becomes a whole line of attracting nodes, ok.

So, this particular case is called as it is like a marginal case; you have only one point and then suddenly you have an infinite number of stable points. So, it is like a marginal case and so these, this line is called as a; I mean it is a line of stable points, ok. It is a line of fixed points, stable points ok; it is not a stable point, it is a line of fixed points.

(Refer Slide Time: 46:37)

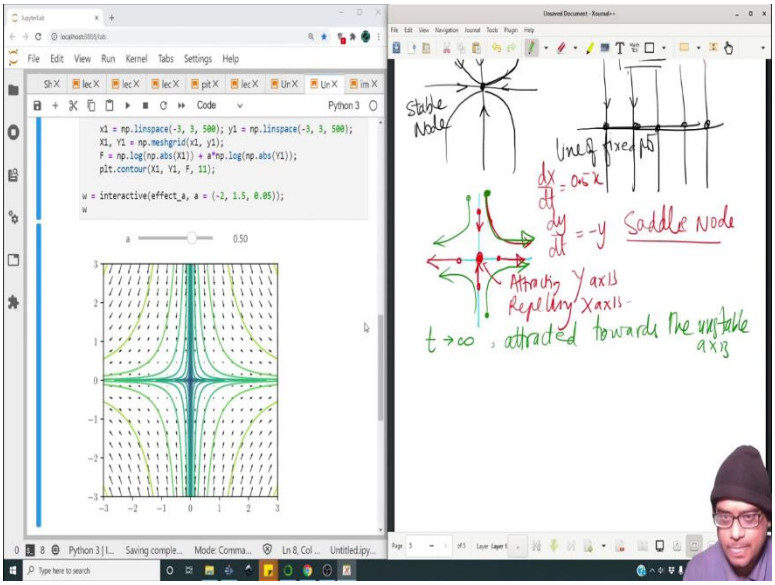


Now, when a group becomes larger than 0, see what happens; $\frac{dx}{dt}$ is some say $0.5x$ and $\frac{dy}{dt} = -y$. So, x is growing exponential in time. So, if I have a point on the x-axis, it is growing exponential in time. So, it is going away from the origin; but a point on the y-axis is nevertheless attracted towards the origin, ok. So, this particular fixed point it is attracting in the y axis, but repelling in the x-axis.

So, this kind of a fixed point is called as a saddle node; because the saddle I mean that you put on a horse, it has a curvature like this in one direction and a reverse curvature in the other direction. So, it is attracting in one direction and repelling in the other direction.

That is why it is called as the saddle node ok, something like this. So, its attracting in this direction, but repelling in this direction, alright. So, you just imagine how saddle of a horse looks like, ok. So, the points when they start over here, they are attracted towards in the direction of y towards the origin and they repel like this, ok. So, it is called a saddle node.

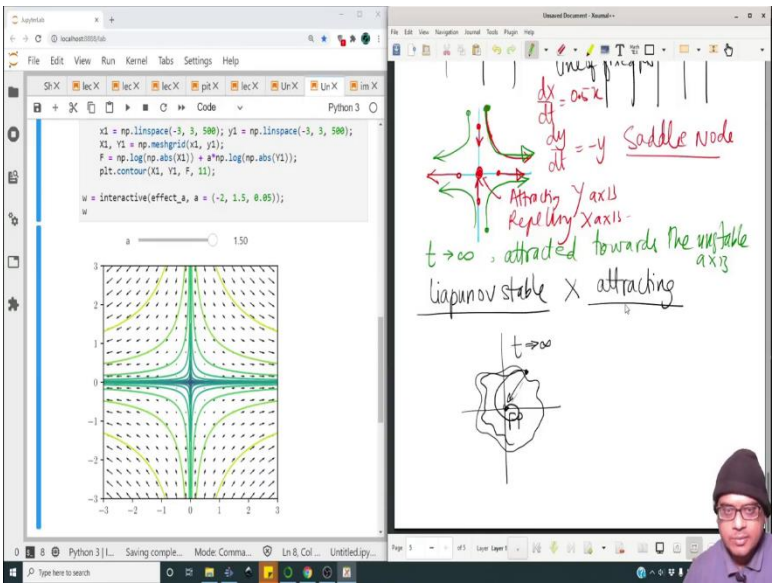
(Refer Slide Time: 48:05)



And the important thing is for t tends to infinity, the point is attracted towards the unstable axis. I mean you would imagine that at time goes to infinity, it would be attracted towards a stable orbit; but no, toward a stable axis, but no. Here it is attracted initially towards the origin in the y direction; but then once it becomes close to y equal to 0, it gets pulled away to infinity along the x-axis, so, ok.

So, it attracted towards the unstable axis; it sounds very counterintuitive, but this is what it is. Here points go like this, points go like this and points go like this ok; this you can see over here as well, ok. So, this is what happens as a keep on growing, it is the same thing.

(Refer Slide Time: 49:01)



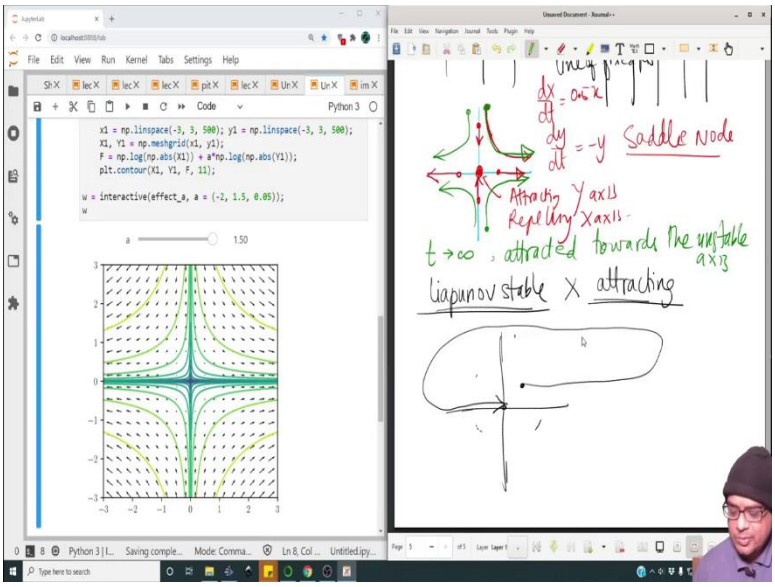
So, as the value of a has changed, the entire behavior of the equations has changed from a stable node to a line of fixed points to something which repels and obviously, this has to do something with the eigenvalues, ok.

So, for this particular system in which only diagonal terms are present, this is one eigenvalue and this is one eigenvalue. This is the x eigenvalue, this is the y eigenvalue, ok. So, we can interpret it like that also and we will do it after this; but before that just a bit of terminology, there is something which is called as a Lyapunov stable orbit.

And the meaning of Lyapunov stable means, if we have a point which originates, which is away at a certain distance from a fixed point. So, if the point remains in the vicinity of the fixed point for all times greater than 0; I mean for all large times, so t tends to infinity if the point remains bounded.

So, it does not have to actually come back to itself. So, if the point remains bounded like this, then we say that the fixed point is Lyapunov or stable; if the point is attracted towards the fixed point, so it goes to the fixed point, then the point is said to be attracting. So, Lyapunov stable is not the same as an attracting point, ok.

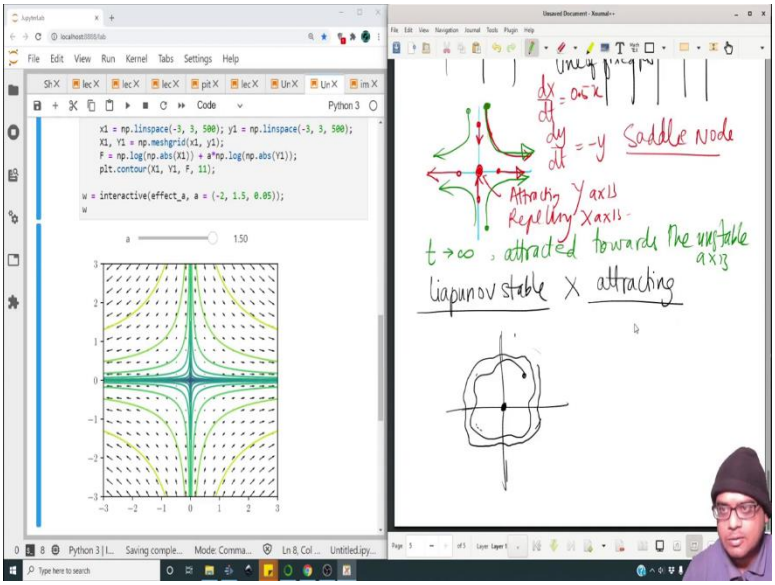
(Refer Slide Time: 50:55)



A point can be attracting; meaning if I start with a point close to a fixed point over here ok, the point may very well go like this and come to the fixed point, it can still attract it.

But it is going a long distance before it is getting attracted. So, it is not Lyapunov stable, but it is attractive, ok.

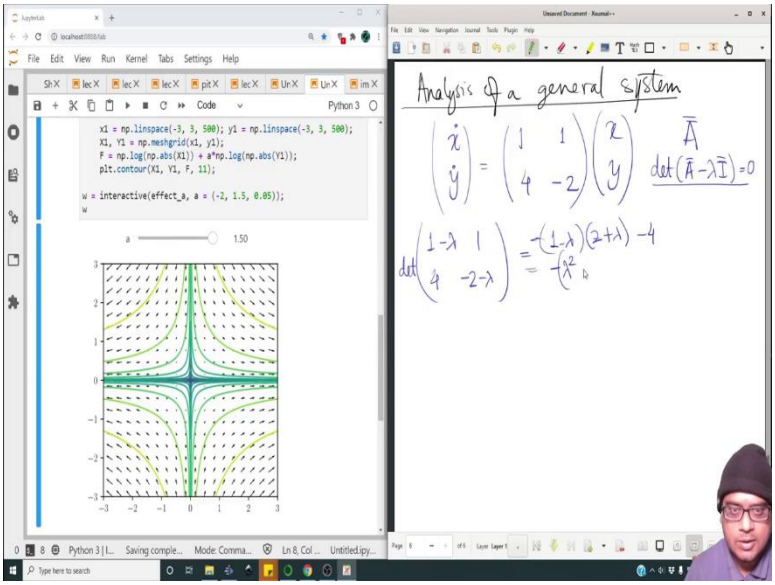
(Refer Slide Time: 51:13)



Similarly, the point can simply meander around the fixed point ok; it can, it will not grow out of one, but it will simply meander around. And if it does not cross a certain threshold, I mean that is in your hands; if it does not cross a certain threshold, then we say that the Lyapunov that the point is Lyapunov are stable. So, this attracting, this fixed point is Lyapunov stable, ok. So, obviously, this is not a course on non-linear dynamics; but I am just trying to throw these Jargons over to you.

So, that when you do study it formally through one of these NPTEL courses maybe; you will know how to go about it, it does not make a lot of difference for us to understand the tools that we can use to understand such systems.

(Refer Slide Time: 52:07)

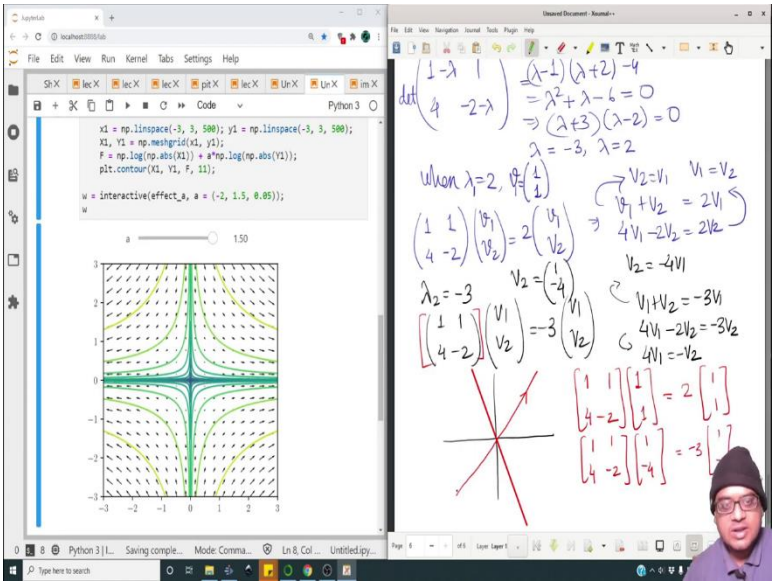


So, with this out of the way, let us move on to the analysis of a general system. Let us consider, I mean it will be clear what the ramifications are of the eigenvalues through this example. Let us consider this particular example, it is a very simple example; but it will bring to the picture what is going on.

So, consider this; let this be a linear system. So, now, we want to solve this, ok. So, we would be looking for the eigenvalues of the system. So, let us actually look into what the eigenvalues of this is, of this matrix are. So, 1 minus lambda, 1, 4, minus 2 minus lambda the determinant of this will be 0.

So, when we have a matrix A, the eigenvalues are found using the determinant of A minus lambda I equal to 0. So, I am not going to discuss elaborately on this; but let us see, I will just give a small proof towards the end.

(Refer Slide Time: 53:22)



So, $(1-\lambda)(-2-\lambda)-4$; what is this going to be, $-2-\lambda+2\lambda+\lambda^2-4$. So, $\lambda^2+\lambda-6=0$, which implies $(\lambda+3)(\lambda-2)=0$. So, $\lambda=-3$ and $\lambda=2$.

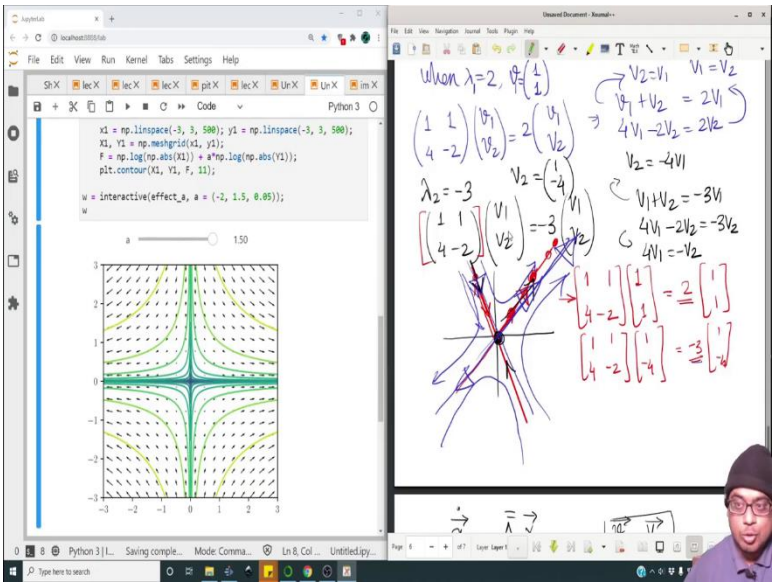
So, when $\lambda=2$. What do we have? So, when $\lambda=2$. What are the eigenvectors? So, the vectors are 1, 1, 4, -2, v_1, v_2 is equal to 2 time v_1, v_2; this implies $v_1+v_2=2v_1$ and $4v_1-2v_2=2v_2$.

So, this implies $v_2=v_1$, this first equation and the second equation also implies $v_1=v_2$. So, the eigen vector will be [1, 1]. So, this is the eigenvector. So, $\lambda_1=2$, the eigenvector is [1, 1]. When the eigenvector eigen when the eigenvalue is -3. So, what do we have; 1, 1, 4, -2, v_1, v_2 is equal to -3 times v_1, v_2; this implies $v_1+v_2=-3v_1$, and $4v_1-2v_2=-3v_2$.

This equation gives us $v_2=-4v_1$ and this equation gives me; what does it give? $4v_1=-v_2$, ok. So, is the same thing, this is good. So, the eigenvector will be [1, -4]. So, in the 2-dimensional plane; so, these are not normalized eigenvectors. So, the 1 comma 1 eigenvector will point in this direction, and the [1, -4] eigenvector will point in this direction, ok.

And what is the meaning of an eigenvector? Here it will be clear. So, an eigenvector means, if this mapping, this particular matrix 1, 1, 4, -2 is applied on this direction; it simply leads to a scaling of that direction ok, it simply leads to scaling of the direction simply. Similarly, when it is applied on this direction; that is 1, 4, 1, -2 times 1, -4, it also leads to a scaling in that direction.

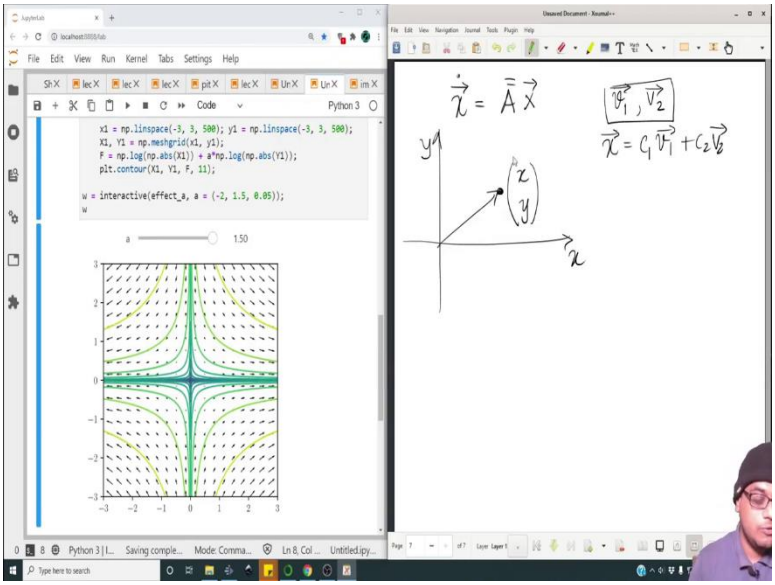
(Refer Slide Time: 56:42)



So, these directions are not altered by the action of this linear transformation, ok. So, if I have a vector like this; then it will be twice of that vector, because of that mapping. It means that, if I have an initial point on this line; an initial point will move somewhere over here at the end of next time step, and it will move somewhere in the next time step like this. But this is minus 3; so if you have an initial vector like this, it will try to contract, ok.

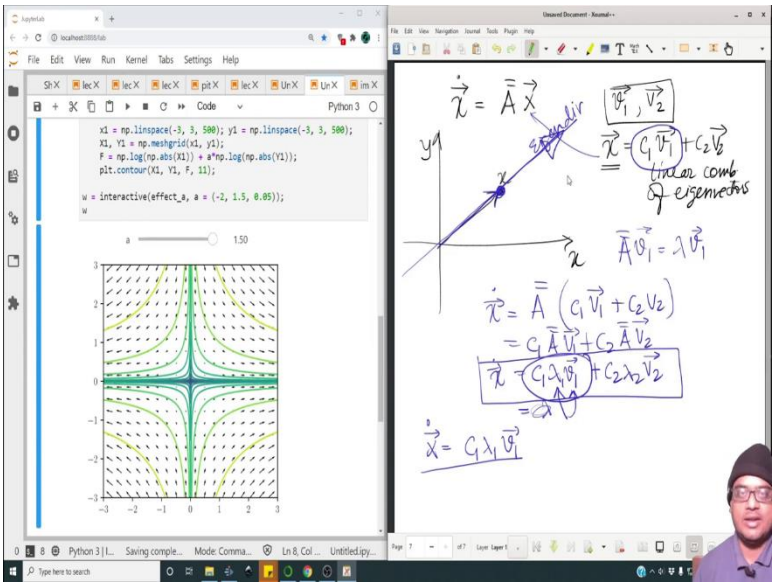
It is shrinking rather than expanding, because the eigenvalue was negative. So, the point is attracted like this. So, we can see that on these special directions; the points are attracted and the points are repelled. So, this particular fixed point, obviously for a linear system the origin is a fixed point; this fixed point is attracting along this eigen direction or the eigenvector and it is repelling along the other eigenvector, ok.

(Refer Slide Time: 57:49)



So, $\dot{x} = Ax$, ok. So, if we know that the space, the two-dimensional space has eigenvectors v_1, v_2; then it means that, any vector in this space can be written in terms of a linear combination of these two vectors. So, *x* can be written as $c_1 v_1 + c_2 v_2$. And this vector is not a velocity vector or something like that; it is the pair of terms *(x, y)* and in that quantifies the vector in the x-y plane, do not be confused with the velocity vector over here, ok.

(Refer Slide Time: 58:38)



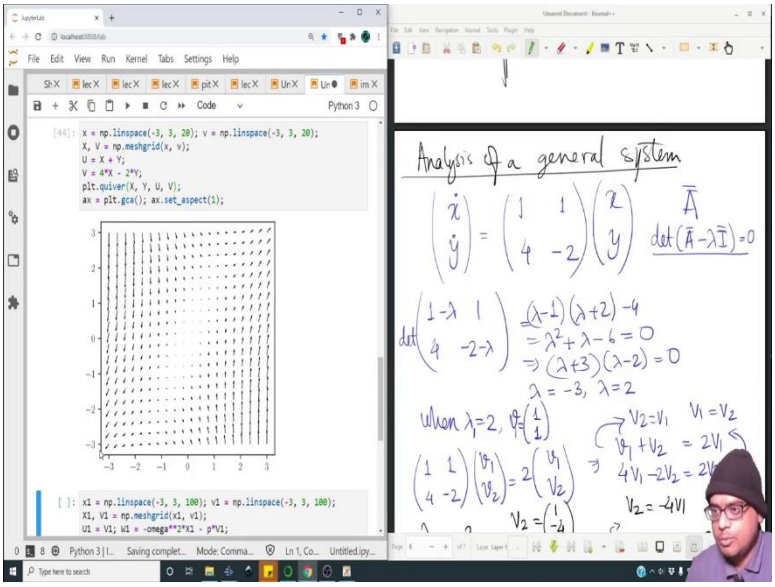
So, now, what about this? If this is true, I can represent any point as a linear combination of these two alright; this is a linear combination of the eigenvectors. Then I can substitute this over here and say $\dot{x} = A(c_1 v_1 + c_2 v_2)$ and this becomes $c_1 A v_1 + c_2 A v_2$. But by definition of being an eigenvector Av_1 will be λv_1, ok. So, this will be $c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2$.

But $c_1 v_1 + c_2 v_2$ was the original vector. So, this becomes equal to $\lambda\, I$ mean. So, we cannot represent this anymore, because λ_1 is distinct from λ_2, ok. So, $\dot{x}$ is therefore dictated by the behavior of the eigenvalue and the eigenvector; we cannot simplify this anymore. So, if we choose the initial point to be aligned with v_1 ok; if the initial point only consists of this eigenvector, then $\dot{x}$ will be $c_1 \lambda_1 v_1$, it will be only this term.

I am just choosing the initial point to lie on that eigen direction; this is that eigen direction and I am choosing the initial point to lie on that. And this means that, I am going to increase this vector continuously along the same eigen direction. So, in this particular case, this particular eigen direction had a positive eigenvector and thus points on this would continuously move away from the origin.

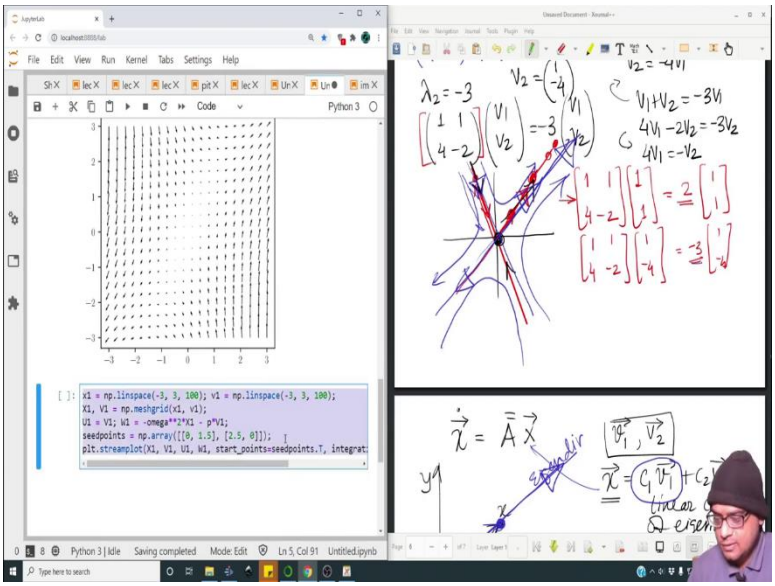
While this eigen direction had a negative eigenvalue and hence the points would be continuously attracted towards the origin. And now we can sort of reconstruct the trajectories as they would look; they would look something like this ok, they would look something like this. So, now, let us go to a computer and actually see whether this is how the trajectories look like.

(Refer Slide Time: 61:33)



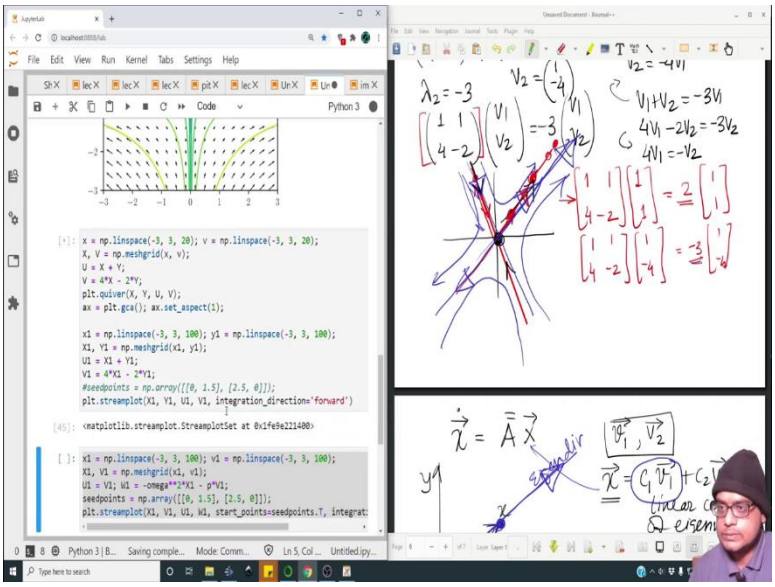
So, fine let me. So, I have plotted first the vector plot of this and just have a look at the vector plot; the points over here are going like this. Let us go to our prediction, the points are going something like this; the points are actually going something like this, the points are headed like this and so on. So, now, let us plot on top of this various stream lines.

(Refer Slide Time: 61:58)

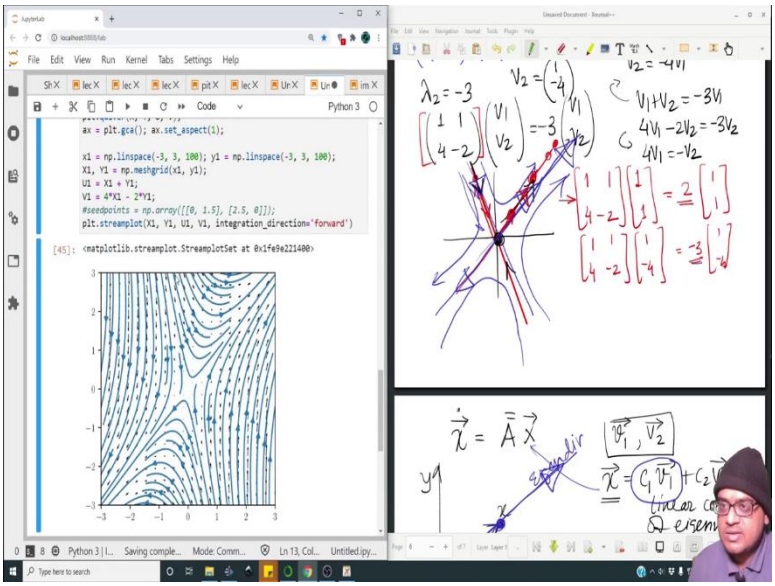


Let me copy this bit of code.

(Refer Slide Time: 62:06)

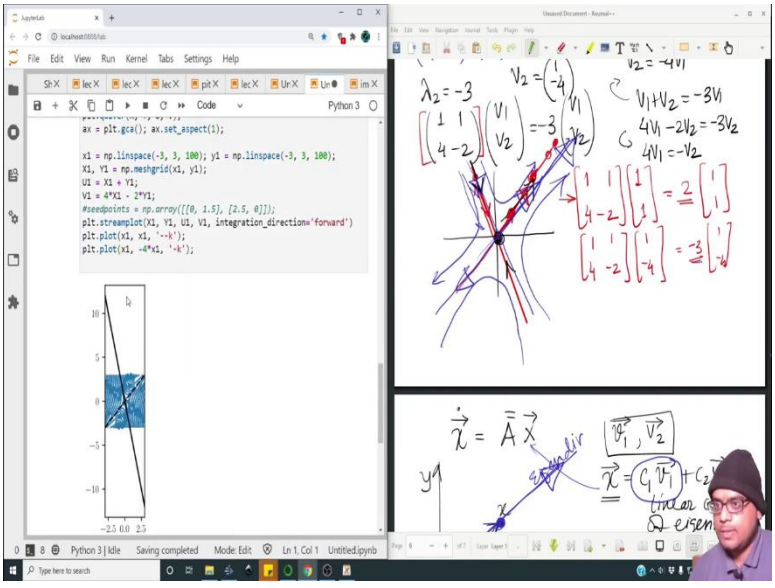


(Refer Slide Time: 62:13)



So, let me, alright. So, I have plotted the stream lines as well; we do not need this cell anymore. So, the streamline makes it apparent; the trajectories come like this and go away like this. The trajectories are flowing like this ok; this is that the trajectories are flowing like this. Now, let us superpose on top of this the true, the two eigen directions, ok.

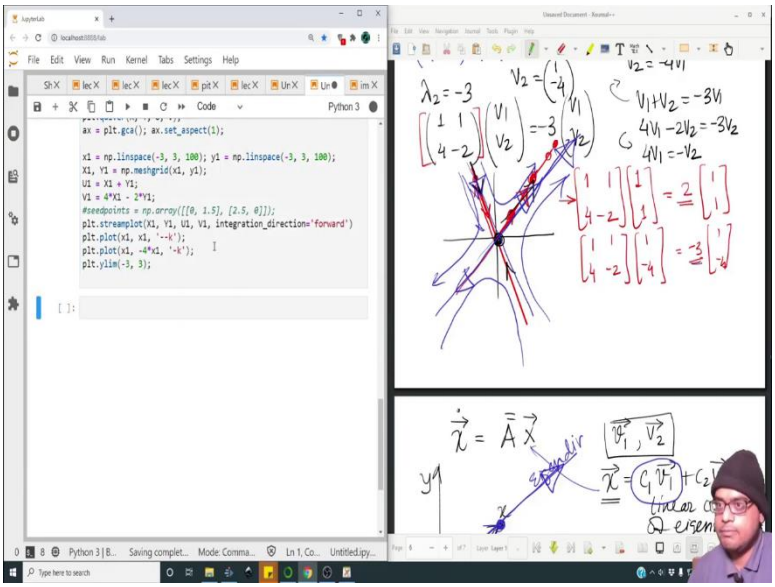
(Refer Slide Time: 62:39)



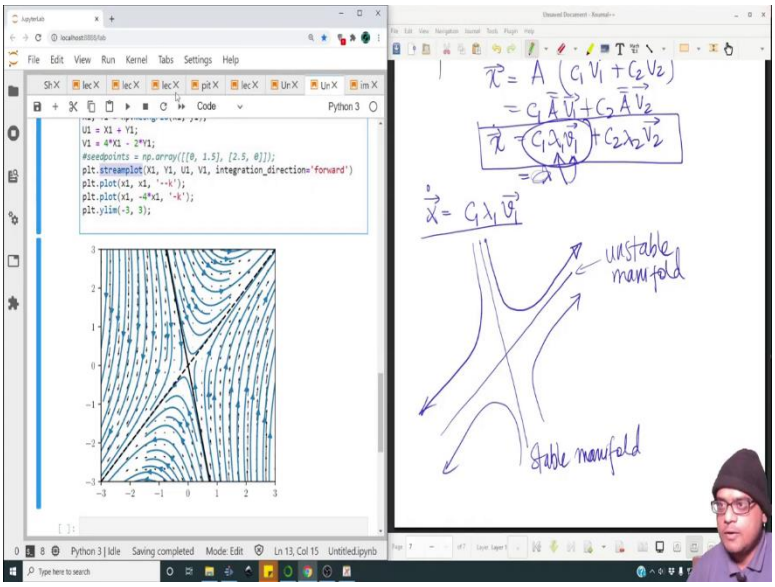
So, plt.plot; so, the first line will be $x = y$. So, (x_1, x_1), let me make this eigen direction as black. So, because it is the unstable axis, we will make it like this, so dotted as a broken

line. And the other line what is the direction, 1, -4. So, it will be $x_1, -4x_1$ and let me put it by a solid line, ok.

(Refer Slide Time: 63:22)



(Refer Slide Time: 63:28)

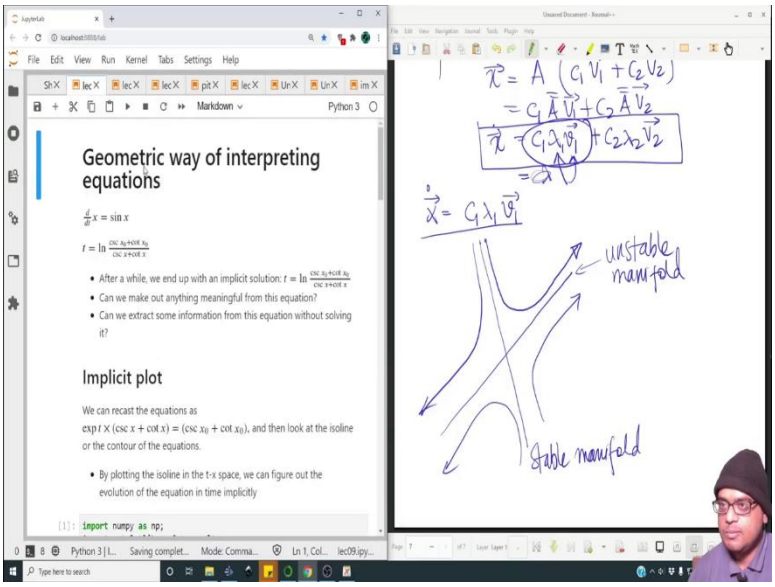


So, let us set the y limits plt.ylin and it will be from -3 to 3, ok. So, the solid line is the attracting eigen direction, the broken line is the repelling eigen direction, ok. And in the language of non-linear dynamics, the repelling direction is called as the unstable manifold; while the attracting direction is called as a stable manifold.

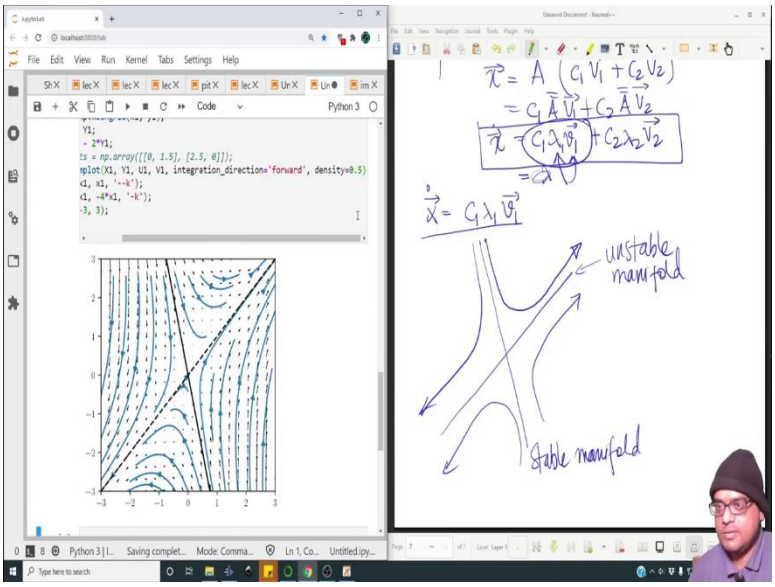
And for long times as time goes to infinity, points in the phase plane will be attracted towards the unstable manifold. And whatever we had arrived at the conclusion that, we had arrived at numeric with just intuition; we could verify all this over here as well, ok.

You can have a look at the various things inside the stream plot in order to make it look better ok; you can change the c map as well, norms, line width, density. You can play around with various; let me change the density, we do not need say 0.5, ok.

(Refer Slide Time: 64:41)



(Refer Slide Time: 64:49)



So, this looks much cleaner if you will. So, this pretty much shows how the flows in the phase plane look like, ok. So, now I request you to have a close look at the homework; because whatever you would need to undertake the assignment, we have done in this particular lecture. And I request you to pay very close attention to some of the things that will be covered in the assignment ok; it will help you in appreciate the structure of the system of linear equations in a better manner.

So, with this we end this lecture over here. In the next lecture, we will begin with the analysis of a system, which has complex eigenvalues and then we will see how we can evolve from there. With this I say goodbye, until then farewell, bye.