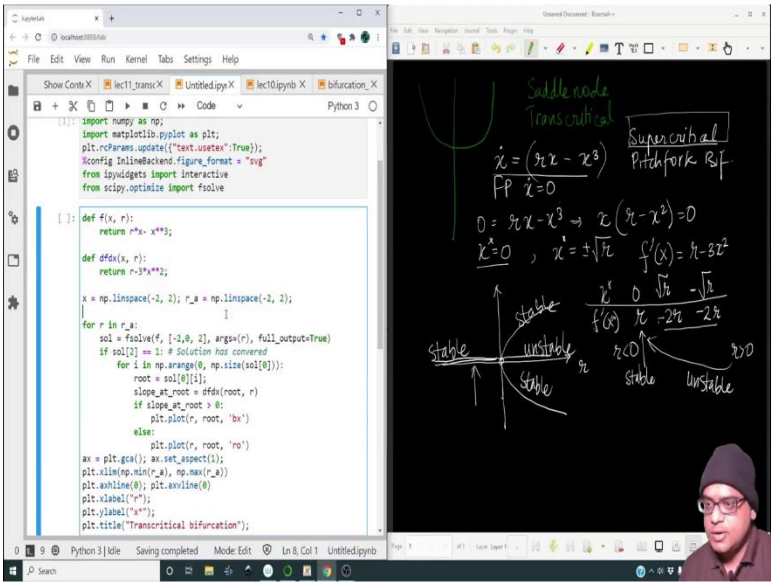
Tools in Scientific Computing
Prof. Aditya Bandopadhyay
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 12
Bifurcations – Pitchfork Bifurcation

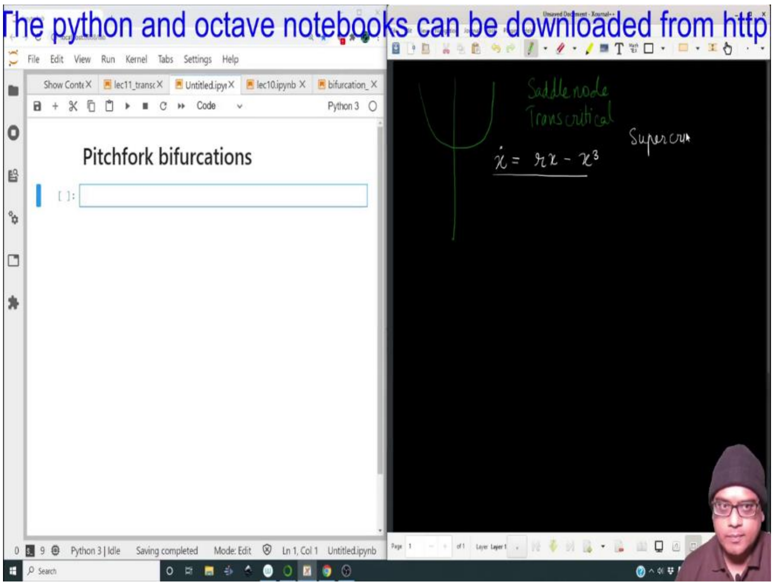
Hello, everyone. In the last lecture, we had a look at first saddle known bifurcations and then transcritical bifurcations. In this particular lecture, we are going to look at the third kind of bifurcation that is called as the Pitchfork Bifurcation.

(Refer Slide Time: 00:52)



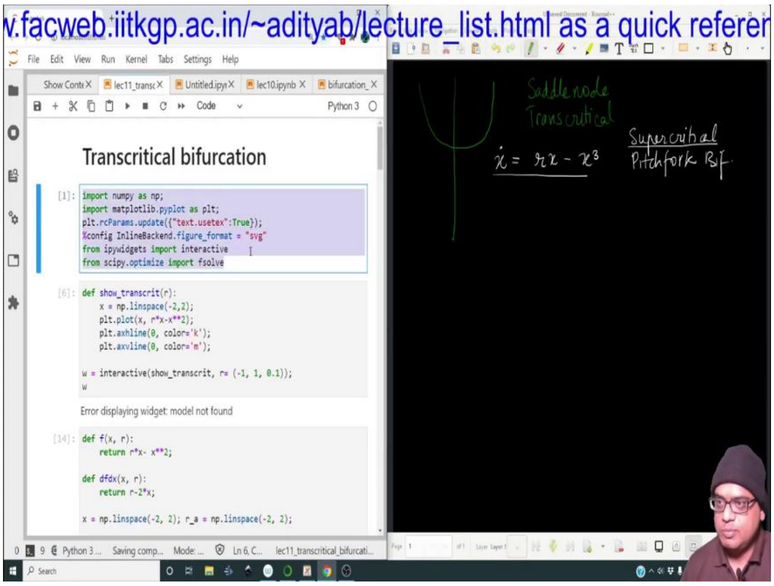
So, a pitchfork is a tool that is one uses and it looks something like this ok and in the Indian context it would resemble something like a trishul. So, the normal forms that we had seen for the saddle node and a transcritical bifurcations they were quadratic ok.

(Refer Slide Time: 01:26)



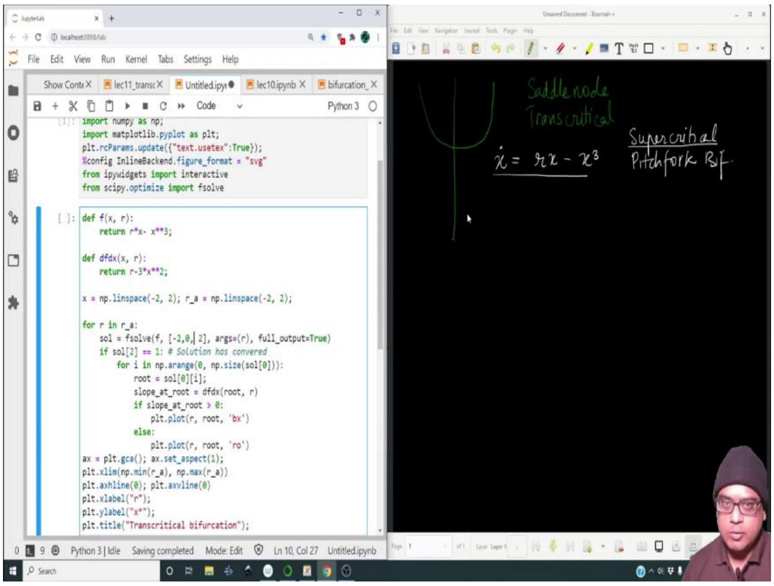
And, in the case of pitchfork bifurcations we will go to a cubic form of the normal form. So, let us consider the governing equation $\dot{x} = rx - x^3$. So, this is the normal form for what is called as a supercritical pitchfork bifurcation.

(Refer Slide Time: 01:46)



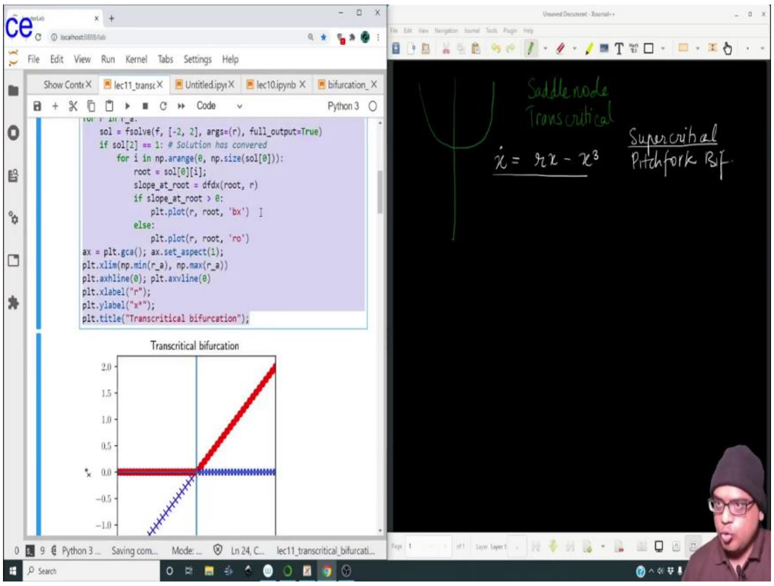
And, it will be cleared later on why it is called as a supercritical pitchfork bifurcation, in particular why it is called as a supercritical bifurcation.

(Refer Slide Time: 02:08)



So, let us begin let us first copy this bit of code which we will use, alright. So, let me execute this.

(Refer Slide Time: 02:23)



So, we will also require this particular code and also make use of this cell where we will simply change the functions that we were using. So, recall that we had used this particular snippet in order to study a transcritical bifurcation and the good thing about using Python is we can reuse various snippets.

I mean we can reuse various snippets of any kind of programming language that you want, but in the case of JupyterLab we can have a very quick look into what is going on. So, we have defined the fx as rx-x^3 and the derivative of it dfdx as r -3x^2 ok. So, once again let us give an additional guess point because it is a cubic equation we expect three roots.

So, even before going into the code we can check out what the fixed points are. So, fixed point corresponds to $\dot{x} = 0$. So, $0 = rx - x^3$ which implies $x(r - x^2) = 0$. So, x= 0 is a fixed point and $x = \pm\sqrt{r}$ is a fixed point. So, these are the fixed points, let us denote them by x^*.

So, when r is negative x = 0 is the only fixed point when r becomes positive apart from x = 0 we have $x = \pm\sqrt{r}$. So, these are the two additional fixed points. What about the stability and the fixed points? If you recall the stability of the fixed points is determined by the derivative of the function on the right hand side with respect to x.

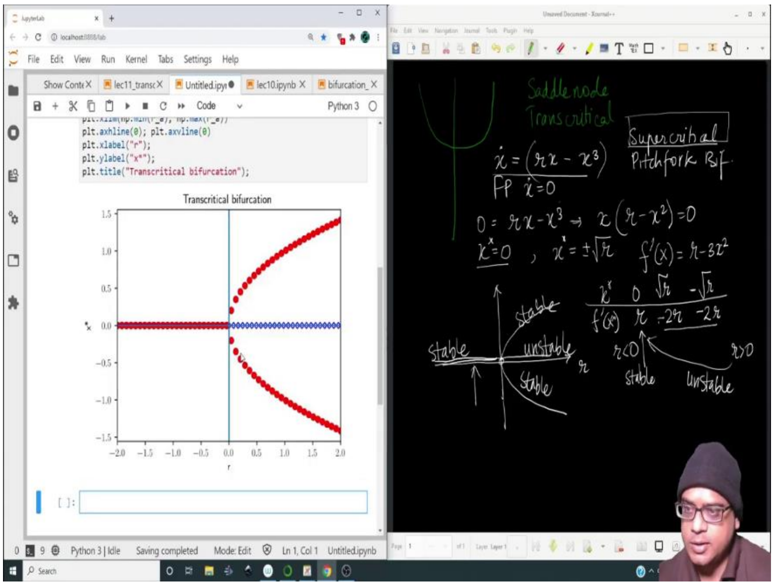
So, this $dfdx = r - 3x^2$. So, when we substitute $x^* = 0$, so, when $x^* = 0$ f`(dfdx) is going to be r. When $x^* = +\sqrt{r}$ this term is going to be $dfdx = r - 3r = -2r$ which is minus. The same thing happens when we choose the root $x^* = -\sqrt{r}$ we again obtain -2r.

So, when r < 0, this particular fixed point is stable because when f`x < 0, we have a stable root. When r < 0 these roots do not exist; when r >0, however, this particular root becomes unstable because r > 0 implies f`at x^* > 0.

However, these two roots are stable. So, thus we have a bifurcation diagram, so r <0 this is the root and this is stable. When r crosses 0 these two branches become stable and this is unstable and this diagram resembles a pitchfork when it is kept on its side and hence it is called as a pitchfork bifurcation.

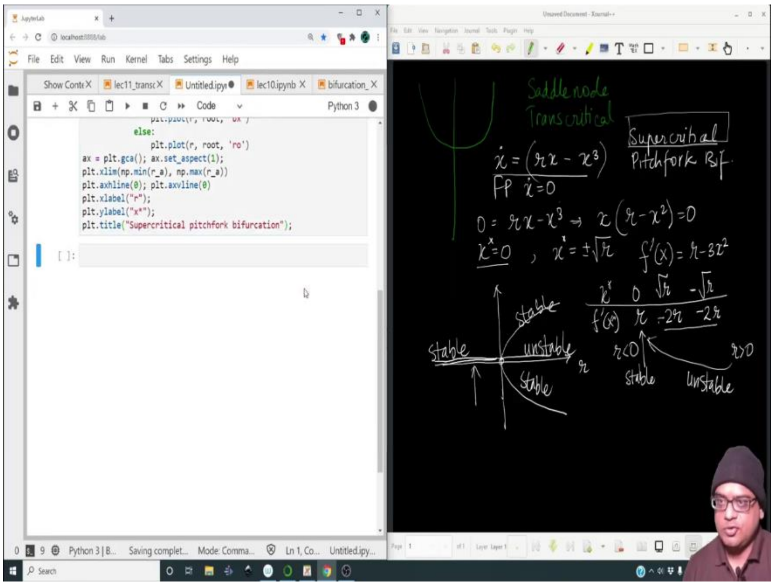
It is called as a supercritical pitchfork bifurcation because it leads to the creation of two stable branches from one stable branch. The original is the root $x^* = 0$ it loses stability. So, we have put in the functions over here and now let us run this.

(Refer Slide Time: 07:05)



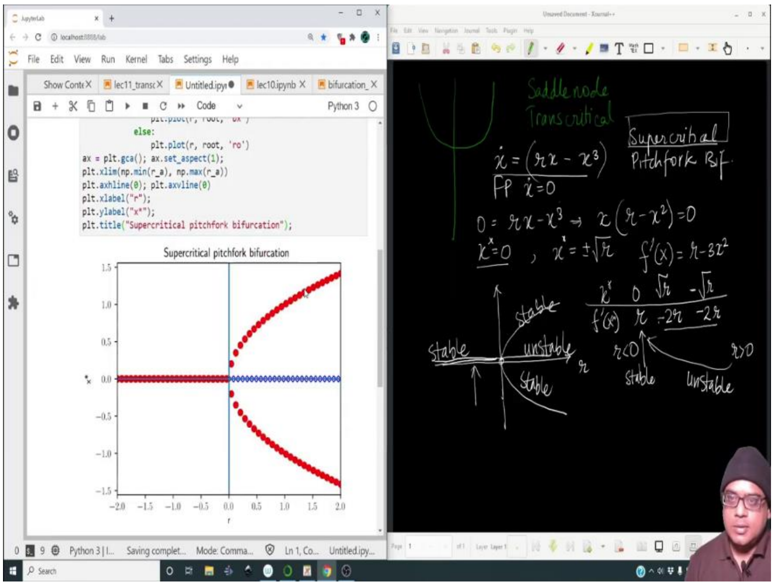
So, it takes a bit of time ok. So, great we do have the trend that we had predicted. These are the stable points; these are the stable points and this becomes unstable.

(Refer Slide Time: 07:33)



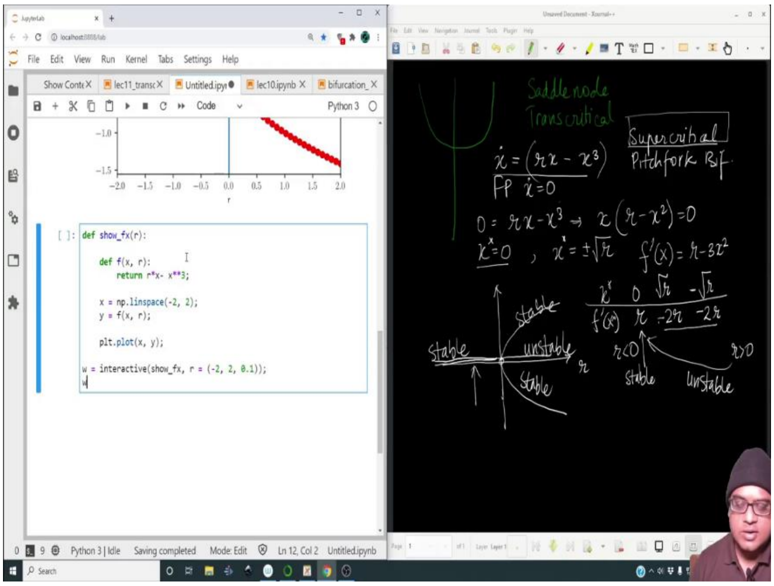
So, let us change the title of the plot to a pitchfork another a Supercritical pitchfork bifurcation alright. So, we have missed this and that should be fine.

(Refer Slide Time: 07:48)



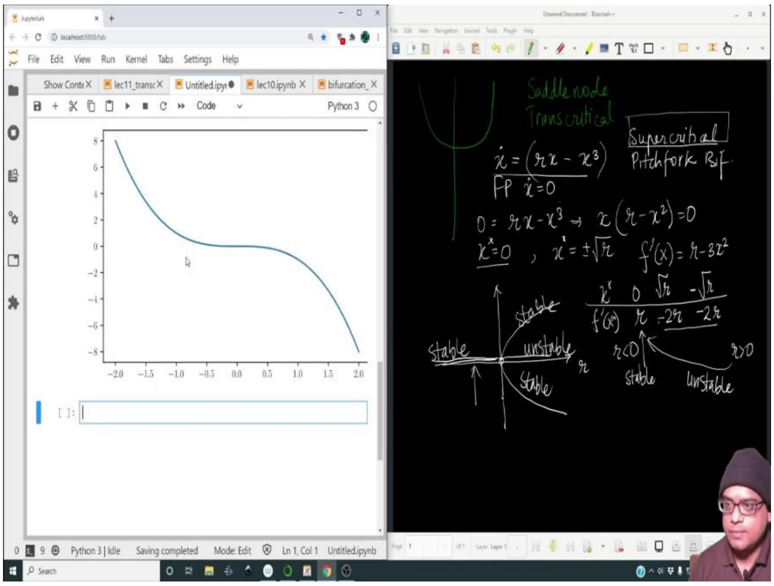
So, let us visualize how this entire thing would actually look like if we were to plot the right hand side of this function as a function of r ok.

(Refer Slide Time: 08:08)

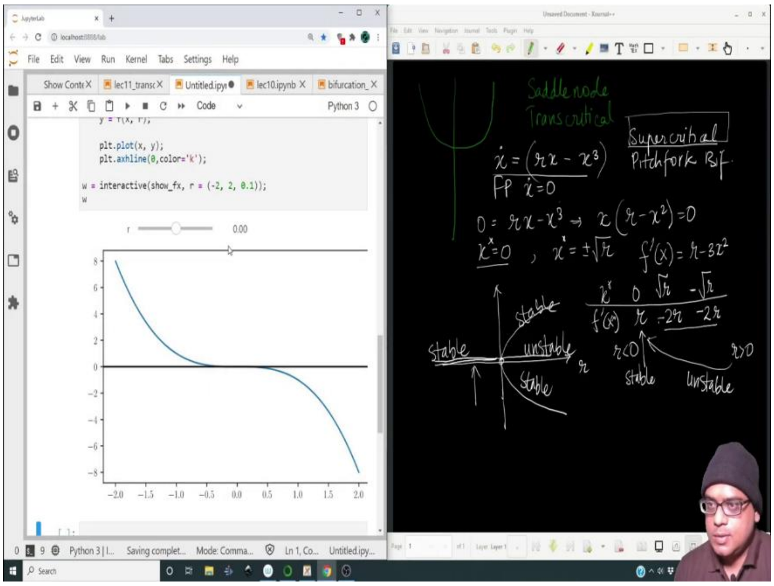


So, let me copy this. Let me make a new cell. So, let us define show pitchfork or rather show_fx(r). Let r be the input to this function. Let us wrap this inside the code and let x = np.linspace(-2,2) and let y= f(x,r). Let us then plot it and let us wrap it inside an interactive widget alright. Let us display the widget as well, let us run this cell.

(Refer Slide Time: 09:13)

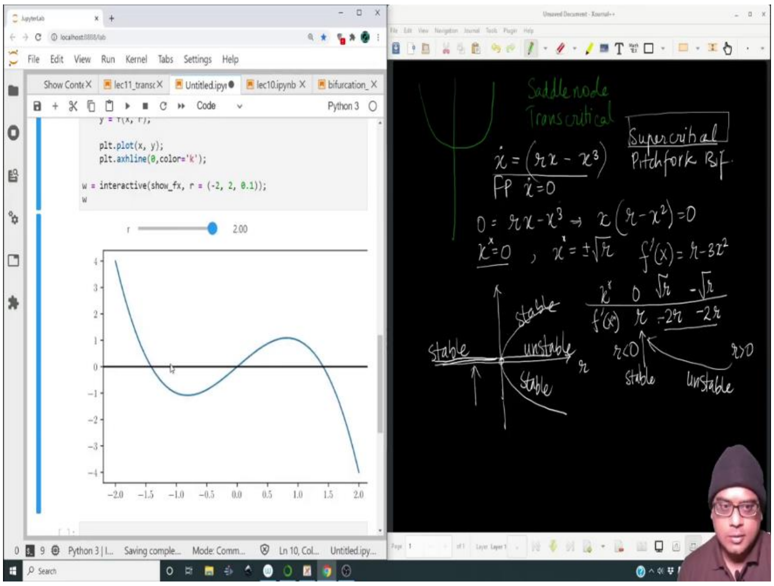


(Refer Slide Time: 09:23)



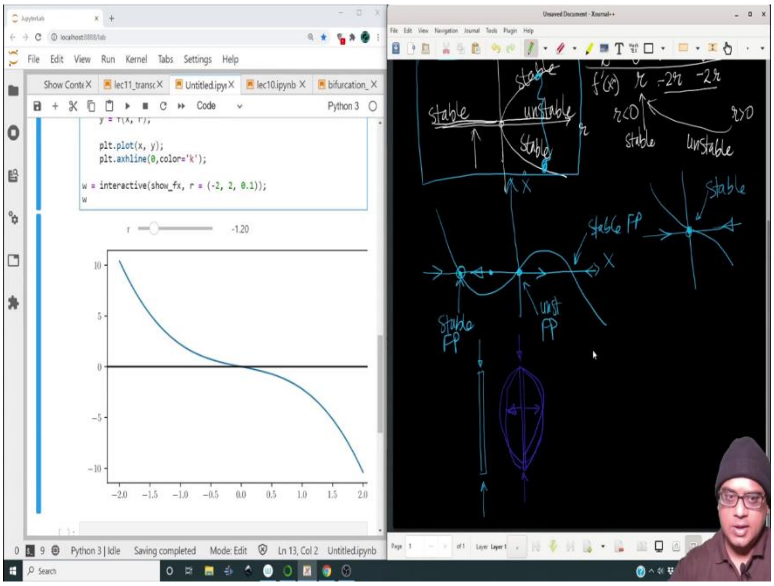
So, let us make the x axis as well. So, when r = 0, there is exactly one root.

(Refer Slide Time: 09:29)



When r becomes positive there are two roots alright. So, there are two roots and let us go quickly to our notebook and see what we have already discussed.

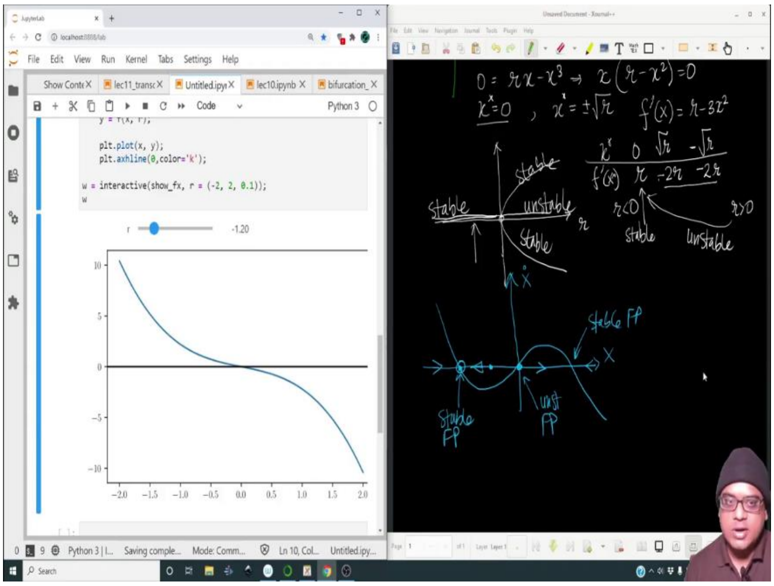
(Refer Slide Time: 09:49)



So, over here the $\dot{x}$ is negative. So, this is $\dot{x}$ versus x. So, over here $\dot{x}$ is negative. So, it is attracting sorry, $\dot{x}$ is positive and so, it is attracting all the points on this line would be attracted towards this point over here $\dot{x}$ is negative. So, all the points would be attracted towards this point. So, this is stable equal or a stable fixed point.

So, over here the points go towards the right because $\dot{x}$ is positive over here $\dot{x}$ is negative. So, points go towards the left. So, again these are stable fixed points and this is an unstable fixed point. And what about the case where r is negative?

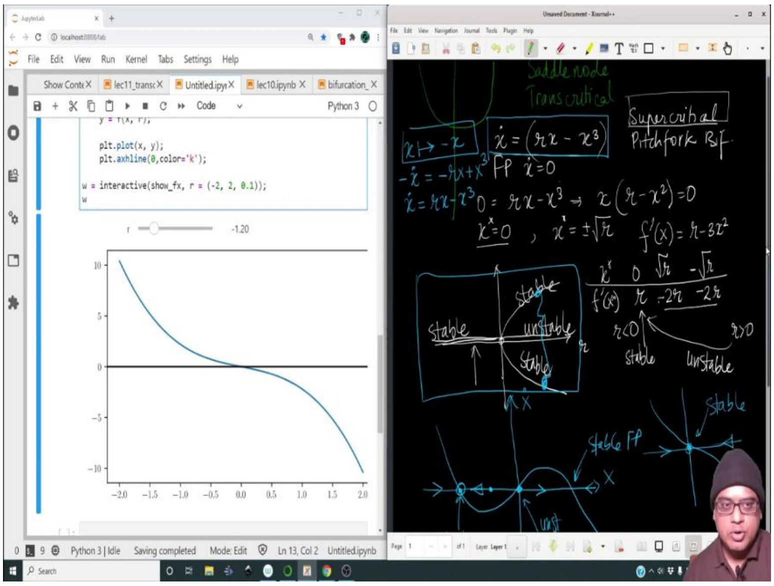
(Refer Slide Time: 10:49)



We only have one intersection point something like this. Over here it attracts, over here also it attracts. So, $x^* = 0$ is a stable fixed point and this eventually manifests itself in this kind of a bifurcation diagram.

So, while this particular algebraic form of the ordinary differential equation is typical for a supercritical pitchfork bifurcation, there can be other kinds of function and the key point is to have cubic behavior and cubic and a cubic behavior is sort of quantified by one root or three roots ok, never two roots and the presence of three roots leads to a shifting of the nature of the point. Apart from that there is something else such kinds of problems are often dictated by a certain symmetry.

(Refer Slide Time: 12:08)

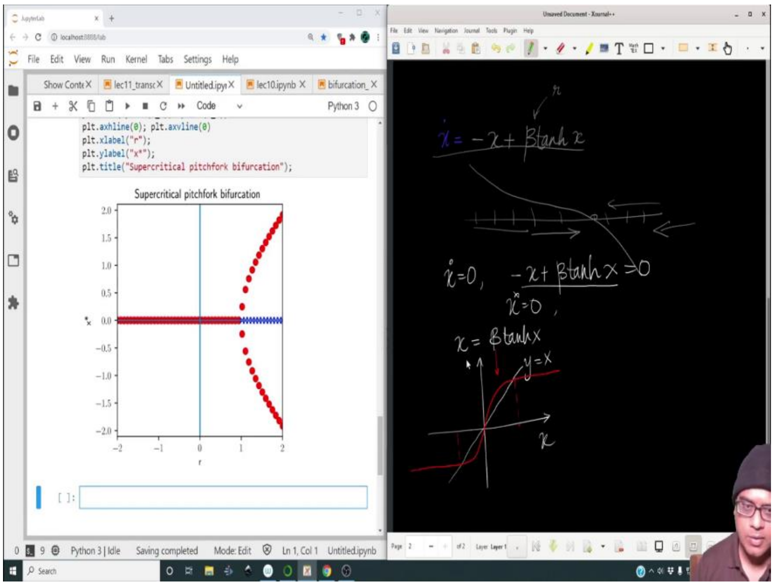


So, this particular problem it is symmetric under the transformation x goes as - x. So, if you substitute all of these so $-\dot{x} = -rx + x^3$ and is the same as $\dot{x} = r - x^3$. So, this equation is invariant under the transformation x as - x.

And, the fact that the problem is symmetric implies that the occurrence of the two roots is also symmetric. The two roots symmetrically occur while one root vanishes. And, this is quite prototypical of a problem. For example, the buckling of a beam if you compress a beam nothing happens, but after a certain point it will tend to either buckle towards the left or it will tend it will tend to buckle towards the right ok.

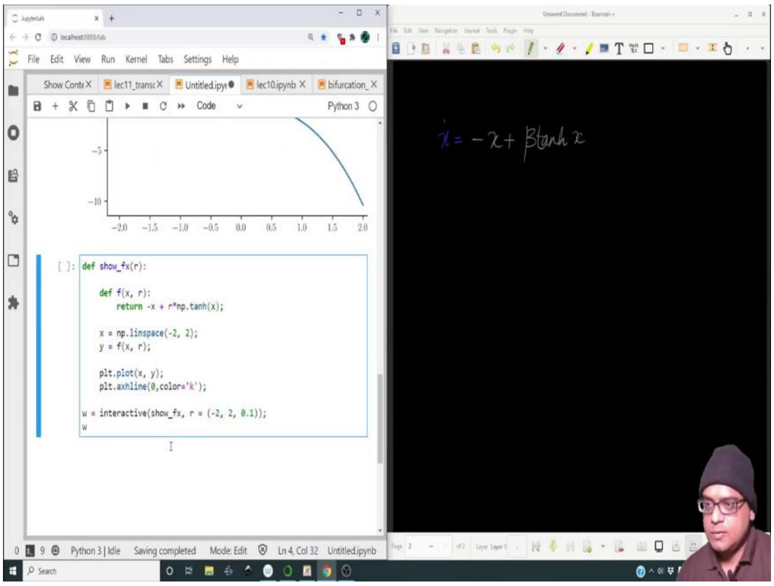
And, whether it chooses to buckle in this direction or in this direction it is quite symmetric there is no nothing sacrosanct about which direction it buckles to. So, such kinds of problems are characterized by the presence of a distinct symmetry in the normal form. Let us look at a simple prototype of this kind of bifurcation. So, it is let me create.

(Refer Slide Time: 13:41)



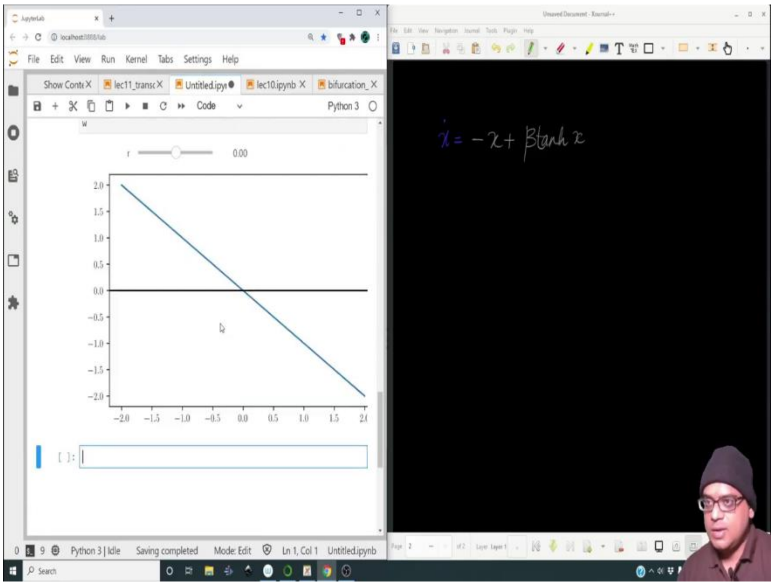
So, $\dot{x} = -x + \beta\tanh(x)$, let me use a different color,. So, first things first let us plot $\dot{x} = -x + \beta\tanh(x)$. Let us see how that thing looks like.

(Refer Slide Time: 14:09)



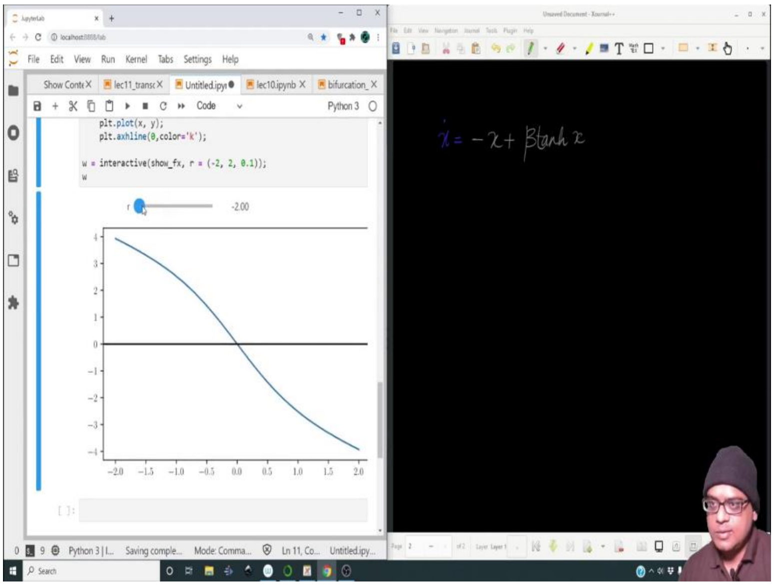
Let me copy this; let me paste it over here. So, fx is -x+ r*np.tanh(x).

(Refer Slide Time: 14:32)

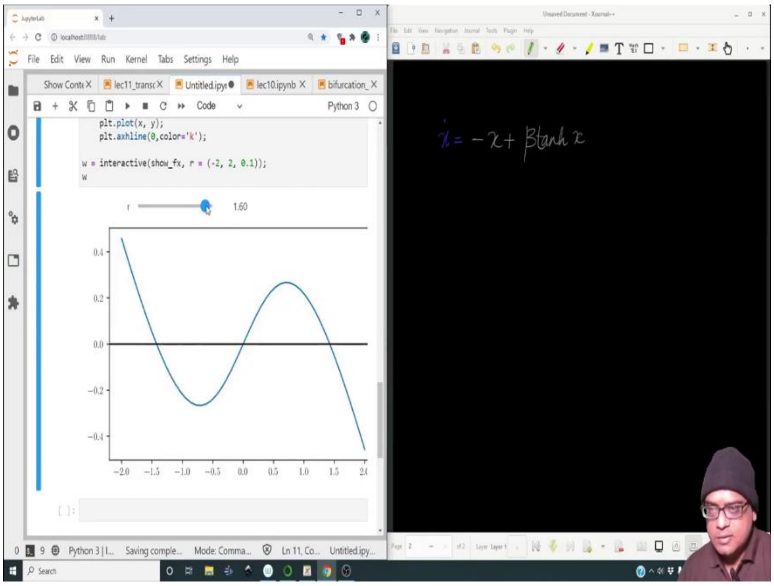


So, let us run this particular cell and see the widget ok.

(Refer Slide Time: 14:35)



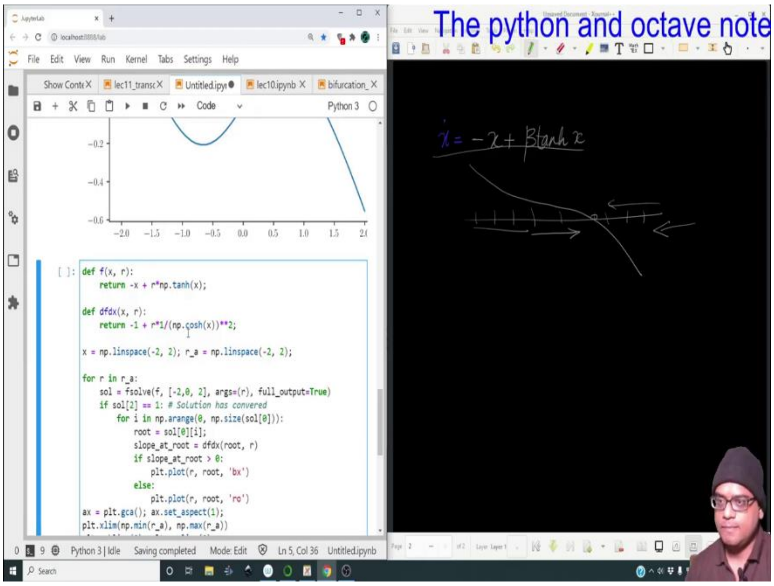
(Refer Slide Time: 14:39)



So, this is a sigmoidal behavior for negative r; hence r becomes positive, we have the occurrence of three roots, 1 2 3 and the occurrence of three roots is in such a way that the fixed point x = 0 changes its nature from being an attractor to a repeller ok. So far it is attracting and then it begins to repel and these are the two fixed points which are symmetric about the y axis.

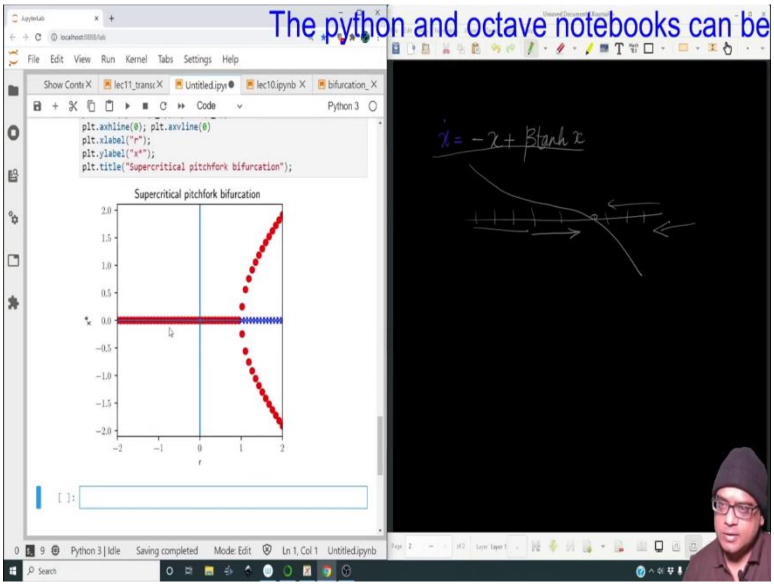
So, let us try to make the bifurcation diagram corresponding to this kind of a vector flow. So, like I have said before this kind of equations are also called as vector flows because this govern how points on a line they flow ok. So, if it is like this all these points would flow towards the right, all these points would flow towards the left ok it is also called as a vector flow because of that precise reason.

(Refer Slide Time: 15:53)



So, let me copy this, let me paste it over here. Let me change the functional form of this. So, the dd so, the derivative of this function will be -1 + r*1/np.cosh(x)**2. So, d/dx of tanh(x) is $sech(x)^2$. So, yeah let us run this cell and see what we get.

(Refer Slide Time: 16:31)

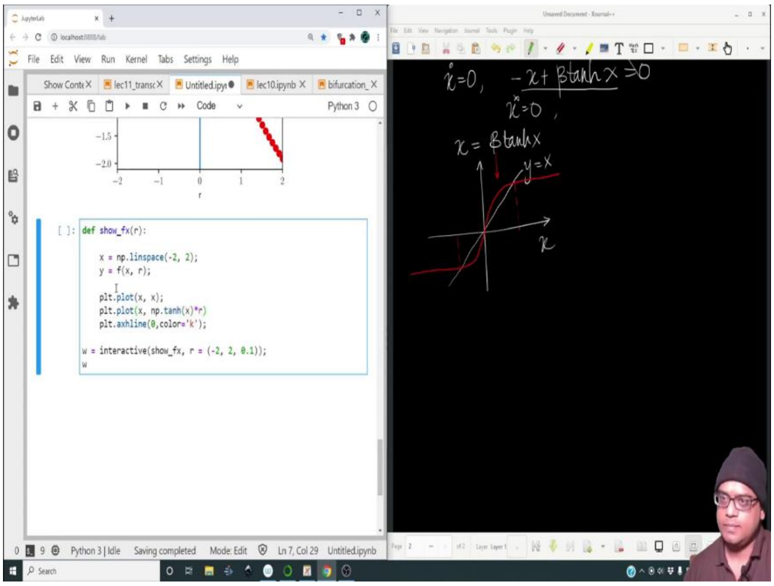


So, we do have that kind of a behavior and notice that at r=1, we have the occurrence of the bifurcation ok. So, this beta is what we are calling it as r that is the control parameter.

So, what are the fixed points? So, the fixed points are corresponding to $\dot{x} = 0$ that is $-x + \beta tanh(x) = 0$ and obviously, x is of x= 0 is a fixed point, but what are the other two fixed points? It is not easy to say because this is sort of a transcendental equation, but we can interpret this as following. It is $x = \beta tanh(x)$.

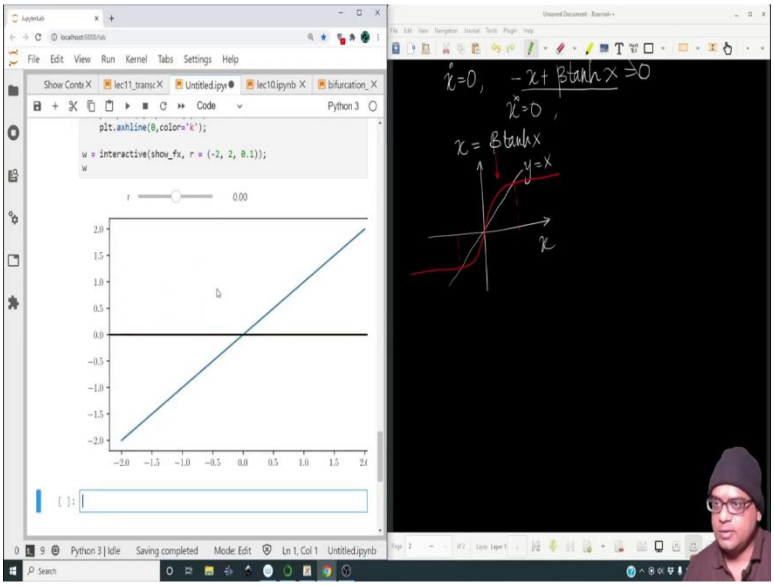
So, if we plot x versus this curve so, this is the function y = x that is the left hand side and tanh function looks something like this. So, depending on the strength of β we will have different functions appearing and that leads to the intersection points which are also the roots. And, obviously, this problem has a very distinct symmetry, alright. So, why does the why does it occur at r= 1, is something which I will let you think about ok.

(Refer Slide Time: 18:22)



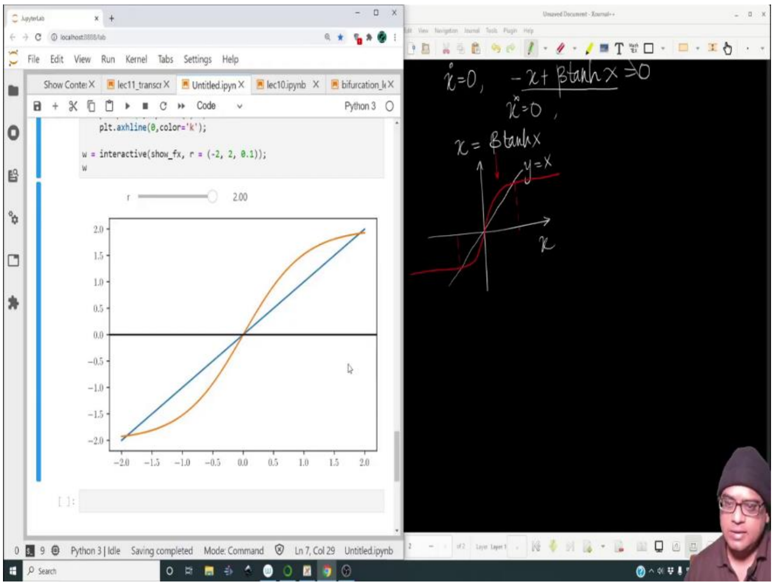
But, just to be just to give you a hint you can analyze, this particular cell to find out what is really going on. So, over here we have we do not need this function for now. Let us plot plt.plot(x, x) and plt.plot(x, np.tanh(x)*r).

(Refer Slide Time: 18:47)



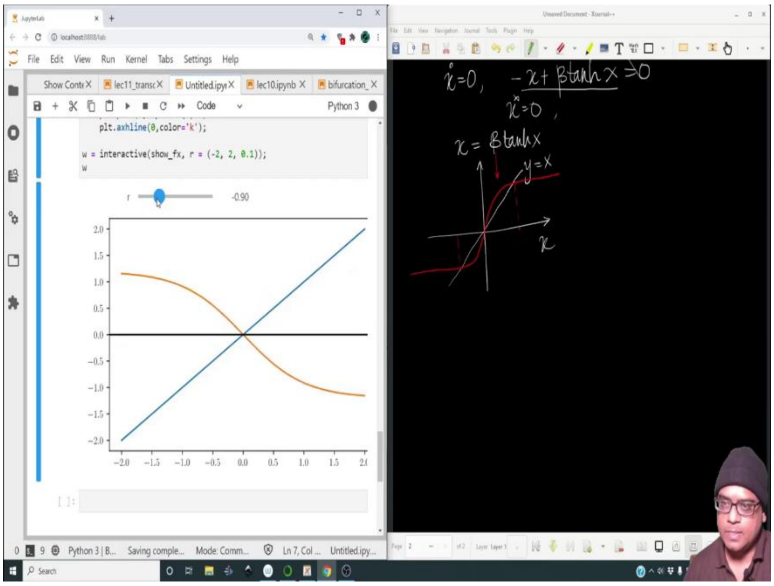
Let us see the interactive widget ok.

(Refer Slide Time: 18:50)



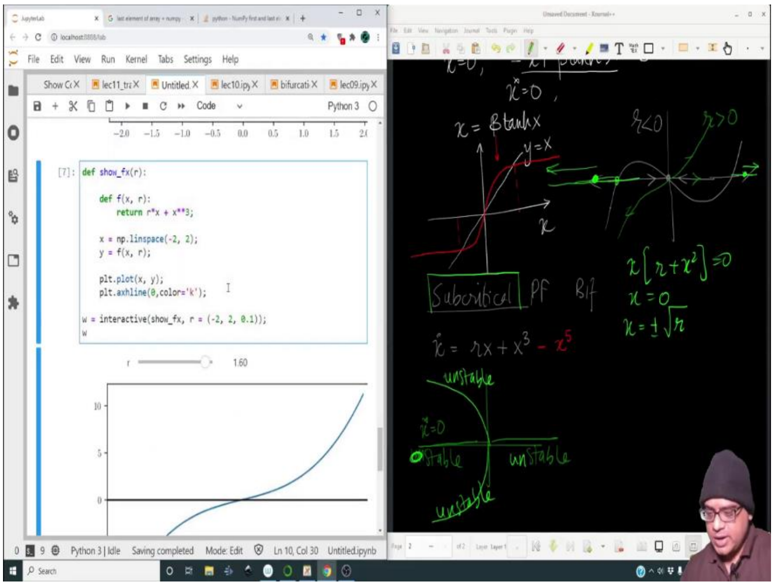
So, two roots once it becomes tangential at r =1 one there is only one root and then obviously, there is only one root ok. So, think about what is going on.

(Refer Slide Time: 19:01)



So, this will help you in formulating some kind of a reason towards assessing by r =1 in particular yields the pitchfork bifurcation. Let us now turn our attention towards subcritical bifurcation.

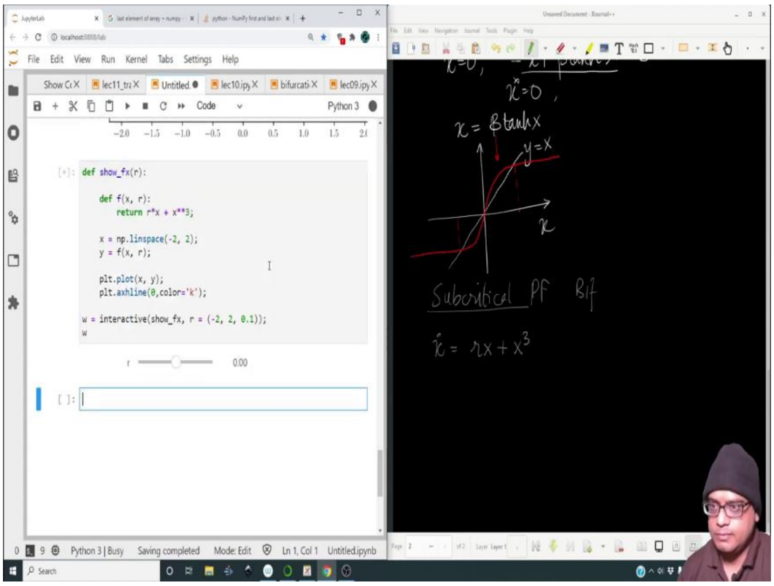
(Refer Slide Time: 19:32)



So, the next thing that we are going to look at is the subcritical pitchfork bifurcation and by subcritical it means that upon changing a certain parameter you will have less fixed points than you had earlier. So, we expect it to be sort of a inverse.

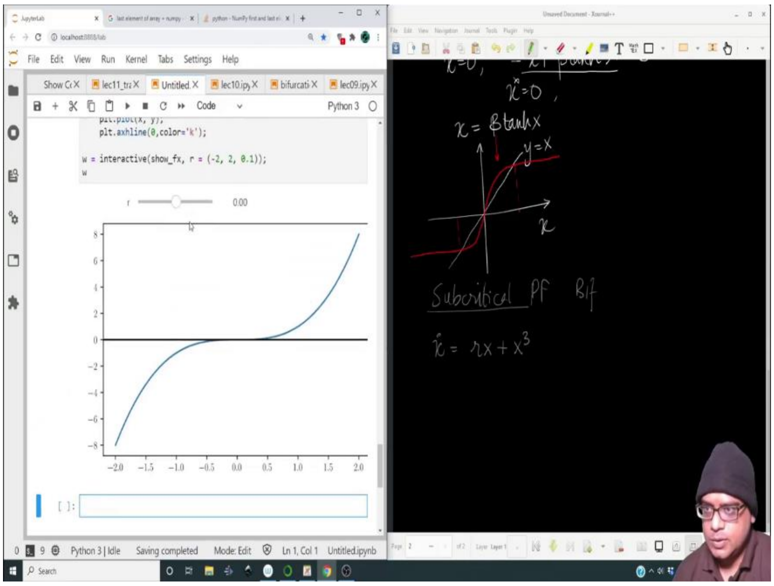
So, super critical had, let us go back from one stable root or fixed point we had two stable fixed points. So, we increase the number of fixed points. In the subcritical we have we will see what we have ok. So, the normal form of this subcritical bifurcation is $\dot{x} = rx + x^3$.

(Refer Slide Time: 20:35)

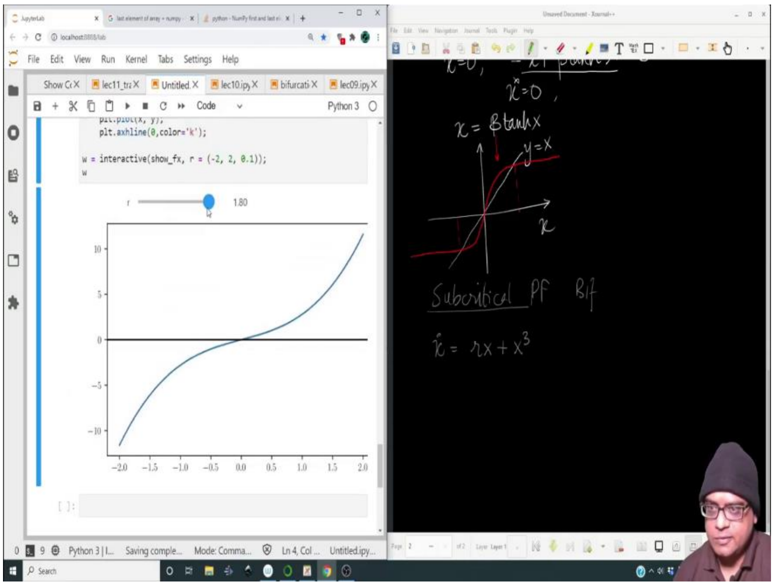


So, let us take this particular code and, let us see how this looks like ok. So, let us change this functional form.

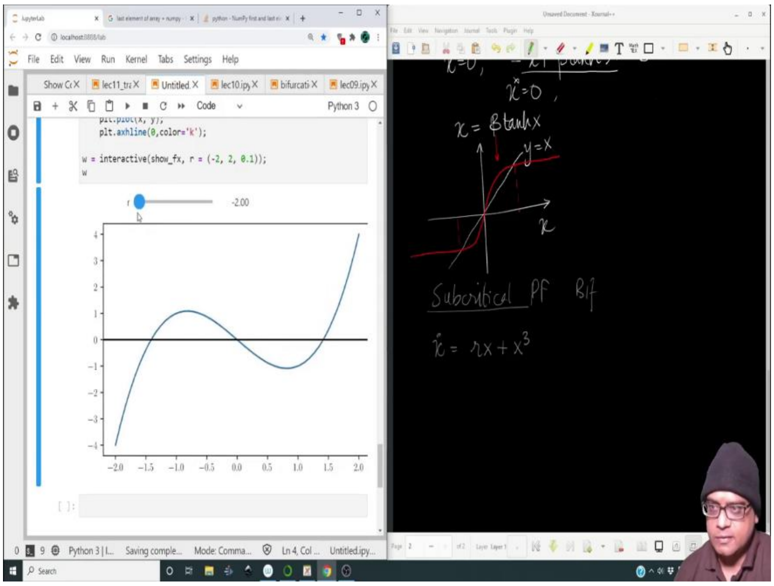
(Refer Slide Time: 20:48)



(Refer Slide Time: 20:51)



(Refer Slide Time: 20:55)



So, this will be $\dot{x} = rx + x^3$ ok. So, when r is positive we have only one fixed root when r is negative we have two fixed three fixed points and what happens what is the stability of the fixed points? You should be able to just look at it and tell that ok this origin is attracting and this is repelling and when.

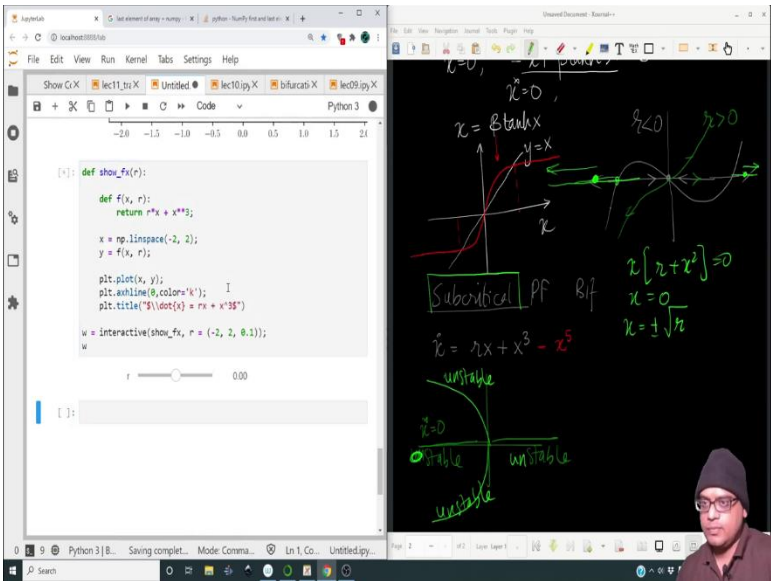
So, this is the case where r< 0. When r > 0, something like this there is only one fixed point. It looks something like this. So, the origin is repelling. So, so, the origin is unstable. So, this fixed point $x^* = 0$ is unstable and then it becomes stable. At the same time what are the roots? So, the roots are $x(r + x^2) = 0$. So, x=0 is one fixed point and $x = \pm\sqrt{r}$ is the other fixed point.

So, this is the curve of $x = \pm\sqrt{r}$ and quite naturally they are unstable; so, this rather when r < 0. So, the origin this is unstable and this is stable, this is unstable and this is unstable. So, we had one stable point which has now become unstable.

So, this is a this is called as a subcritical pitchfork bifurcation ok. So, what happens when we have an initial condition over here? The vectors the flow on this particular line will be such that it will keep on continuing towards minus infinity and when the point is over here it will keep on going towards plus infinity.

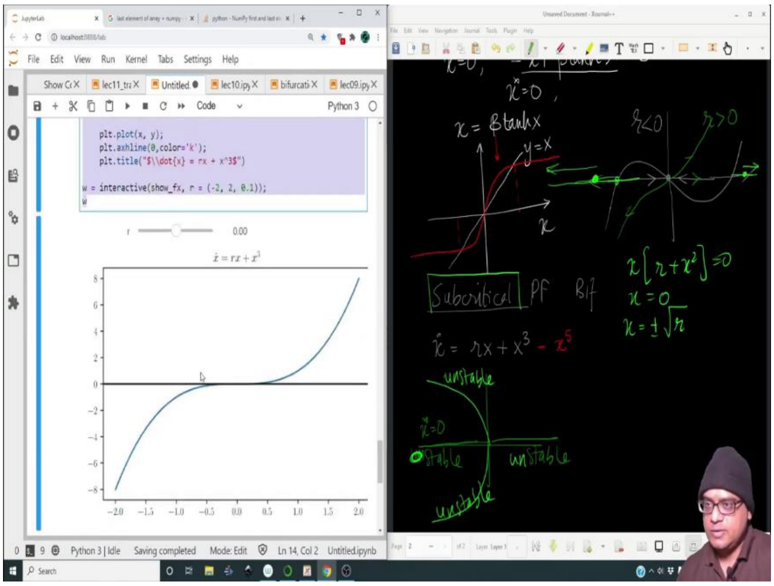
Consequently what this means is any point any initial condition over here is bound to blow up or it is going to grow to say to infinity or minus infinity. And, usually in order to avoid that there are some additional terms in the normal form. So, a very easy way of controlling this is simply doing $\dot{x} = rx + x^3 - x^5$.

(Refer Slide Time: 23:42)



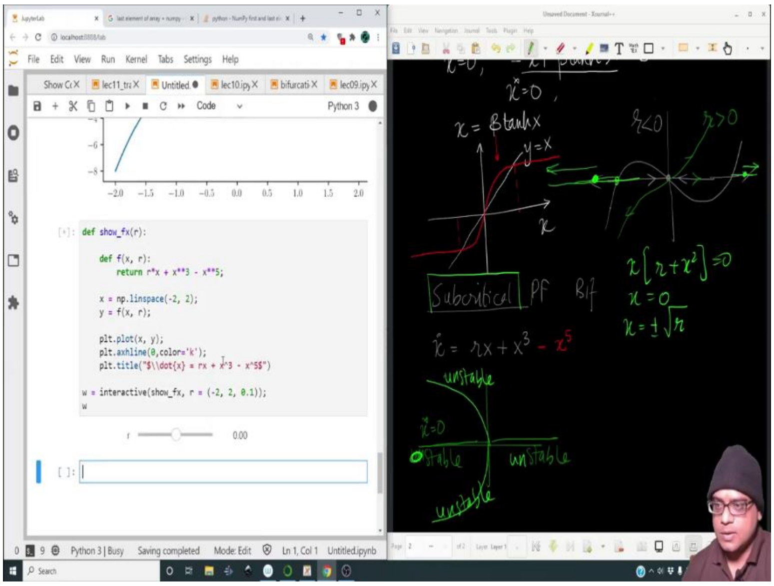
So, this was the let me put a title. So, this normal form is dot x forgotten to put quotes. So, $\dot{x} = rx + x^3 - x^5$.

(Refer Slide Time: 24:08)



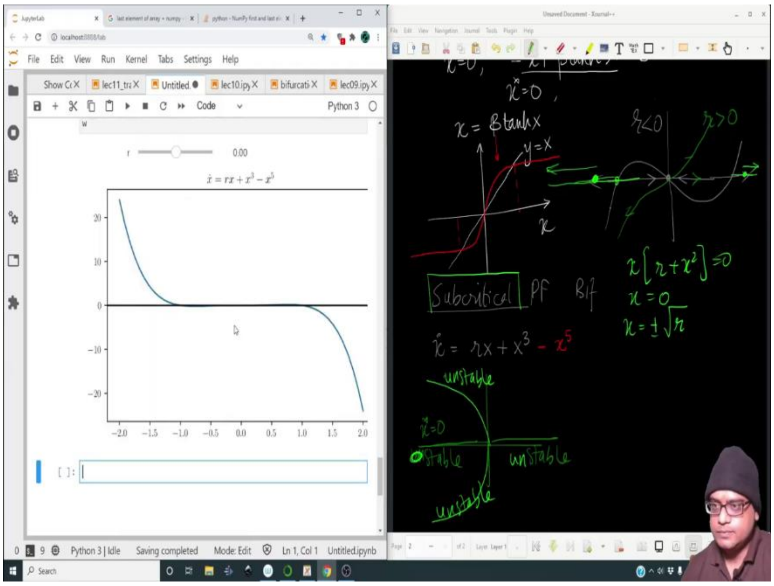
Let us copy this.

(Refer Slide Time: 24:11)

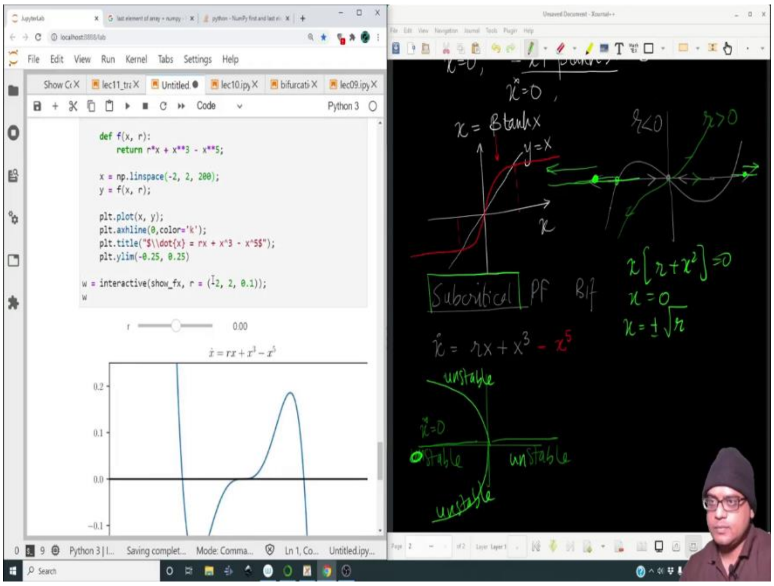


And, let me show you what the presence of the x^5 term brings into picture ok. So, we have done this, essentially, we are plotting this.

(Refer Slide Time: 24:30)

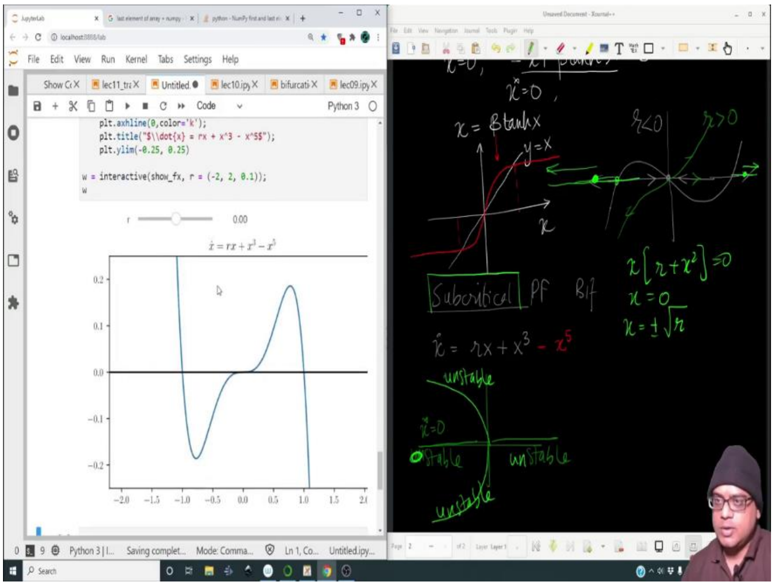


(Refer Slide Time: 24:35)



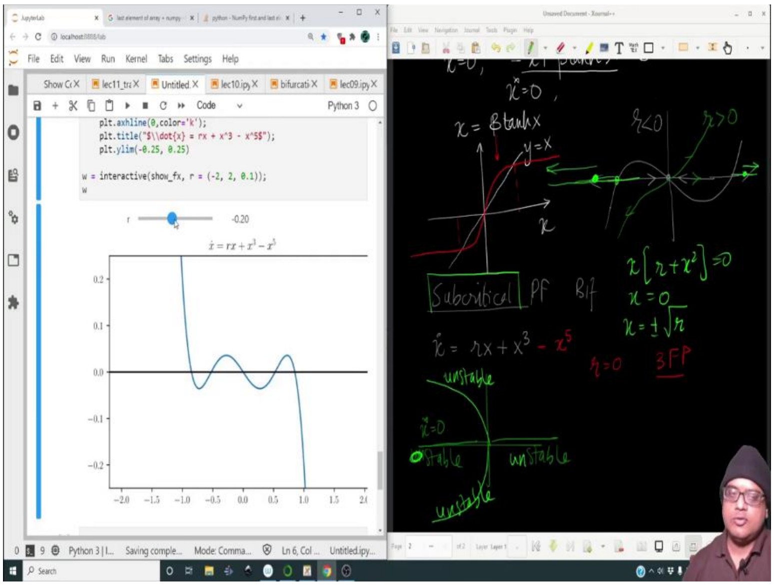
Let me change the y limit to something more manageable. Let me increase the number of points ok.

(Refer Slide Time: 24:45)



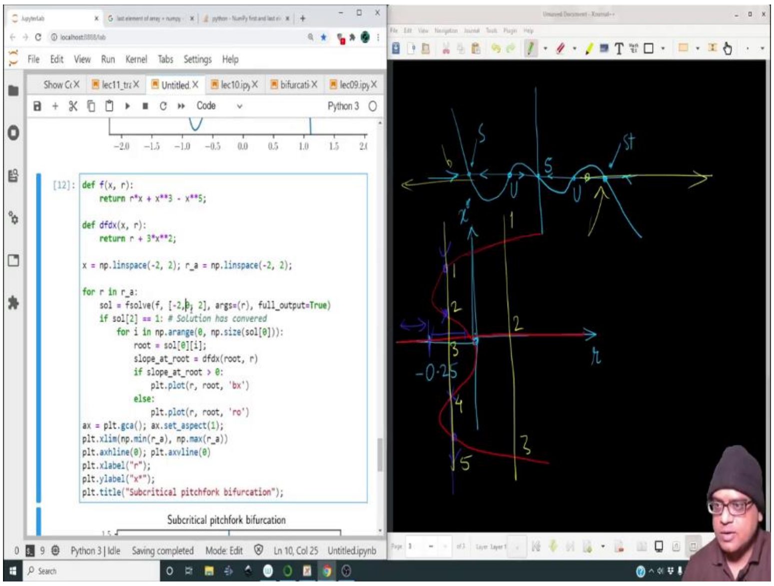
So, when r= 0 we have three roots 1 2 3. So, r = 0, three fixed points and at r = 0. This thing is obviously, attracting because the vector points over here will try to go towards the right and the vector points over here will try to go towards the left. This is attracting, this is ok. So, it needs to be figured out what this is, while over here also it is attracting. So, it has three fixed points, two end points are attracting. Let me make the r< 0.

(Refer Slide Time: 25:29)



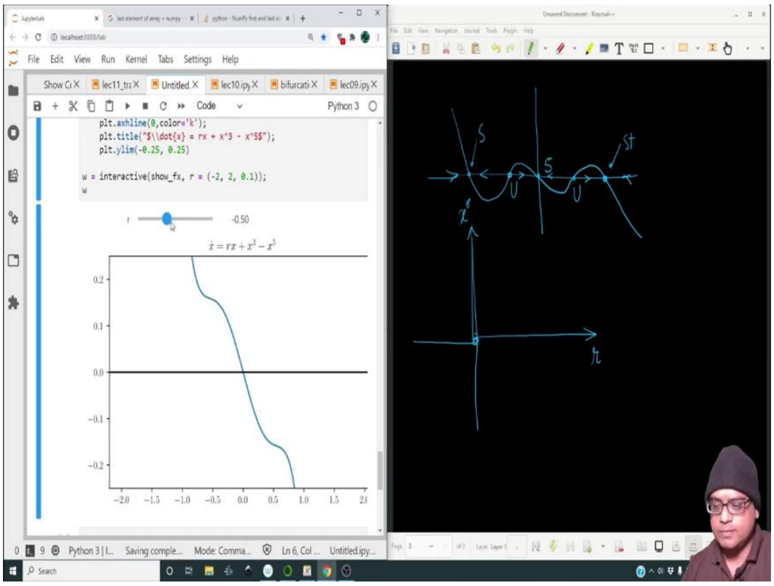
So, r < 0 gives us 1 2 3 4 5 fixed points ok.

(Refer Slide Time: 25:48)



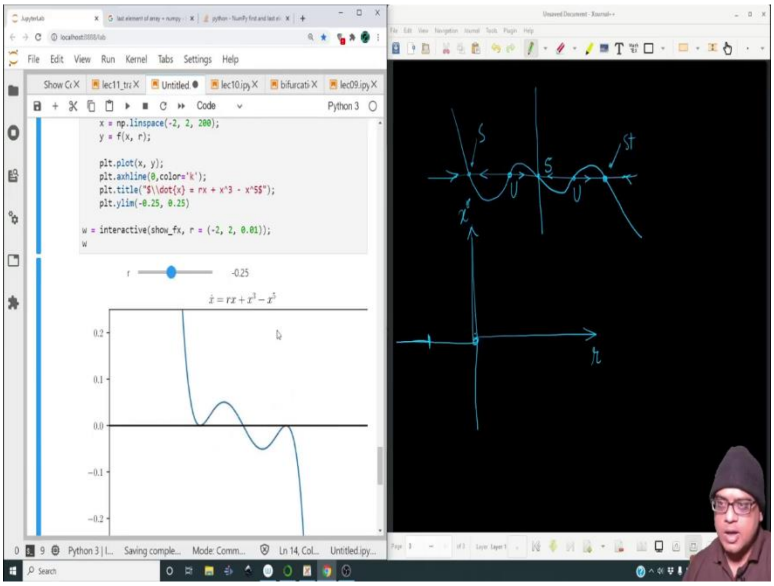
So, let us analyze what those fixed points behave like. So, we have something like this and this point is obviously attracting; this point is repelling; this point is attracting; this is repelling and this is attracting. So, this is stable, this is also stable, this unstable, unstable, stable.

(Refer Slide Time: 26:42)



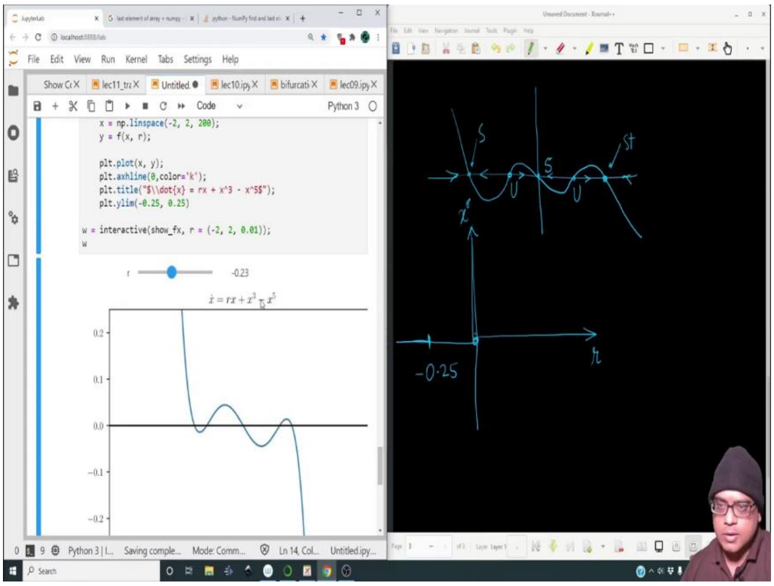
So, we can sort of imagine the bifurcation diagram to be something like this. So, on this axis we have r. So, on this axis we have the fixed points when $\dot{x}$ or rather when r is negative, there is only one fixed point ok. So, there is only one fixed point up until a certain negative r. So, from - 0.3 to -0.2, we are having the occurrence of multiple roots.

(Refer Slide Time: 27:07)



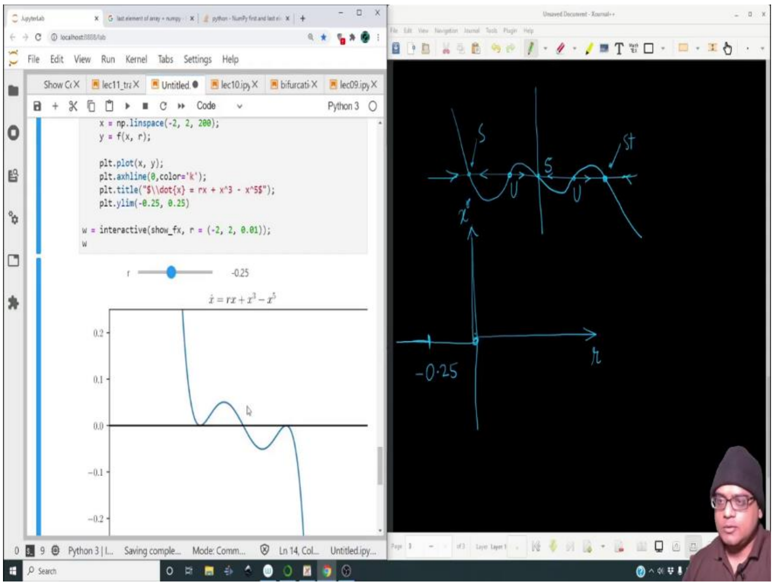
Let me increase the resolution of this slider. So, at r = - 0.25, we have 1 2 3 roots ok. So, r less than this we had only one root and r =0.25, we have the occurrence of three roots ok.

(Refer Slide Time: 27:47)



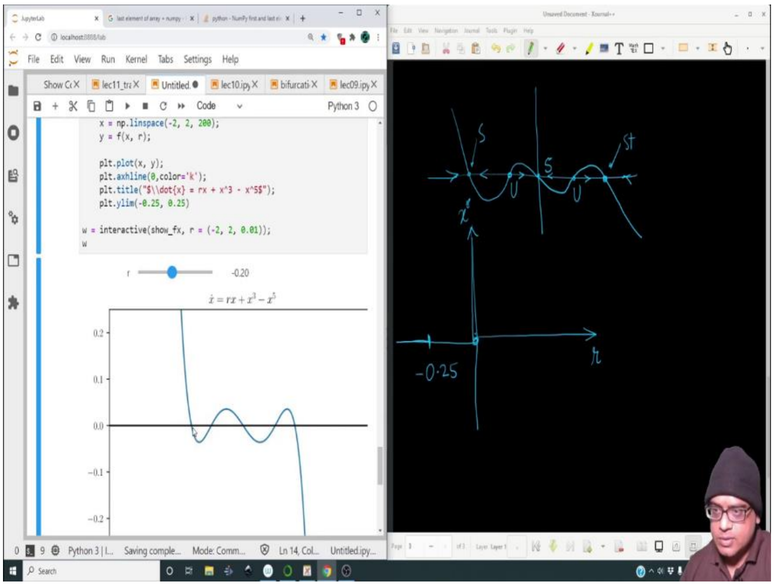
And, when r crosses this so, when that happens what is the stability of the two of the three points? Yeah.

(Refer Slide Time: 27:59)



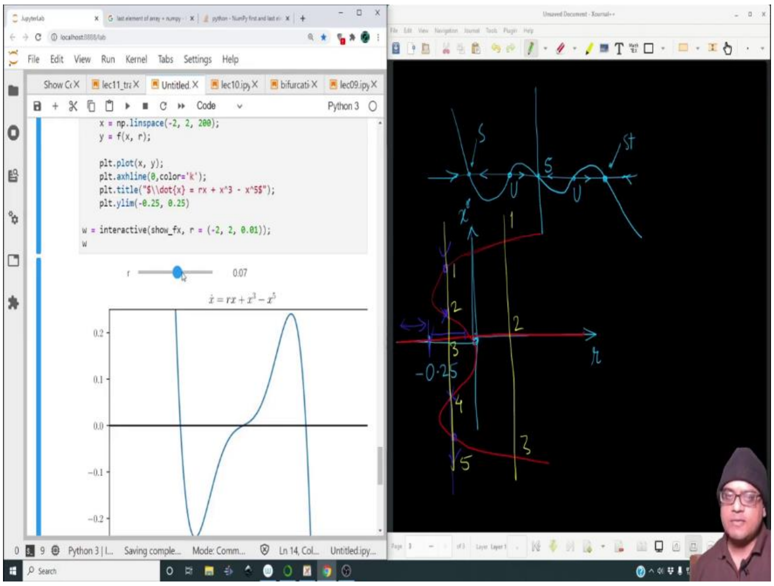
So, what is the stability of this point? So, obviously, this is a half stable point, this is also a half stable point. The trajectories are all going towards the left. So, origin is still stable ok, the trajectories will go towards the left, towards the right. So, the origin is still stable and what about this point? They are half stable.

(Refer Slide Time: 28:24)



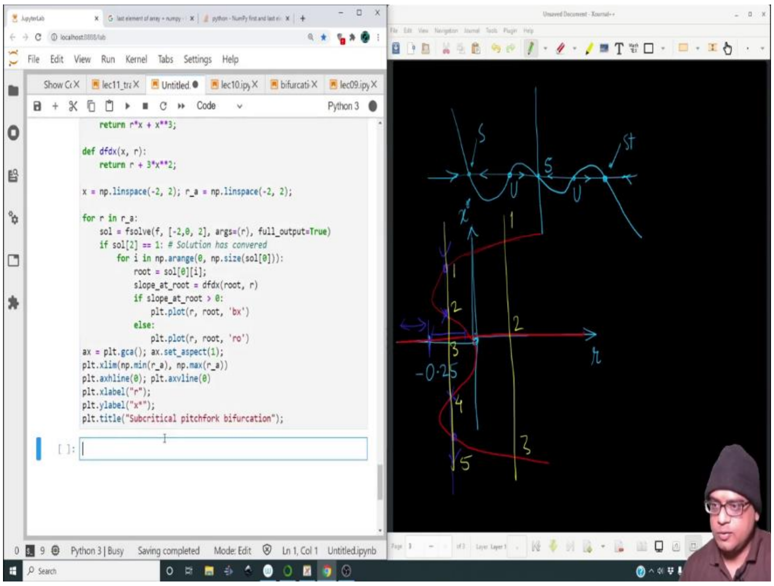
But, when we increase it further, then these end points they are stable, but these intermediate points they are unstable ok. So, the curve actually looks something like this. So, this these are all the fixed points till here. The origin is stable, then over here also the origin is stable, but these roots these particular roots they are unstable, but these roots are stable. So, we have five roots over here on this axis we have five roots; so, 1, 2, 3, 4, 5 ok. So, 1, 3, 5 are stable 2 and 4 are unstable beyond that we have 1, 2, 3.

(Refer Slide Time: 29:36)



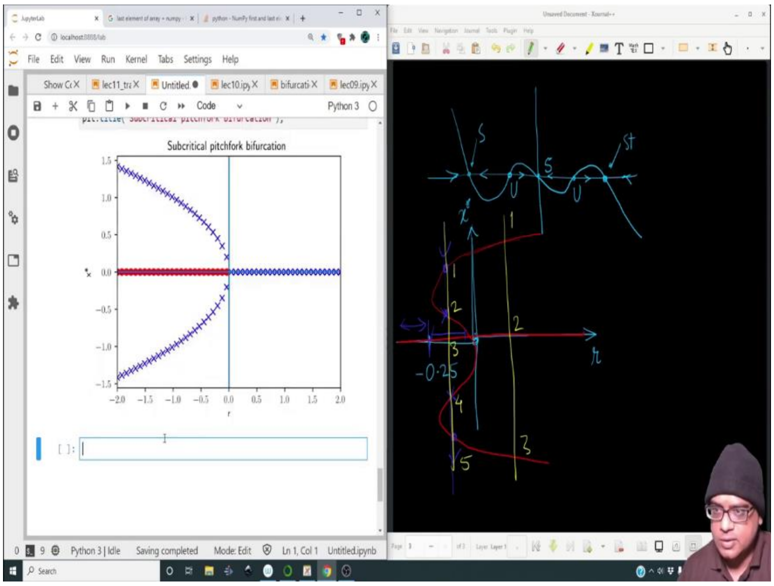
Let me increase r and let us see what happens. When you increase r we have only three roots and this is stable, this is stable and this is unstable. So, the origin becomes unstable. So, we have a whole lot of things going on. Let us find out the bifurcation diagram numerically. For this we can simply reuse the same snippet, no harm done.

(Refer Slide Time: 30:01)



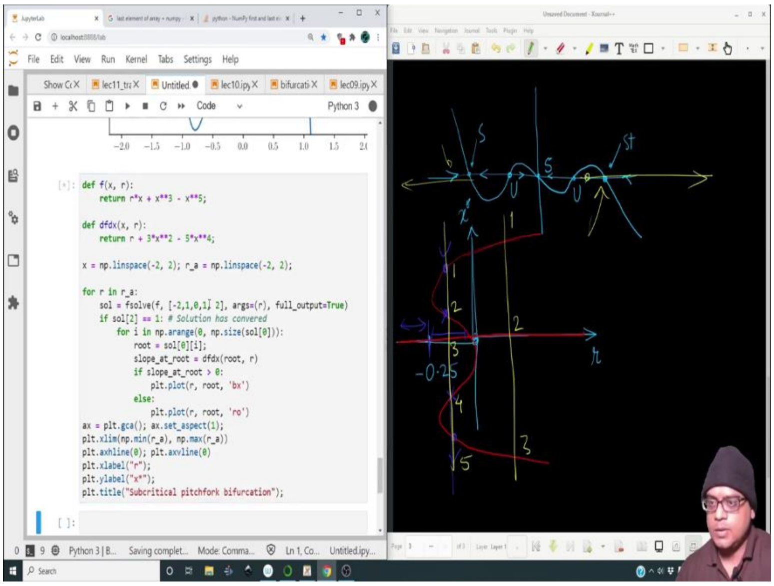
Let me go over here. Let me change the functional form. So, this is $rx + x^3$. This is going to be $r + 3x^2$. So, this is going to be a subcritical pitchfork bifurcation, there you go.

(Refer Slide Time: 30:30)



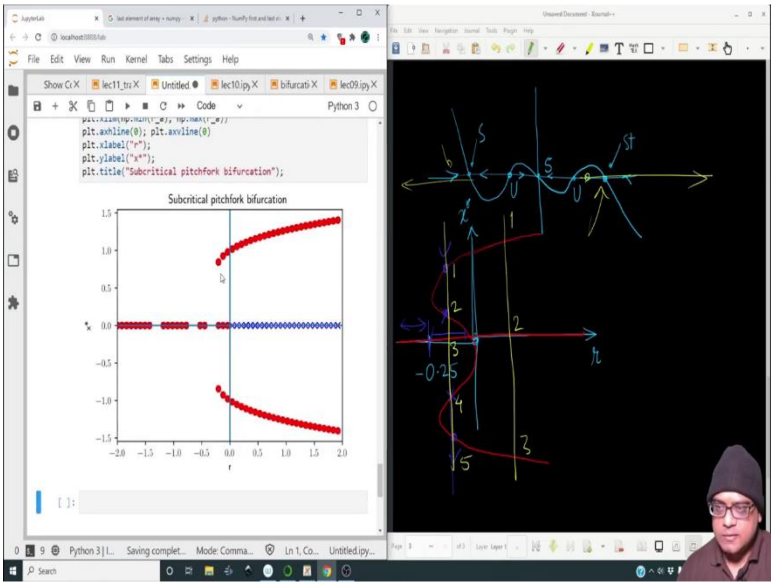
So, there was stable point it become unstable and these are the two other unstable branches.

(Refer Slide Time: 30:41)

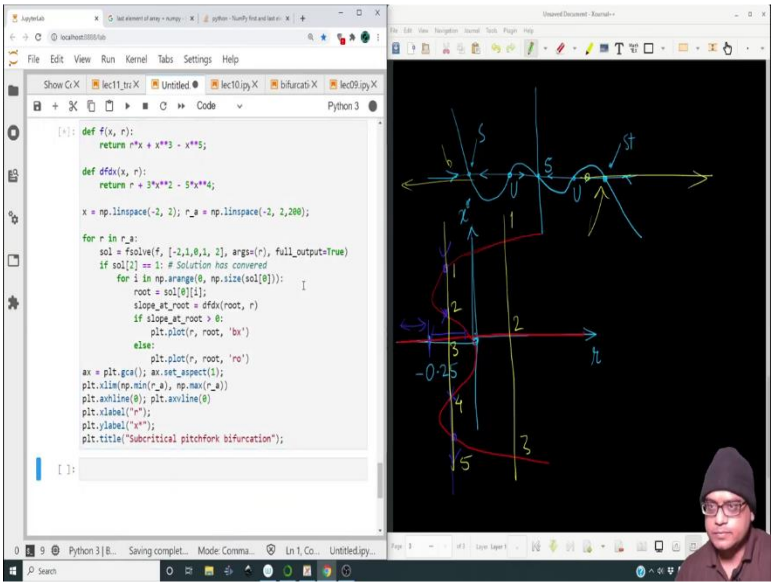


What happens when we put the additional term? This term is basically to make some far boundaries well behaved because of this x^5 behavior, roots which would fly off to infinity and minus infinity they are now bounded ok. If these roots were not there then obviously, the initial condition would fly off to infinity and fly off to minus infinity ok. So, once we have done this we know that there is going to be five roots. Let me put a few more guess points, let me run this. In fact, I think we have forgotten to do this ok.

(Refer Slide Time: 31:33)

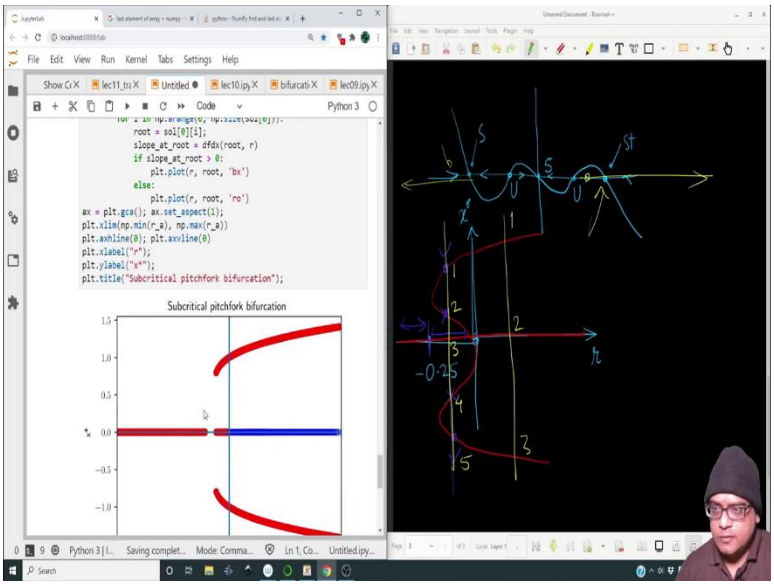


(Refer Slide Time: 31:46)



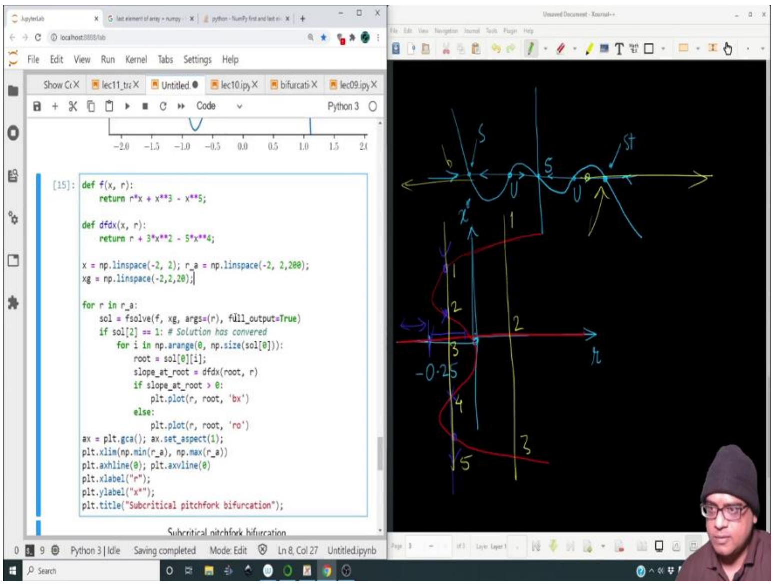
So, there is something which is going on let us use more points r has to have a sufficient resolution. So, r_a must have say 200 points now.

(Refer Slide Time: 31:51)



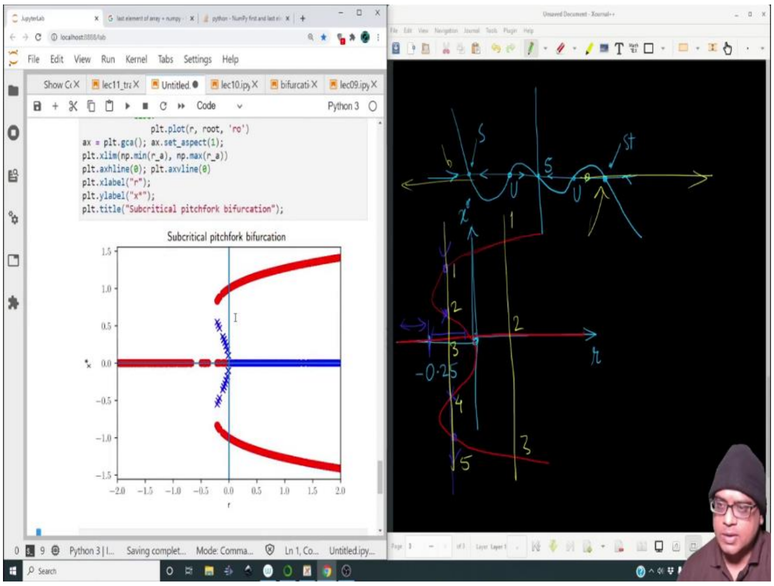
So, obviously, we are not getting those roots let us put some greater better guesses.

(Refer Slide Time: 32:04)



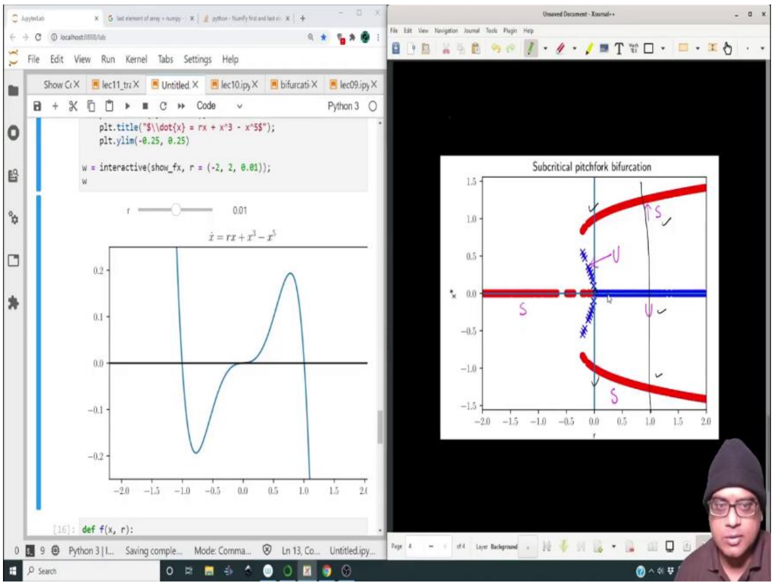
So, let me create a variable xg and xg = np.linspace(-2,2, 20), let us take 20 guess points. Let us run this.

(Refer Slide Time: 32:20)



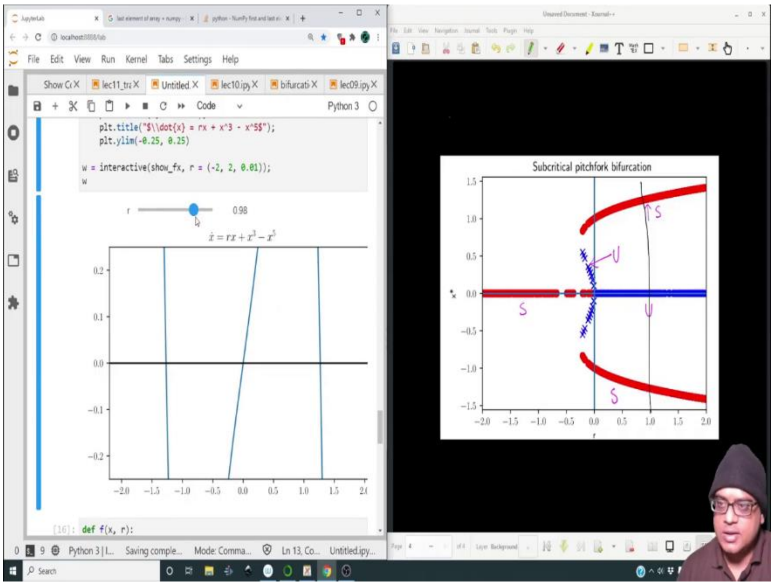
Great. So, there you have it. What does this diagram mean? What does this diagram mean? Let me open the snipping tool.

(Refer Slide Time: 32:44)



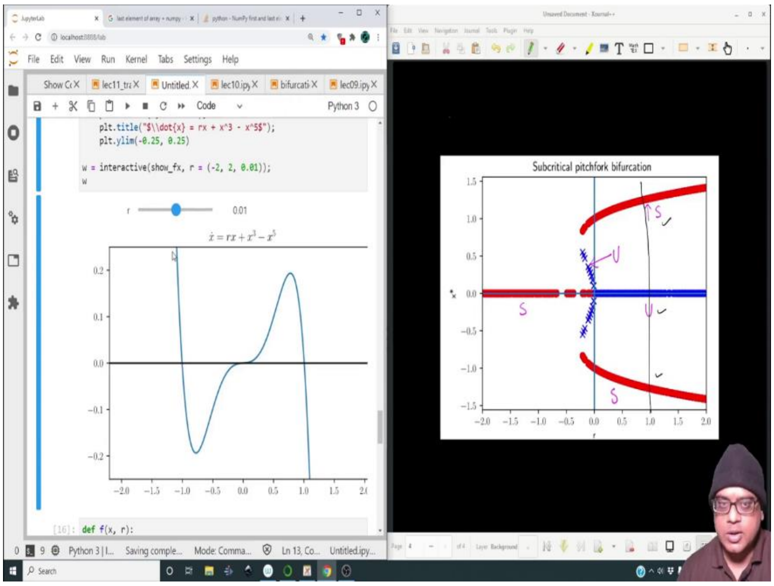
Let me take it over here. So, these are the stable fixed points, these are the unstable fixed points, these are the stable branches, this is unstable, this is stable and this is also stable. So, if I have this parameter r, suppose I choose this parameter r so, suppose r = 1 let us go over here oops let us go over here and set r = 1.

(Refer Slide Time: 33:26)



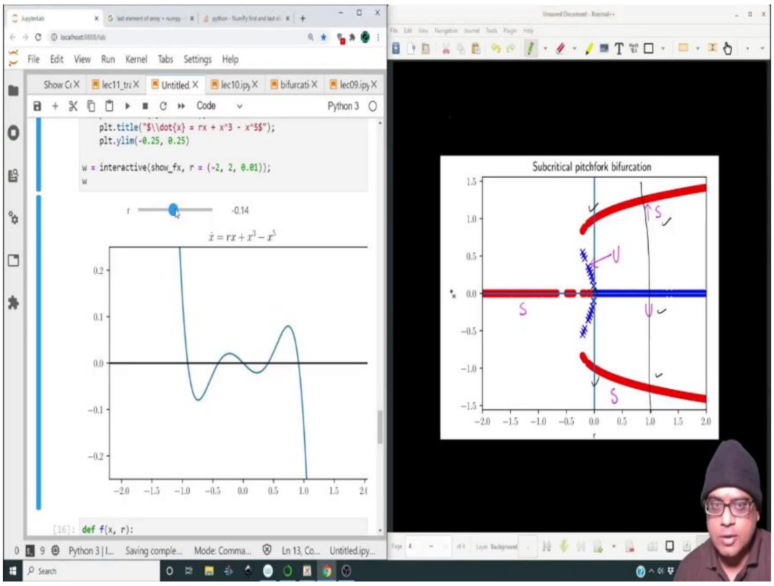
Now, at r = 1 quite obviously, the far roots are the stable ones while the origin is unstable. So, this makes sense.

(Refer Slide Time: 33:48)



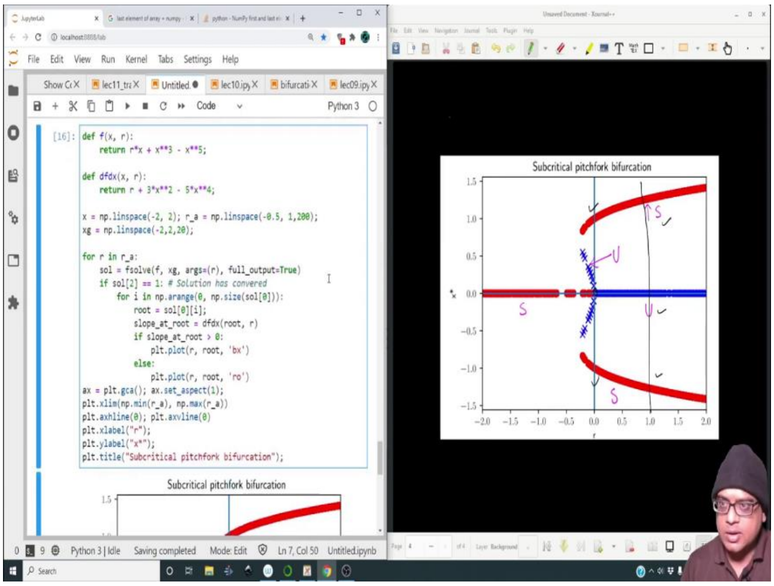
Let us now choose r = 0. At r = 0, the far points are still stable while the origin is sort of half stable ok. So, this is stable, this is stable, origin is half stable.

(Refer Slide Time: 34:00)



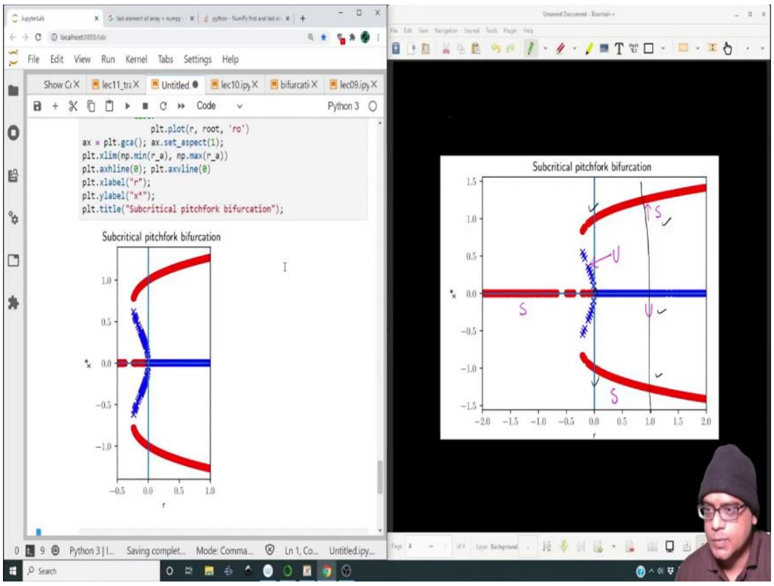
Let me reduce this. So, now, we have zones over here in fact.

(Refer Slide Time: 34:12)

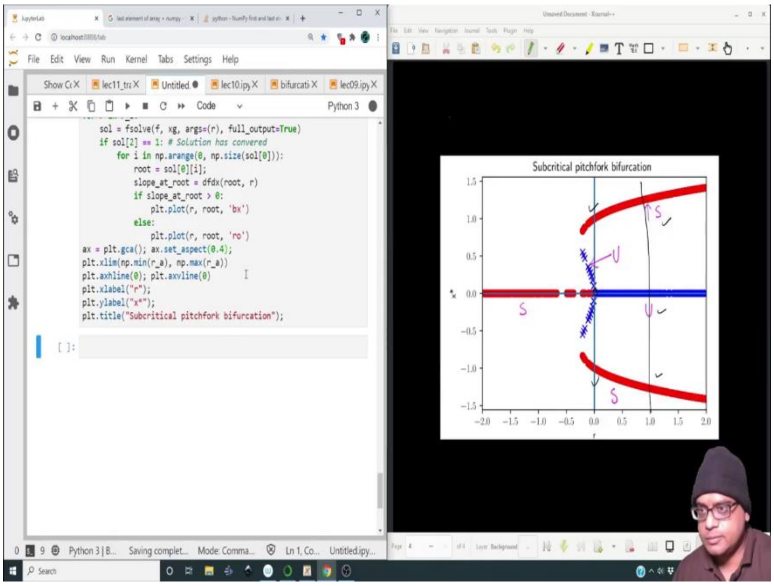


Let me reduce the r range from - 0.5 to 1 we do not need so many control parameter points.

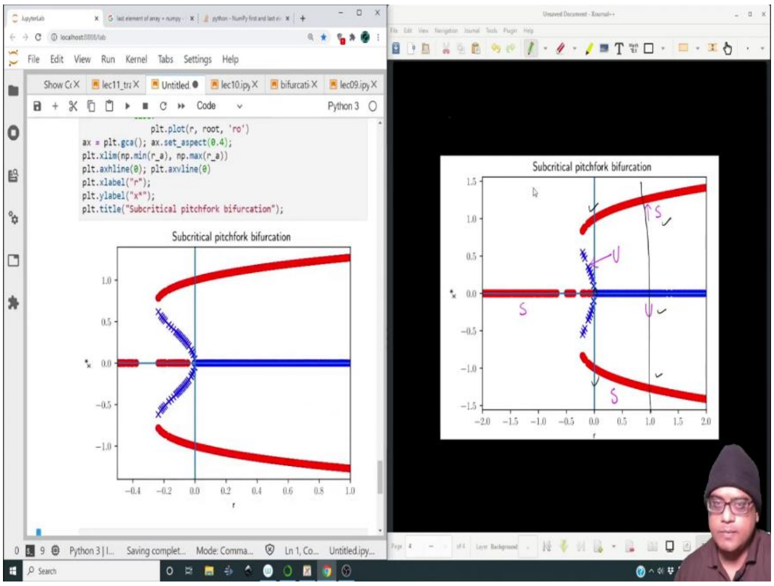
(Refer Slide Time: 34:24)



(Refer Slide Time: 34:31)

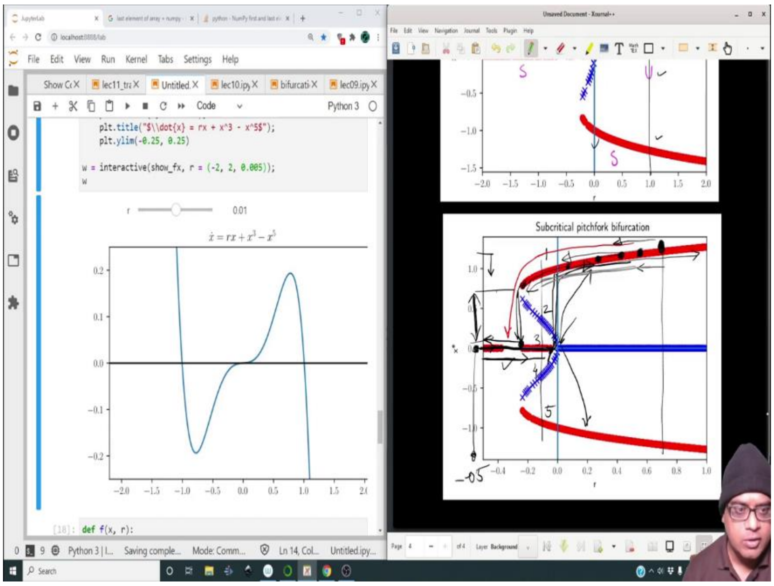


(Refer Slide Time: 34:37)



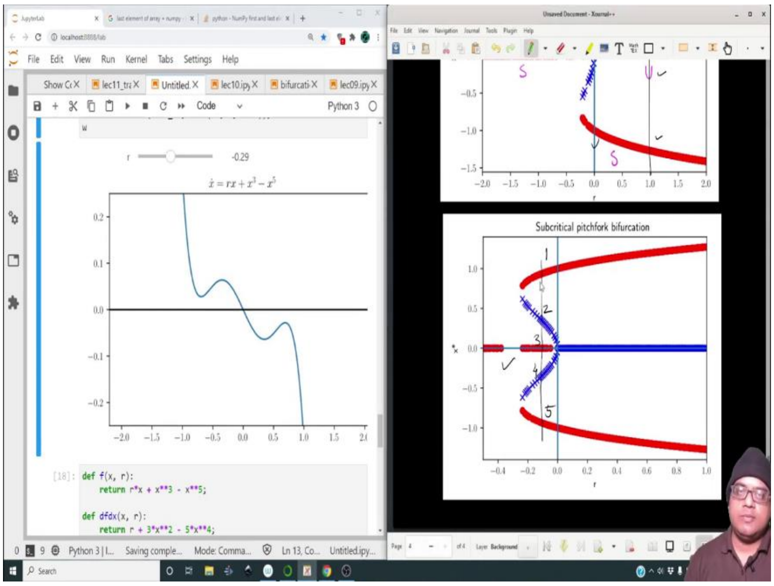
Let me change the aspect ratio to 0.4, yeah.

(Refer Slide Time: 34:49)



So, let me create a new snip so that we can discuss this better. Let me copy this yeah ok. So, over at these points so, when r is between - 0.25 and 0, we have five fixed points 1, 2, 3, 4 and 5 ok; So these are the five fixed points and out of that the far branch is stable, the inner branches are unstable while the origin is stable. The origin is stable over here. You can easily figure out and this is what it looks like.

(Refer Slide Time: 35:35)

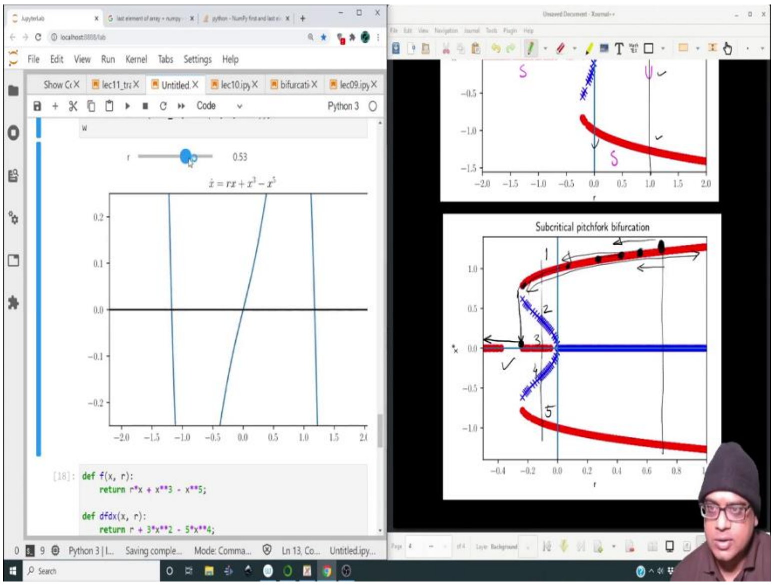


Now, when we keep reducing r then again we have only one root and the origin is stable. So, if we were to look at this branch if we were to look at this branch in its totality, if we were to have this stable point and if we were to change the control parameter from next stable point what would happen to this fixed point not the stable point to this fixed point?

So, if we have a this particular value of r so, the fixed point is over here. If I now reduce the value of r, the fixed point will still slide along this particular branch. This will be the fixed point, then if I reduce the value of r further this will be the fixed point, this will be the fixed point. So, when we are already on some fixed point branch ok, when we are already using a solution in which the initial condition gives us this particular fixed point ok.

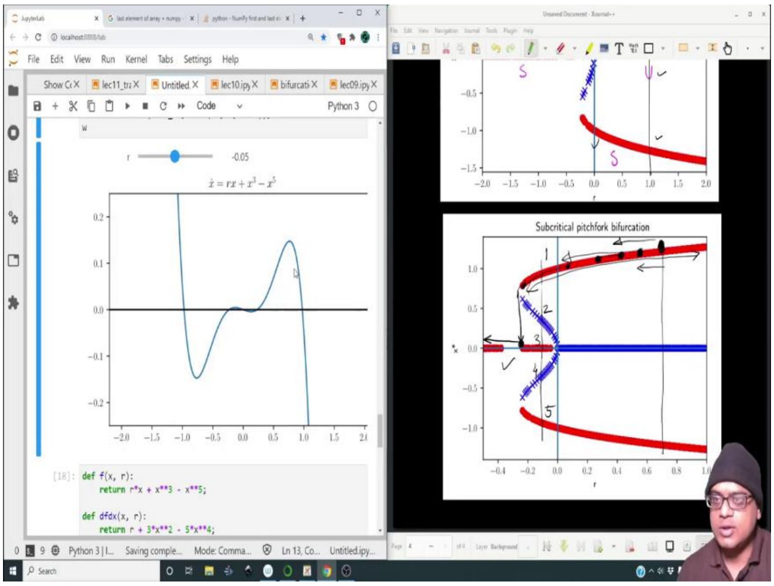
We have this particular fixed point and when we are reducing the value of r, we are sort of sliding on this particular branch up until this point. Now, at this point when we reduce the value of r further, it will simply jump from this branch to this stable branch ok.

(Refer Slide Time: 37:08)



And, then it as we reduce r further it will move along this let us again take a look when r is large the origin. So, this suppose we focus on this point ok. This is that stable solution this is the fixed point which corresponds to this equation.

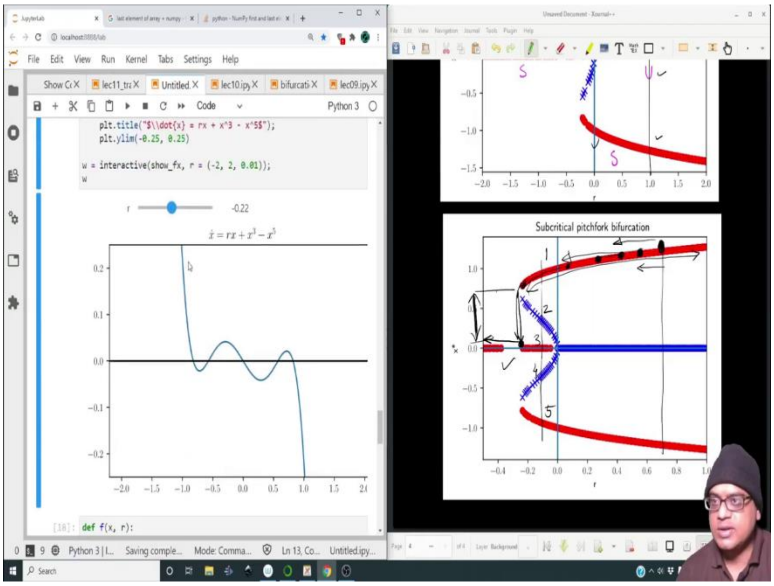
(Refer Slide Time: 37:25)



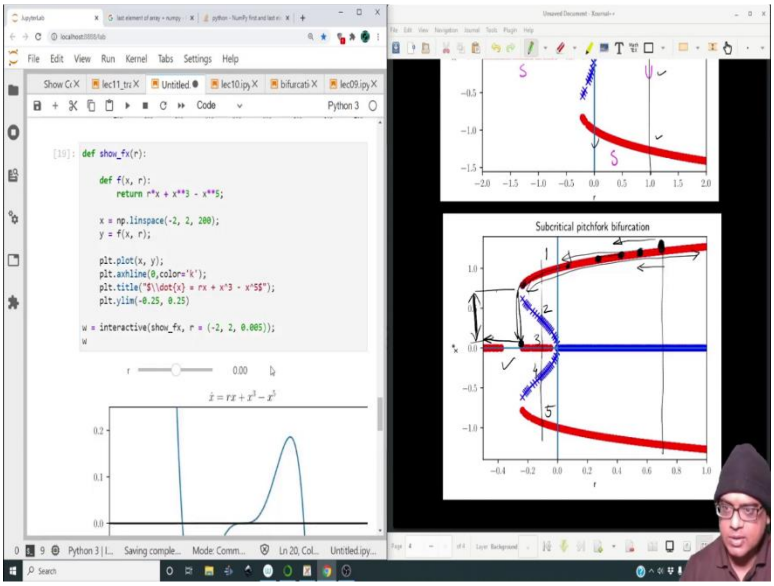
As r is reducing look at the stable point how it is shifting. It is reducing, it is reducing what I mean is look at focus on this particular point it is moving. So, the value of the fixed point is reducing ok. Let me go again. So, it is reducing, reducing, reducing, reducing and then suddenly now it becomes half stable and then it suddenly whatever fixed point was here it suddenly jumps to the origin. This is what is happening.

It is suddenly jumps to the origin. It does not have access to these fixed points ok. It never goes to the unstable branch ok. Once again you just focus on the right fixed point this. It is reducing, reducing.

(Refer Slide Time: 38:21)

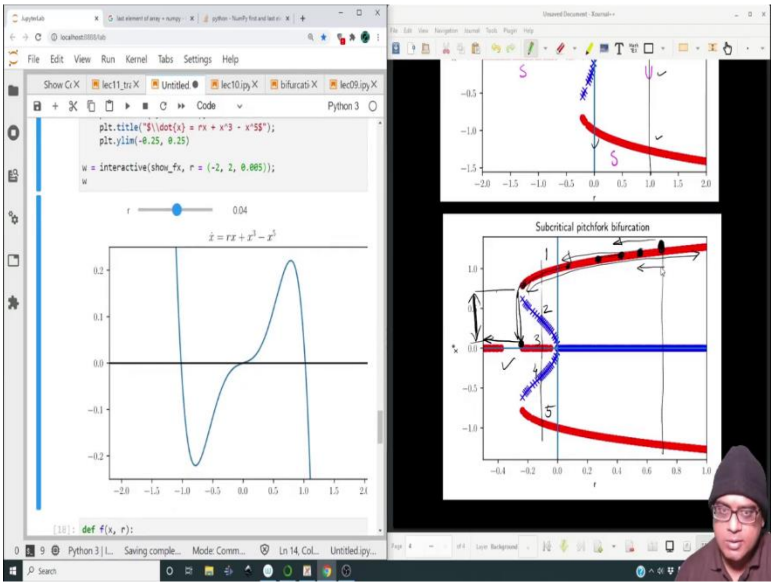


(Refer Slide Time: 38:26)



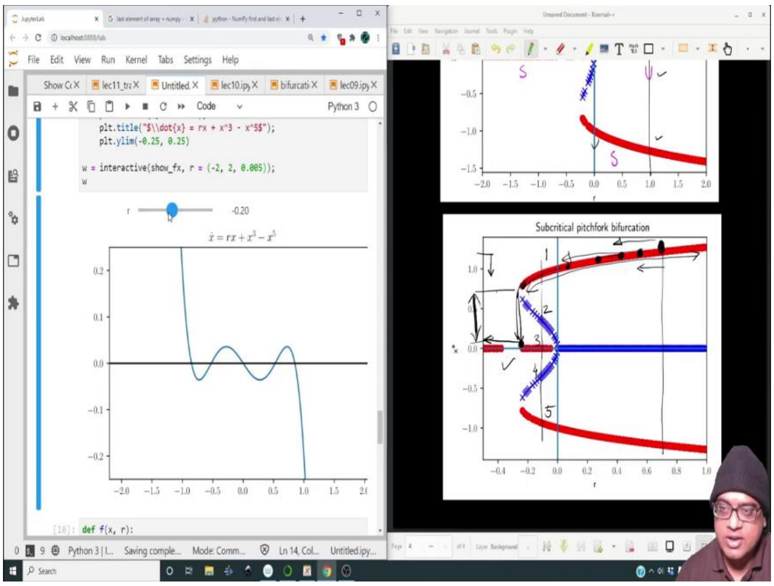
And, at this point let me increase the resolution of r further let me make it something like this ok. We start off with this; the fixed point is shifting towards the left meaning that the fixed point is reducing.

(Refer Slide Time: 38:35)



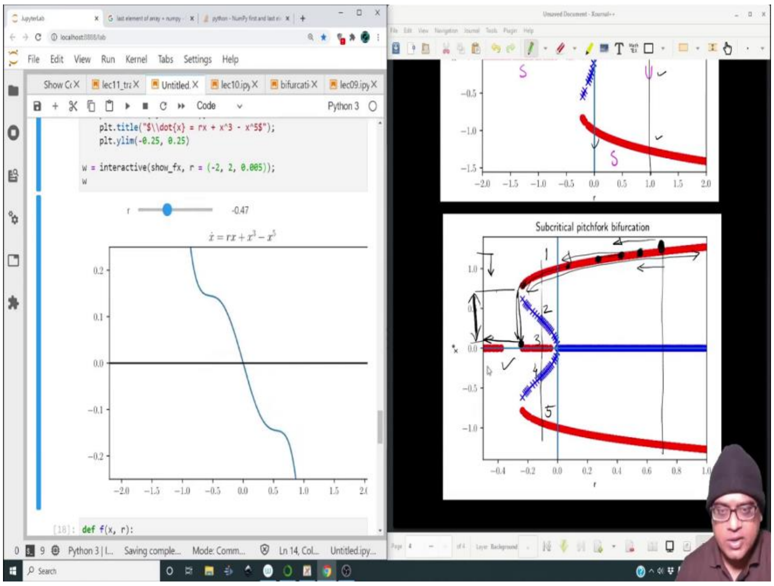
So, it is reducing the fixed point magnitude is reducing.

(Refer Slide Time: 38:48)



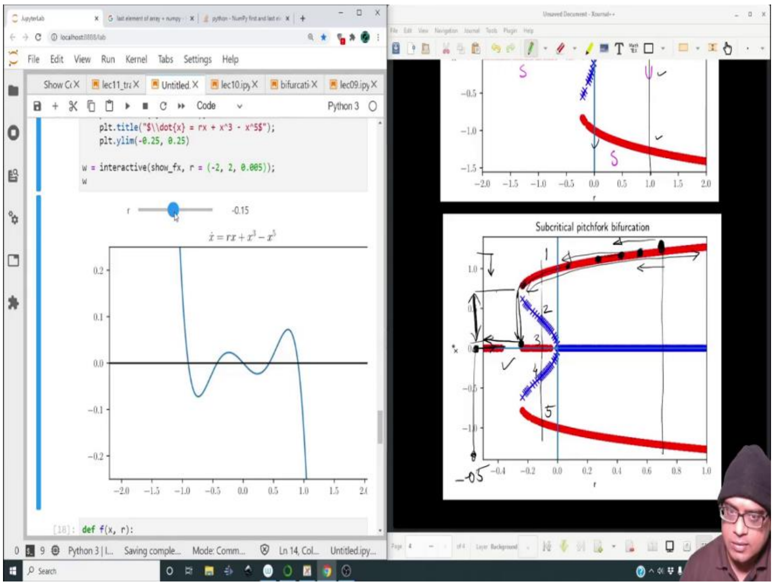
And, at a certain point over here, it undergoes a transition it becomes half stable and there is no more root and root particularly vanishes and suddenly jump towards the origin.

(Refer Slide Time: 38:58)

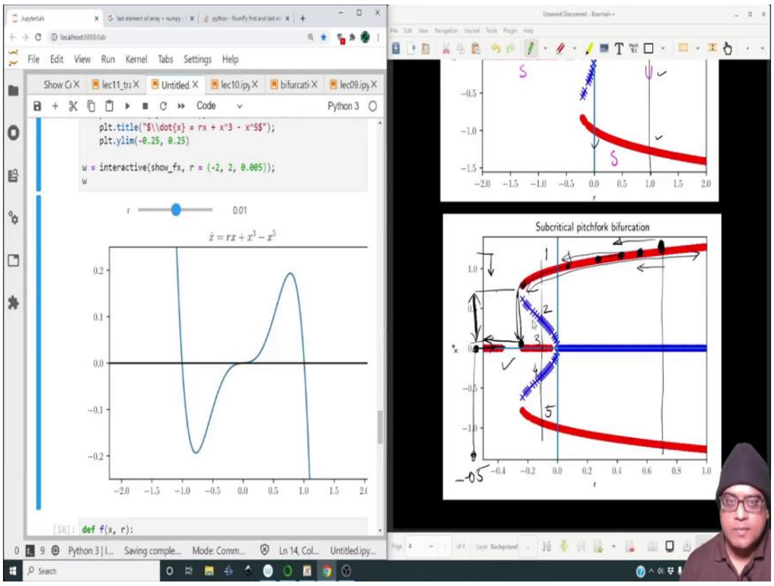


Inversely, if we start at this particular parameter say - 0.5 and we increase the value of r we are originally sitting at this branch; so, 0 0 0. So, focus on this point focus on this one. It is still 0s that is the only stable branch. Now, as we increase further oops, as we increase further suddenly 0 it is still attracting towards 0 ok.

(Refer Slide Time: 39:33)



(Refer Slide Time: 39:39)

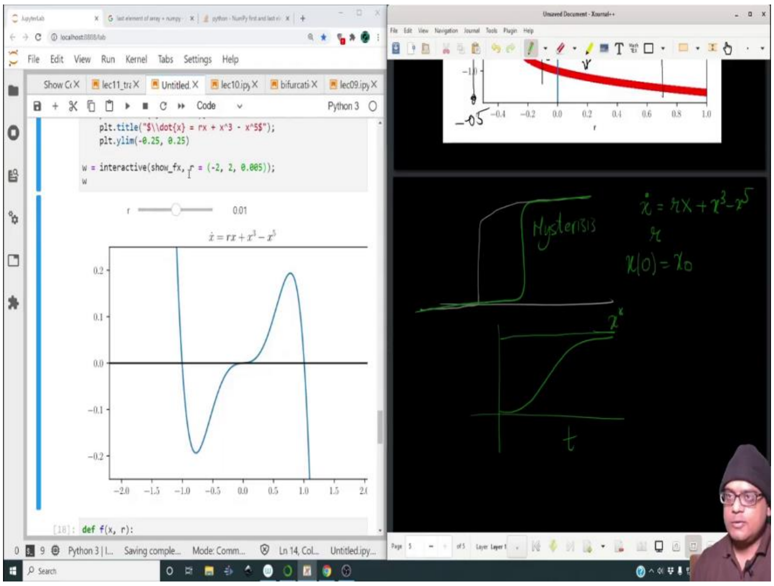


0 is still attracting and now 0 is repelling. So, it will move along this branch up until this point that fixed point will still be 0 and suddenly we will have a jump from this fixed point to these fixed points. It means that if we have initial conditions of 0, it would obviously, stay on 0.

I mean if we had a solution which would converge to 0 and we changed the value of r it would obviously, lie on this branch because 0 is not unstable, 0 is stable. So, it would stay on this branch up until this point, after this 0 becomes unstable. So, when the parameter crosses the value of 0 the x^* goes on to one of these two branches. And, these two branches are obviously, symmetric and they are characteristic of this subcritical pitchfork bifurcation.

So, now I have told you how this entire thing looks like and this is what is called as a catastrophe it suddenly goes from this stable thing and suddenly it jumps onto another branch ok. And, this particular thing is hysteresis because by increasing by reducing the parameter I am going on a curve which looks something like this.

(Refer Slide Time: 41:02)

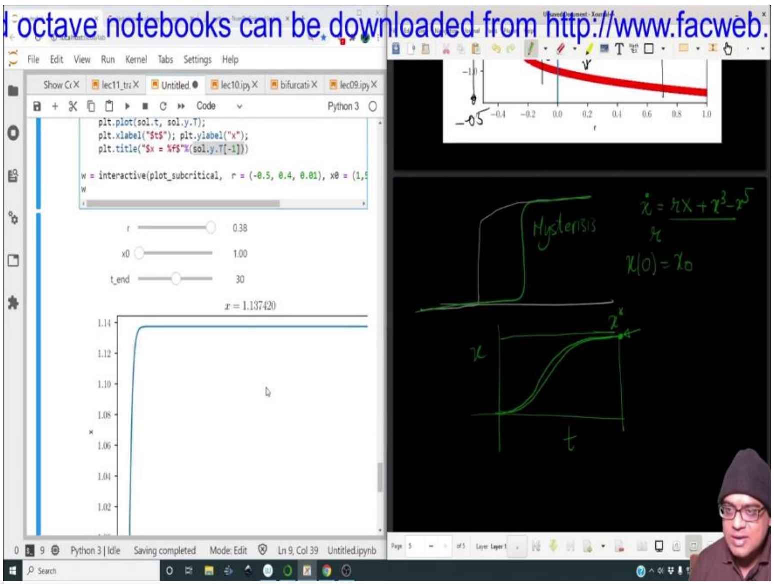


I am reducing the parameter it is going something like this, but when I am increasing the parameter it looks something like this.

So, the fixed point is jumping around branches, but it is not doing so in a reversible fashion and this thing is called as hysteresis. So, I have said this many times that the evolution of the equation goes on to a fixed point $\dot{x} = \mathrm{rx} + x^3 - x^5$ for some value of r with some initial condition what do we mean that it goes to a fixed point?

It means that in time it settles on to some value of x and this is the fixed point. So, now we should be able to solve this initial value problem and assess how that jump actually happens we should be able to see that jump, let us see let us find out. So, for this we have to solve the differential equation.

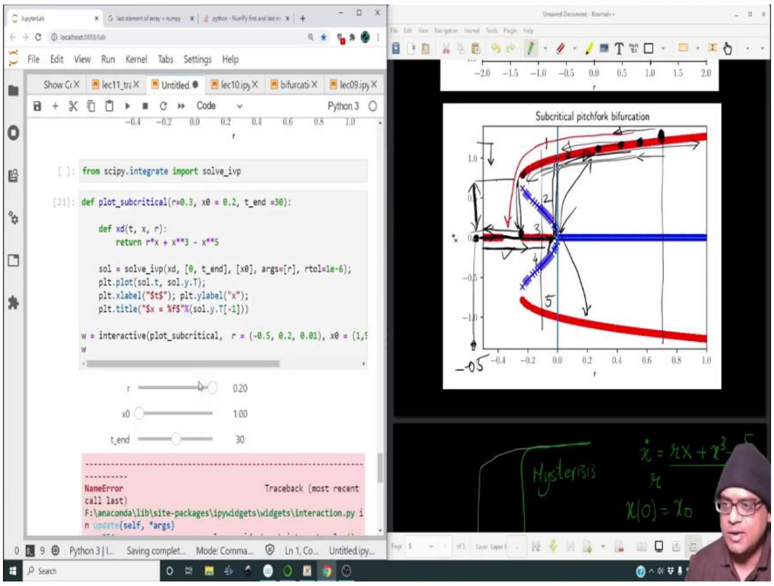
(Refer Slide Time: 42:19)



And, we have already written an ivp code. So, I have gone ahead and made this particular function it is based on the logistic map that we had done in a previous lecture. I have simply modified that function, so that it takes the parameter r and the initial condition and the time at which we should integrate this.

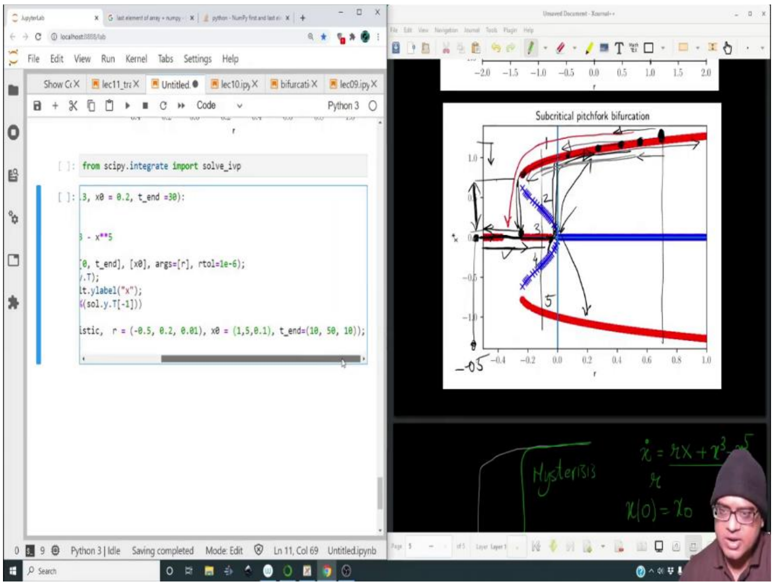
So, t end is this particular time. I am defining the right hand side as this. So, this function is called as xd and I am passing that to the solve ivp function. So, solve ivp solve ivp function is something which integrates the differential equation. It helps us in solving initial value problems.

(Refer Slide Time: 43:06)



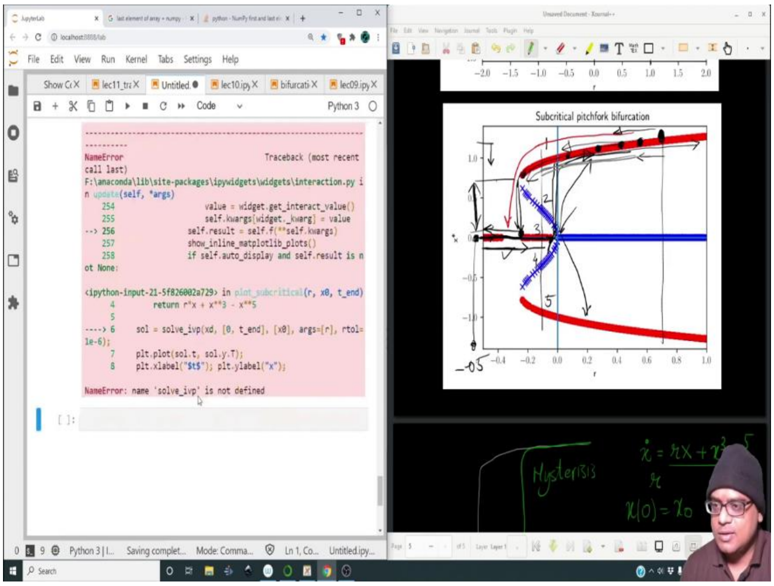
So, before this we have to import solve ivp from scipy. integrate ok. So, this is the this is how we pass the arguments to this. So, the slider it contains a slider for r let me go from -0.5 to 1 or say 0 point and we can go to 0.5 also because by 0.5; in fact, we can go to 0.2 as well, it makes lot of big difference; let me take 0.01 ok. Let us take initial conditions going from minus 0 point or let us take initial conditions from 1 to 5 in steps of 0.1 ok.

(Refer Slide Time: 44:13)



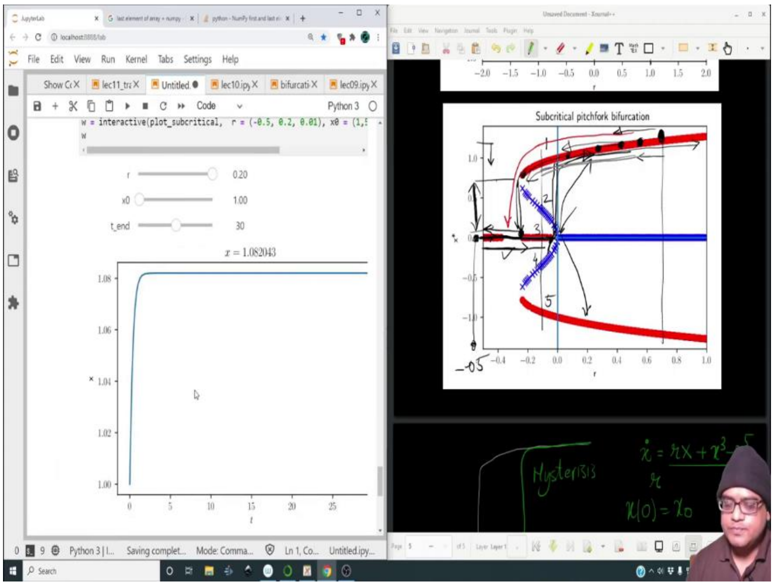
t end takes steps from 10 to 50 in steps of 10. So, let us run this.

(Refer Slide Time: 44:17)



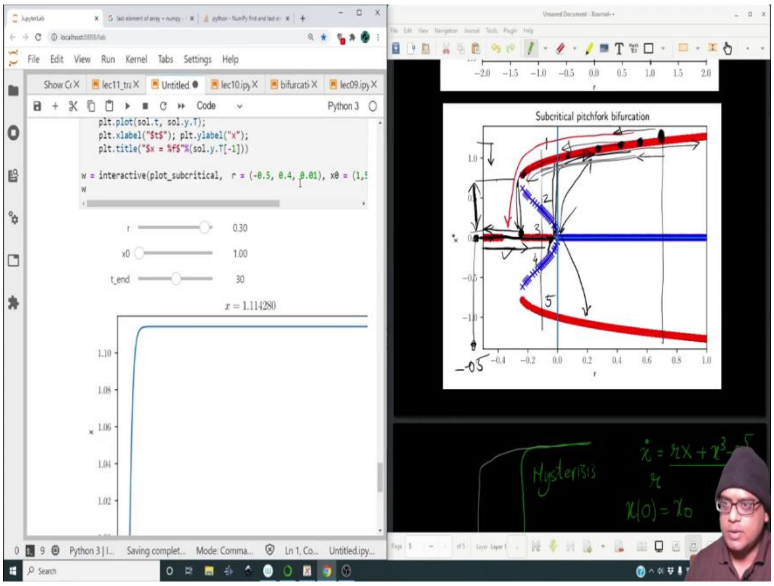
Oops. So, this has to be plot subcritical. So, that is the danger of reusing code. If you make changes in one name make sure you make changes everywhere. There is still some error solve we have not yet run the cell.

(Refer Slide Time: 44:43)

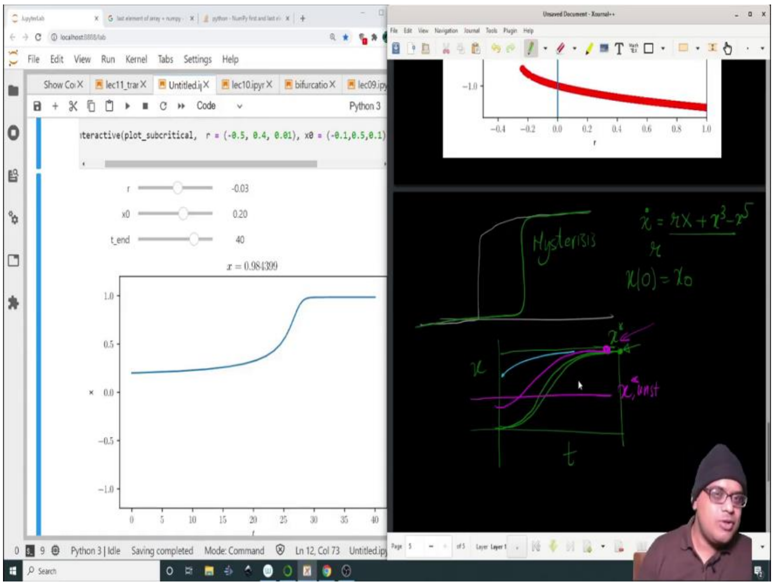


So, we have not imported solve ivp ok. So, now, we have it. So, let us look what happens. When we have an initial condition of 1, if function for this particular value of let me make it 0.4 ok.

(Refer Slide Time: 45:02)



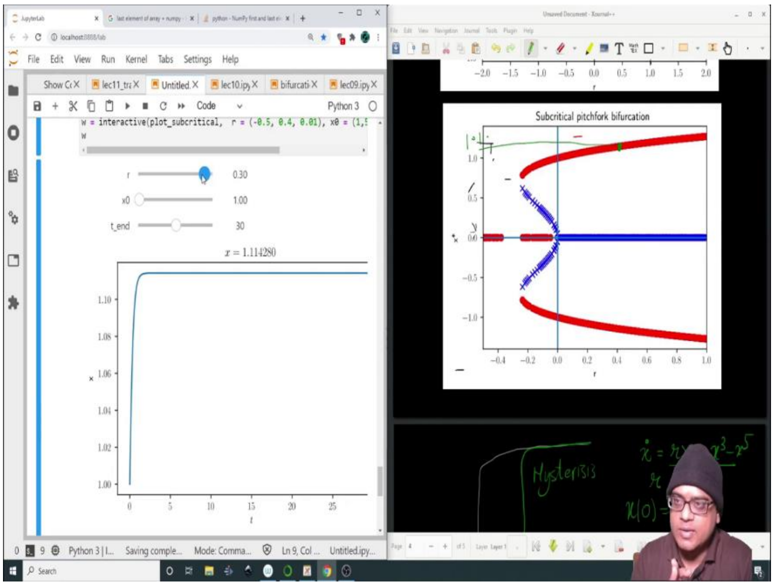
(Refer Slide Time: 45:10)



So, when r is 0.4 when r is 0.4 they I am I am plotting the last I am plotting the last point that the solution returns. So, by that I mean that when the solution returns me this entire array, I am fetching this last value and I am plotting it because that last value if it is if it asymptotically converges towards that point then it is the fixed point.

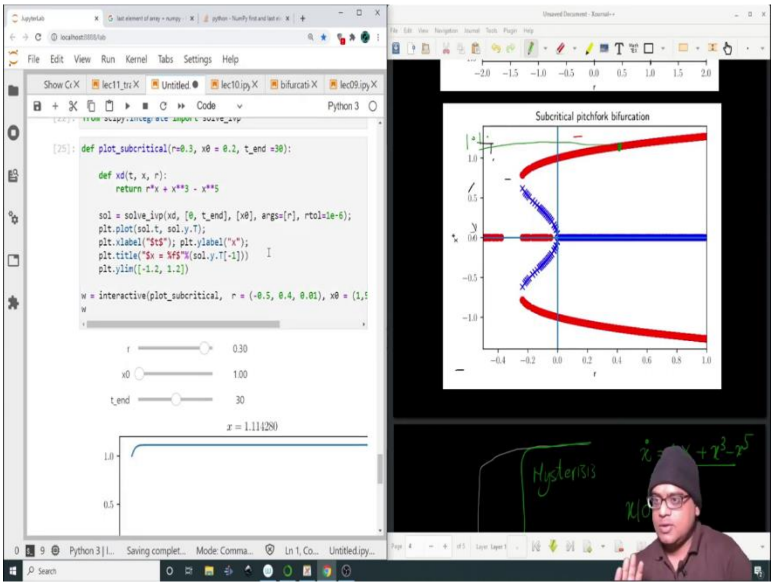
Essentially this entire value of x is the fixed point. So, that fixed point is 1.137420 ok. So, I am plotting that as well. Now, with that fixed point I am able to see that when x = 1, I am reaching that fixed point. So, let me erase some of these lines.

(Refer Slide Time: 46:18)



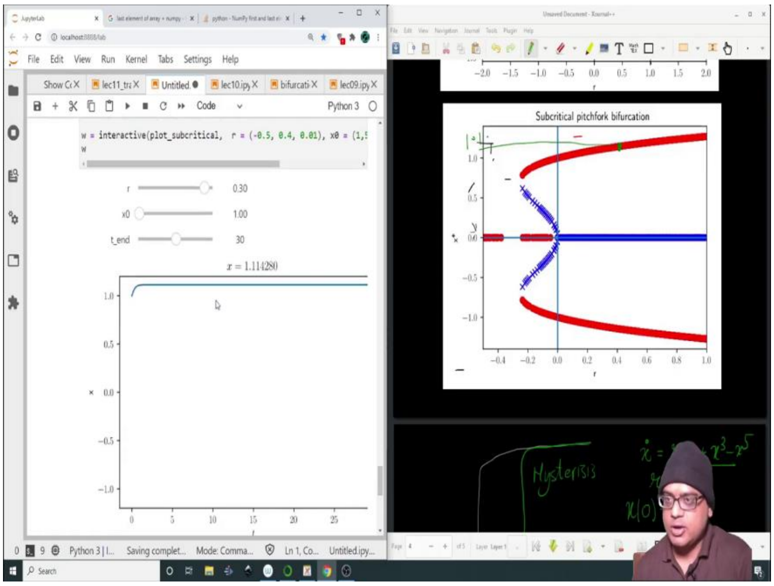
So, when r is 0.4, I am reaching this fixed point and obviously, this fixed point is 1.1 something which checks out. Now, let me slowly reduce the value of r. Look the value of the fixed point is also reducing.

(Refer Slide Time: 46:31)



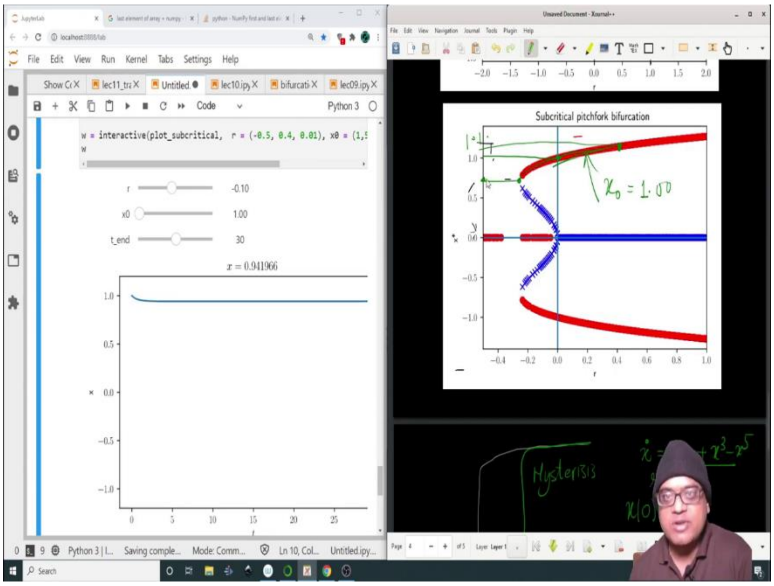
In fact, to better assess that let me fix the y-axis or let me fix it from - 1.2 to 1.2; so, then that the y-axis is locked ok.

(Refer Slide Time: 46:41)



So, when it is 0.4 this is the value when we reduce the value of the fixed point is reducing ok. It is reducing, it is reducing.

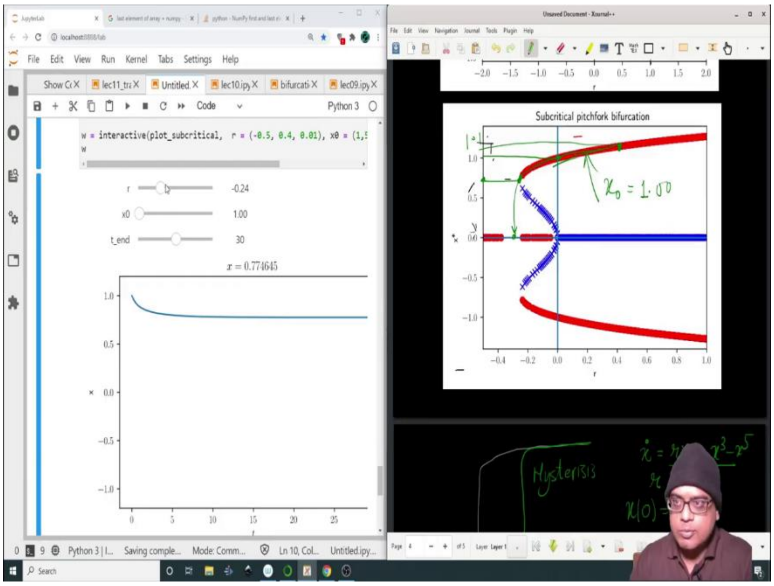
(Refer Slide Time: 46:48)



Now, we are reaching r= 0. No problem, we are still reducing, reducing, reducing ok everything is fine till now, it is still reducing. So, now, what is happening? We are sliding along this curve, we are obtaining fixed points. So, all these points are obtained using an initial condition of 1 ok, but we are still reducing r the fixed point is still reducing. Look this is the $x^* = 1$ line.

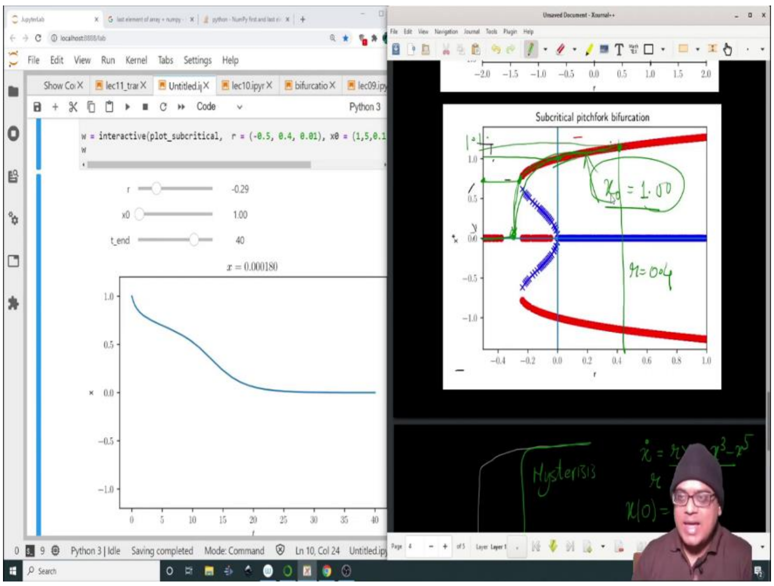
So, we are still reducing r. We are at - 0.1 ok somewhere over here we will still reducing and we expect to reduce up until 0.75. Let us see whether that happens or not; still reducing and still reducing, still reducing.

(Refer Slide Time: 47:50)



We are coming close to the fixed point of 0.75. Now, what have happens I mean now what happens? We have we have reached this point. Now, we expect that once we reduce r further we will suddenly jump onto this curve. We should suddenly see the fixed the fixed point of this system to jump to 0 ok.

(Refer Slide Time: 48:18)

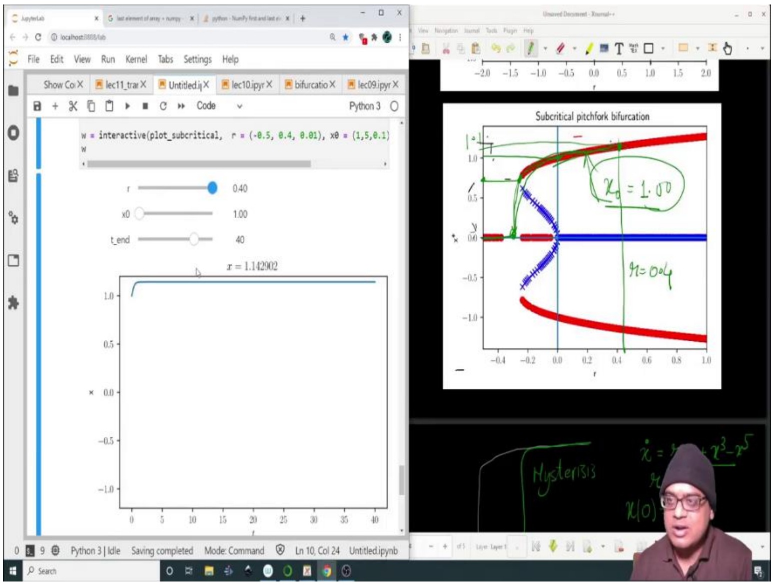


Look it has I have reduced r further now it is taking this path and it is asymptotic to 0. So, the final value of this equation the asymptotic value of this equation is going towards 0. Let me increase the size further ok. If at all I think that it has not reached the asymptote let me increase the t end. Increasing the t end will clearly show that it is trying to go towards the asymptote.

So, we have successfully shown that this particular branch jumps to this particular branch in a catastrophic manner ok. You can see how the curve asymptotically goes to 0 just by sliding the parameter. Remember that is entire branch we have obtained by using an initial condition of 1.0.

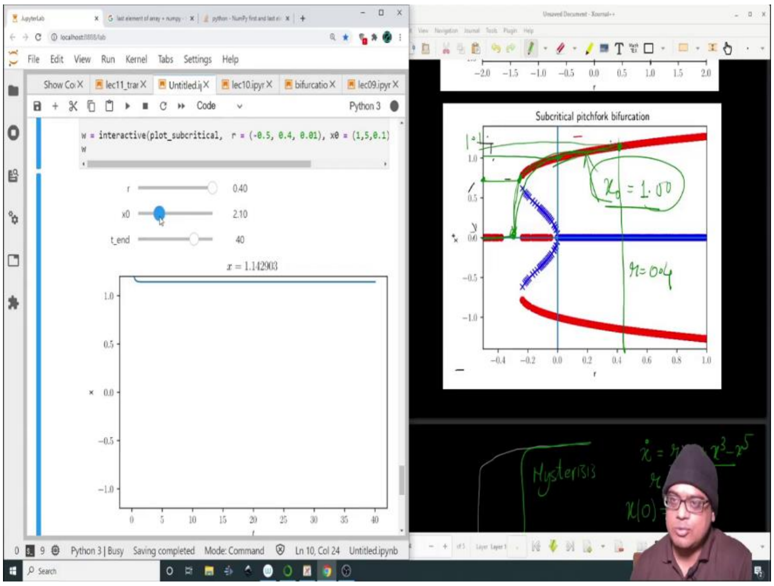
Because when we started the set of simulations we started with a parameter of r = 0.4 and we started off with an initial condition which gave us that final value of x that is the fixed point for this value of the initial condition.

(Refer Slide Time: 49:32)



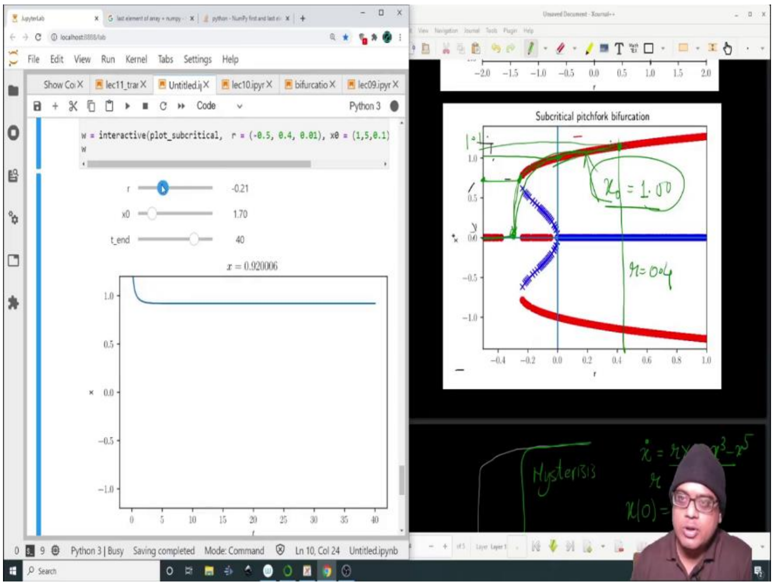
Actually, let us do this again. It should not depend on the initial condition at all. Let me change this ok. So, it is independent of the initial condition even if I change this asymptotic value is independent because those initial conditions sort of settle on asymptotically to that fixed point ok.

(Refer Slide Time: 49:36)



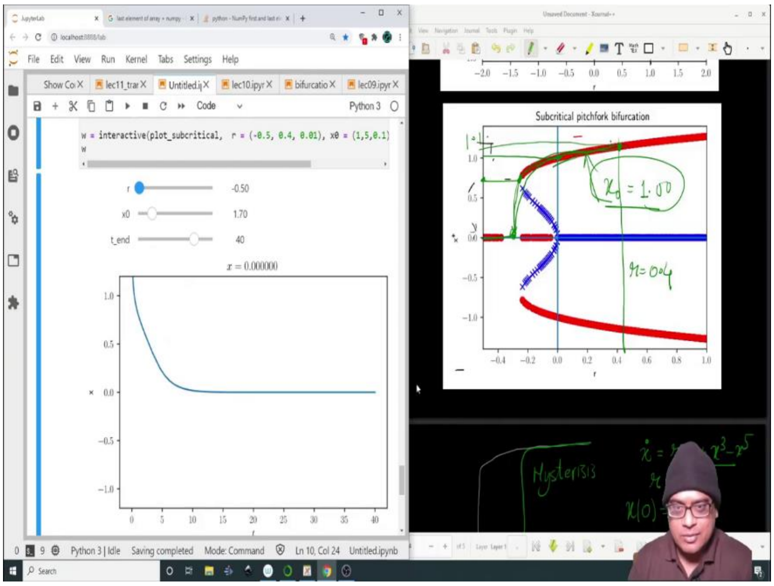
So, let me start with a different initial condition, let me reduce this. So, you see it is again doing that thing the fixed point is reducing that is the asymptotic value of x is reducing.

(Refer Slide Time: 50:03)



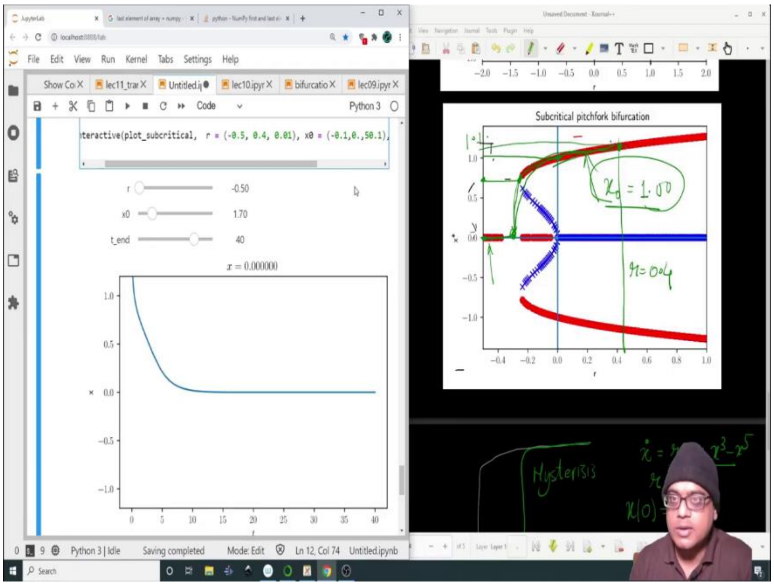
And, at a certain point it will catastrophically drop to 0.

(Refer Slide Time: 50:07)



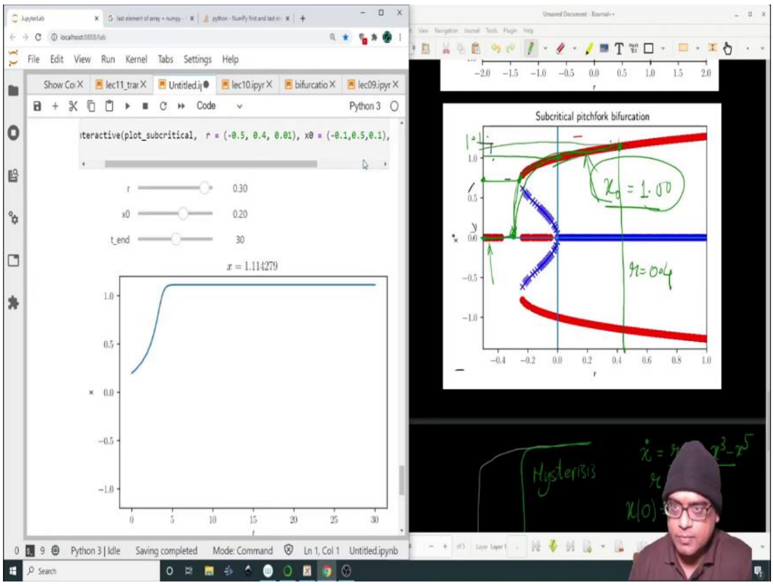
Is reducing, reducing catastrophic drop to 0 ok. I hope it is clear. Now, let us do the inverse. Let us start with an initial condition over at this point which starts at 0 and let us increase the value of r, let us see what happens.

(Refer Slide Time: 50:31)

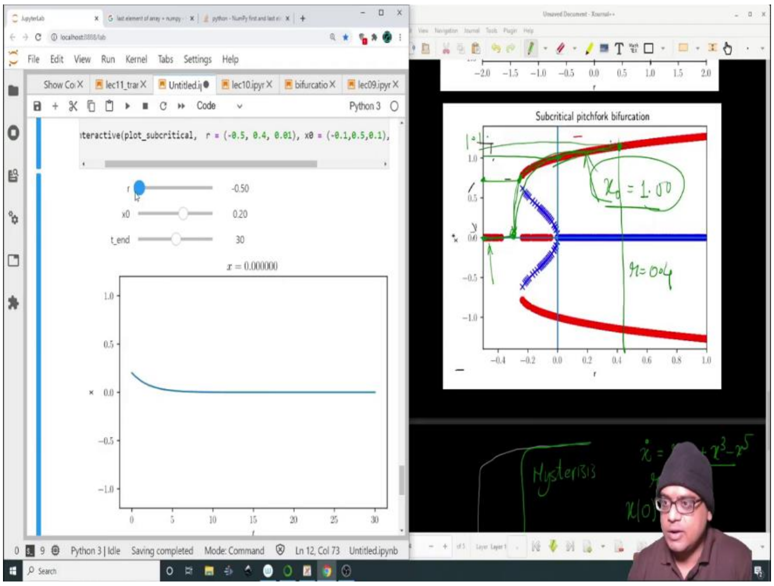


So, let me choose let me in fact, get these initial conditions from - 0.1 to 0.5 ok. So, I what I am trying to do is to find out the other hysteresis branch ok.

(Refer Slide Time: 50:41)



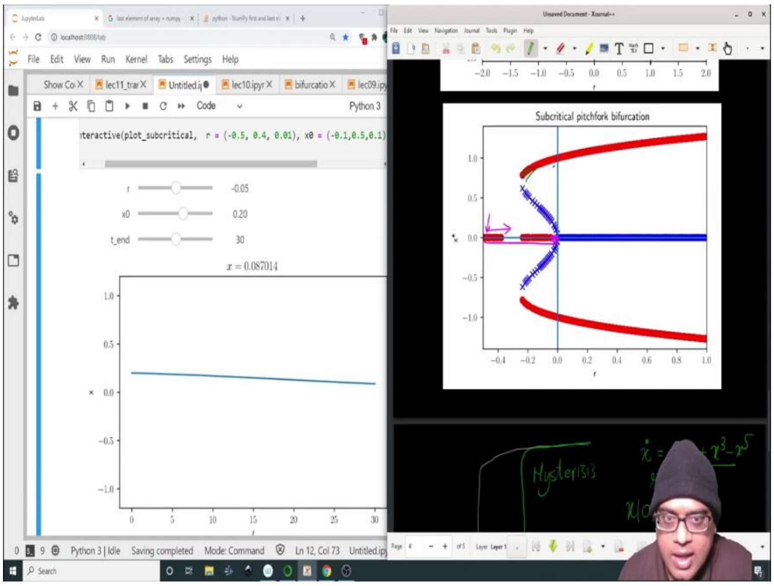
(Refer Slide Time: 50:46)



Let me choose some negative value of r and x0 as 0.2. So, now, we will start with the other branch. We will start so, let me rub this off and if you are not convinced over about whatever is going on, you can try all these things you can make the sliders on your own and what the other thing that you can do is watch this video again and get a feel of it because this is quite important, understanding hysteresis is quite important.

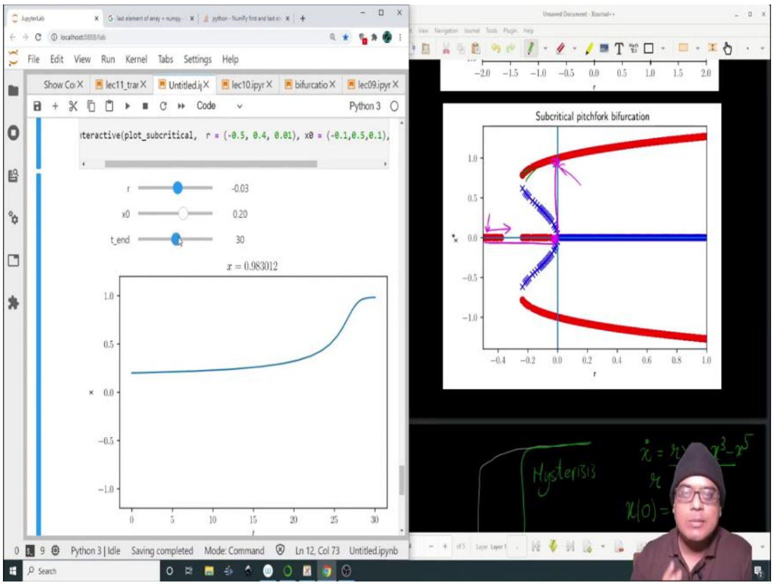
So, now we start over here for a negative value of r for initial condition of 0.2 we have an asymptotic value of 0. Now, what we expect? Keeping the same initial condition when we increase the value of r we should stay on the same branch. Let us see whether the equation still asymptotes to 0.

(Refer Slide Time: 51:40)



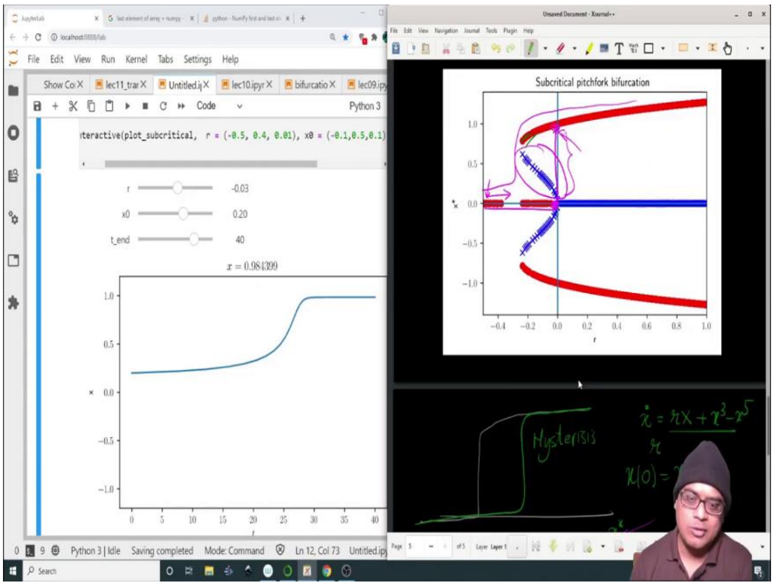
We are increasing r it still asymptotes to 0, still asymptotes to 0, still asymptotes to 0 and at r = 0 something happens.

(Refer Slide Time: 51:51)



So, at so, we have reached this point and now suddenly we expect that this stable point should jump all the way to some value over here and let us see. 0 0 ok sorry, yeah 0 and then suddenly we see that there is a jump in the asymptote and if you are not convinced whether a t end of 30 seconds is able to capture that asymptote, we can increase the slider further.

(Refer Slide Time: 52:20)



So, now, you see that at for this particular value of the parameter there is a sudden jump and this signifies this particular jump, ok. I repeat. The fixed point that you see in whatever discussion is like the asymptotic value of x that the ODE will give us over very long time ok. It is the fixed point meaning whatever the condition is it will go towards that point. So, this is the fixed point.

And, we have shown through integrating explicitly the differential equation how that fixed point shows that jump catastrophe and hysteresis. This is wonderful because Python allows us to change the sliders at will and see how the transition actually happens.

Just saying this that when you move on from this to this, it is not entirely convincing to some people we are completely missing out on this unstable branch and why would the point go to the unstable branch that unstable branch means that those set of fixed values will never be reached when we integrate the differential equation.

Suppose, this is the unstable x star our differential equation will never try to go towards here because it is unstable. Our differential equation will try to go towards this asymptote always. So, it will never go to that unstable branch. So, this is what all the physical meaning of whatever is going on ok.

The fact is python or even octave they give us they are enabling us to analyze these equations in a much user friendly manner than what was previously available ok. It helps us in visualizing the solutions in a more obvious fashion. And, with this I end this wonderful lecture and I will see you again next time with imperfect bifurcations. Bye.