Advanced Concepts In Fluid Mechanics
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Lecture - 09
Reynolds Transport Equation

I. Reynolds Transport Theorem

In fluid mechanics and transport phenomenon, conservation equations form the majority of the governing equations. Conservation equations can include mass conservation, momentum conservation, energy conservation. Any of these conservation equations can be written in one of two forms – differential form and integral form. The differential form expresses the particular conservation equation at a particular point and is useful in obtaining generic governing equations. On the other hand, in many engineering situations when one is concerned with the gross behaviour of flow through a device rather than in pointwise detail, it is beneficial to express the conservation equations in their integral form.

So, to write the integral form of conservation equation, what we essentially use is the Reynolds transport theorem which converts the control mass based conservation equation to its control volume counterpart, in an integral sense. We now obtain the formal expression for Reynolds Transform Theorem.

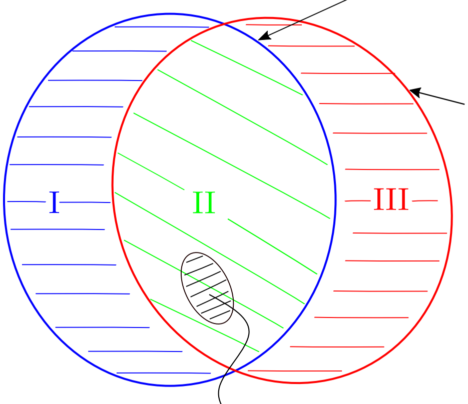

Figure 1: A Control Mass System as it flows over time Δt .

Let us consider a control mass system as illustrated in Figure 1. We consider the transport of a particular extensive property N , 'extensive' implying depence on the extent of the system, i.e. magnitude of N is dependent on the total mass of the control mass system. Extensive property per unit mass is called as intensive property or specific property, and is denoted by n . This extensive/intensive property could be a scalar or a vector. Our objective is to obtain an expression for $\left.\frac{dN}{dt}\right|_{\text{system}}$ in terms of the corresponding changes with respect to the control volume. The utility of doing this derivation lies in the fact that setting N to be the mass of the system gives us the equation of conservation of mass in the Eulerian description, setting it equal to momentum gives us the momentum conservation equation, and so on.

In translating the system conservation laws from control mass to control volume, we will require some correction terms, which will be established through the Reynolds transport theorem. To obtain these expressions, consider the control mass shown in Fig. 1. This control mass occupies the space I+II at time t, and flows to occupy the space II+III at time $t+\Delta t$. We consider the common region of the two configurations, I+II and II+III, i.e. region II as our region of interest, i.e. the control volume For a vanishingly small Δt, the two configurations, I+II and II+III overlap sufficiently to consider the two to be approximately the same, i.e. the control mass system and control volume converge together with a little bit offset regions (which are essentially the correction terms in the translation from control mass to control volume).

Therefore, $\left.\frac{dN}{dt}\right|_{\text{system}}$ can be written as

$$\left.\frac{dN}{dt}\right|_{\text{system}} = \lim_{\Delta t\to 0}\frac{N_{t+\Delta t}^{II+III}-N_t^{I+II}}{\Delta t} = \lim_{\Delta t\to 0}\frac{N_{t+\Delta t}^{II}-N_t^{II}}{\Delta t}+\lim_{\Delta t\to 0}\frac{N_{t+\Delta t}^{III}}{\Delta t}-\lim_{\Delta t\to 0}\frac{N_t^{I}}{\Delta t}. \tag{1}$$

In equation (1), the LHS is the time-rate of change of the property N for the control mass and in the RHS, the first term is the time-rate of change of the property N for the control volume, with the second and the third term being the correction terms for the translation of conservation equation from control mass to control volume. The second term is essentially the time-rate of outflow of the property from the control volume and the third term is the time-rate of inflow. Taking the limits, equation (1) is re-written as,

$$\left.\frac{dN}{dt}\right|_{\text{system}} = \left.\frac{\partial N}{\partial t}\right|_{CV} + \text{Rate of Outflow of } N - \text{Rate of Inflow of } N. \tag{2}$$

In terms of the intensive property, n, the equation (2) transforms to,

$$\left.\frac{dN}{dt}\right|_{\text{system}} = \frac{\partial}{\partial t}\int_{CV}\rho n dV + \int_{CS}\rho n\vec{v}\cdot\hat{\eta} dA. \tag{3}$$

In deriving equation (3) from equation (2), we have considered a small volume element dV inside the control volume, which has a mass of ρdV. Thus, the extensive property N is related to the intensive property n as $N=\rho n dV$. The first term on the RHS is converted straightaway by integrating over all the volume elements that constitute the control volume. However, to obtain the second term of equation (3), consider a small segment of the surface of control volume, also called control surface (CS), through which fluid is entering, illustrated as the left segment in Fig 2. Time-rate of volume of fluid entering through this surface segment is $-\vec{v}\cdot\hat{\eta}dA$, and therefore the time-rate of mass of fluid entering through this surface element is $-\rho\vec{v}\cdot\hat{\eta}dA$ and finally time-rate of N entering through this surface element is $-\rho n\vec{v}\cdot\hat{\eta}dA$. Similarly, the time-rate of N exiting through the surface element on the right is $\rho n\vec{v}\cdot\hat{\eta}dA$. Therefore, combined together and integrated over the complete control surface, CS, the net rate of exiting the control volume is $\int_{CA} \rho n \vec{V} \cdot \hat{\eta} dA$, which appears as the second term on the RHS of equation (3).

A subtle point missing in the discussion about equation (3) is what does $\vec{v}$ represent. Let us say that this control volume is moving at a velocity $\vec{v}_0$. If the fluid is also flowing with the same velocity $\vec{v}_0$, it means that the net flow of fluid in/out of the control volume is zero, even though the fluid is flowing. Hence, we deduce that it is not the absolute value of the fluid's velocity that matters for transport of a property N across the faces of the control volume nut the relative velocity with respect to the control volume. That is, $\vec{v}$ is the velocity of the fluid relative to the control volume, and can also be denoted as $\vec{v}_r$, the subscript r standing for 'relative'. Summarily, the Reynold's transform theorem is,

$$\left.\frac{dN}{dt}\right|_{\text{system}} = \frac{\partial}{\partial t}\int_{CV} \rho n dV + \int_{CS} \rho n \vec{v}_r \cdot \hat{\eta} dA. \tag{4}$$

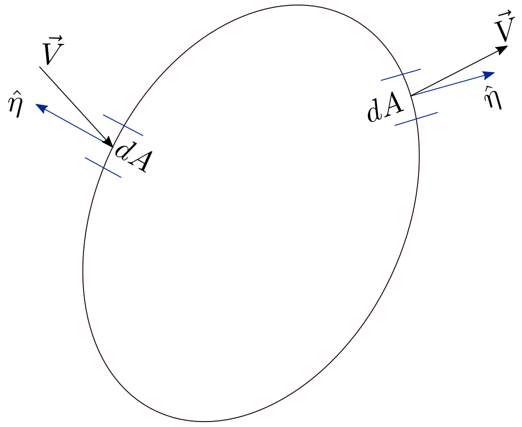

Figure 2: Fluid entering and leaving the control mass through small segments (on the left and on the right) on the control surface (CS)

II. Derivation Conservation of Mass or Continuity Equation

If we consider the extensive property N to be the mass of the system m, then equation (4) can be utilized to derive the equation for conservation of mass.

Therefore, taking the extensive property N as the mass of the system m, the corresponding intensive property becomes 1. Also , under the classical mechanics paradigm which we dealing with in this course, the mass of a particular system remains conserved, i.e., is $\left.\frac{dm}{dt}\right|_{\text{system}}$ zero. Hence, equation (4) becomes,

$$0 = \frac{\partial}{\partial t}\int_{CV} \rho dV + \int_{CS} \rho \vec{v}_r \cdot \hat{\eta} dA. \tag{5}$$

Equation (5) is the most general expression for conservation of mass under Eulerian description. Consider the special case of non-deformable control volume, i.e. V is not a function of t. This implies the temporal derivative expression switches in and out of the first integral term on RHS of equation (5) without requiring any modifications, i.e.,

$$0 = \int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CS} \rho \vec{v}_r \cdot \hat{\eta} dA. \qquad (6)$$

As a side note, if the control volume were allowed to deform, we would still be able to take the temporal derivative inside the integral but we would need to use the Leibniz rule for differentiation under integral sign to obtain the requisite correction terms. We will see later that the Leibniz rule and Reynolds transport theorem are equivalent, an elegant analogy between mathematics and physics.

If we further assume the control volume to be stationary, then $\vec{v}_r$ is simply the fluid velocity $\vec{v}$. Then, we utilize the divergence theorem to convert the second term on RHS of equation (6) from an area integral to a volume integral, i.e.,

$$\int_{CS} \rho \vec{v}_r \cdot \hat{\eta} dA = \int_{CV} \nabla \cdot (\rho \vec{v}) dV. \qquad (7)$$

As a consequence, equation (6) becomes,

$$0 = \int_{CV} \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] dV. \qquad (8)$$

Now, equation (8) has a one-dimensional analogy as $\int_{x=x_0}^{x_1} f(x) dx = 0$. This one-dimensional integral doesn't necessarily imply that $f(x)$ is zero for all points between $x = x_0$ and $x = x_1$. For instance, if $f(x)$ is $\sin(x)$ and $x_0 = 0$ and $x_1 = 2\pi$, then even though $\int_{x=x_0}^{x_1} f(x) dx = 0$, $f(x)$ is not zero for all points between $x = x_0$ and $x = x_1$. However, if the choice of x_0 and x_1 is arbitrary, then is necessarily zero for all points between $x = x_0$ and $x = x_1$. Since the choice of control volume is arbitrary, analogous argument implies,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0. \qquad (9)$$

Equation (9) is the conservation of mass equation (or continuity equation) that has been derived twice earlier with the control mass approach and the cuboidal control volume approach.

So far today, we have studied a general integral form of conservation equation and its special application in deriving conservation of mass or continuity equation. We have also learnt a technique of converting an integral form of conservation equation to the corresponding differential form by converting the control surface integral term to into volume control integral term using the diversion theorem, and then requiring the integrand to be zero as a consequence of the choice of control volume being arbitrary.