Advanced Concepts In Fluid Mechanics
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Lecture – 08

Examples of Bernoulli's Equation

I. Torricelli's Equation

Torricelli's equation is the equation for drainage flow speed out of a small hole in a tank filled with water upto a given height. The setup is illustrated in Figure 1.

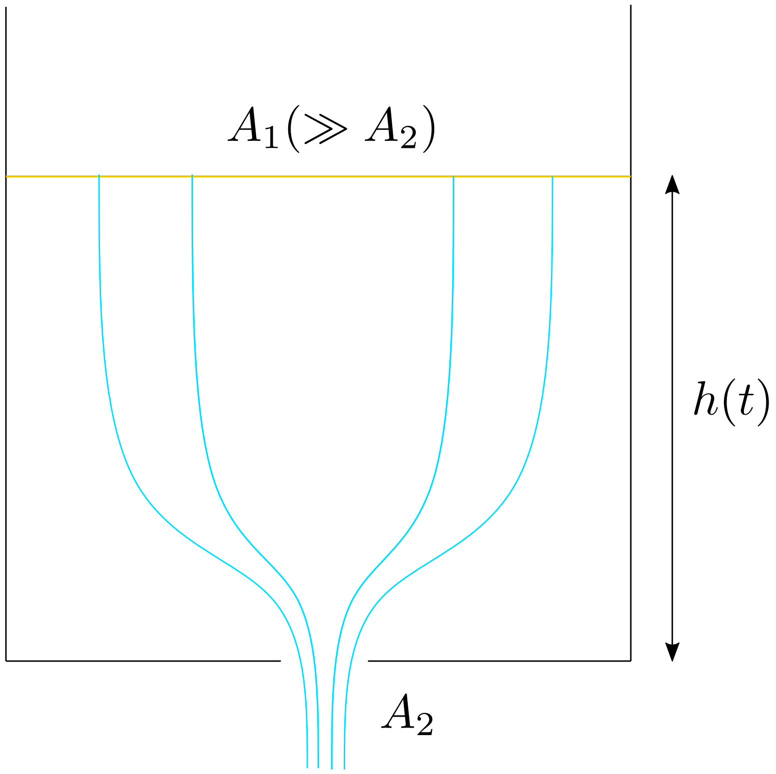

Figure 1: Water draining from the small hole at the bottom of a tank. The surface area and uniform velocity at the free surface are A_1 and V_1, the area and uniform velocity at the drainage hole are A_2 and V_2.

We study this problem as the first problem in this lecture, with focus on the kind of complexities in the analysis and to what extent this problem is popularly oversimplified. Uniform purely-vertical velocity is assumed at the top free surface as well as the drainage hole at the bottom. The flow is assumed to be inviscid.

Clearly, because the top surface of the fluid is coming down as the fluid drains, the flow is unsteady and even if all other assumptions of Bernoulli's equation are justified, we cannot assume the flow to be unsteady. Hence, we are required to keep the unsteady term and assess when the unsteady term can be neglected. Therefore, we will consider the unsteady version of the Bernoulli's equation along considering a streamline (representative streamlines illustrated in Fig 1),

$$\frac{p_2 - p_1}{\rho} + \frac{V_2^2 - V_1^2}{2} + g(z_2 - z_1) + \int_1^2 \frac{\partial v}{\partial t} ds = 0\,. \tag{1}$$

Equation (1) is written considering two points 1 and 2, at the top free surface and at the bottom drainage hole respectively. Since the flow at these two areas is considered uniform, the velocity at point 1 equals the velocity of the top free surface and the velocity at point 2 represents the velocity for the complete area of the drainage hole. We also recapitulate that in writing equation (1), the assumption of constant ρ is crucial and is often implicitly taken. To elucidate this, we recall the continuity equation that must be satisfied. The continuity equation is,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0\ . \tag{2}$$

This equation, under the consideration of uniform velocity at the free surface and drainage hole, simplifies to,

$$A_1 V_1 = A_2 V_2 \tag{3}$$

given either (i) the flow is steady, or, (ii) ρ is constant. Clearly, for the problem at hand, this second condition is satisfied.

Proceeding with equation (1) now, we note that apart from the usual terms, there is another term, i.e. the last term that appears due to unsteadiness, which is crucial. For a major fraction of the height of the tank starting from the top, the streamline can be assumed to be vertical, and it only bends for a small fraction at the bottom to 'squeeze' out of the drainage hole. This implies that the vertical velocity for a major fraction of the streamline is V_1. Therefore, we can approximately express the last term of equation (1) as,

$$\int_1^2 \frac{\partial v}{\partial t} ds \approx \frac{\partial V_1}{\partial t} \int_1^2 ds = h \frac{\partial V_1}{\partial t}\,. \tag{4}$$

This approximation implies that the primary contributor to the integral in the last term of equation (1) is the velocity within the large tank. Hence, proceeding with equation (1), we have,

$$\frac{p_{atm} - p_{atm}}{\rho} + \frac{\left(\frac{A_1 V_1}{A_2}\right)^2 - V_1^2}{2} + g(-h) + h \frac{\partial V_1}{\partial t} = 0\,. \tag{5}$$

In this equation, V_1 is $\dfrac{dh}{dt}$. Clearly this is a complicated non-linear differential equation in h, which is not an easy equation to solve. A simplified scenario arises when the unsteady term, $h\dfrac{\partial V_1}{\partial t}$, is much less than the other terms. If we further consider $A_1 >> A_2 \Rightarrow V_1 << V_2$, equation (5) gets solved as,

$$\frac{V_2^2}{2} - gh = 0 \Rightarrow V_2 = \sqrt{2gh}\,. \tag{6}$$

This expression for V_2 is the popularly known Torricelli's equation, and our analysis above have acquainted us with the complexities involved in obtaining this equation and the assumptions and simplifications that we had to take.

II. Streamwise and Cross-Streamwise Coordinates

Now we discuss the Euler equation in streamwise and cross streamwise coordinate. This coordinate system consists of a streamline and a line normal to it. The objective behind studying this co-ordinate system is that because the flow velocity is which is tangent to the streamline, flow in a streamline coordinate system is essentially one dimensional problem, i.e., dimensionality of the problem gets reduced. We consider a curved cylindrical fluid element along a stream line as illustrated in figure 2.

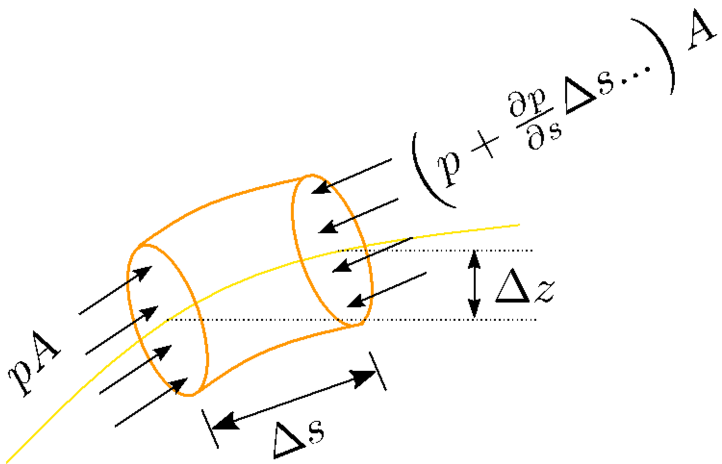

Figure 2: A curved cylindrical fluid element (orange coloured) co-incident with a streamline (yellow coloured), the area of the two faces where fluid enter and leave is A.

Now, we write the force balance for this fluid element. We first examine the force balance along the streamline:

$$\sum F_s = (\Delta m)\, a_s$$

$$\Rightarrow pA - \left[p + \frac{\partial p}{\partial s}\Delta s + ... \right] A - \rho A \Delta s g \sin\theta = \rho A \Delta s a_s = \rho A \Delta s \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} \right) \qquad (7)$$

$$\Rightarrow -\frac{\partial p}{\partial s} + \rho g \sin\theta = \rho a_s = \rho \left(\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial s} \right).$$

Equation (7), particularly the third form, is the Euler's equation along a streamline.

Although the flow is along the streamline, the cross-streamline force balance is also important because the curvature of the streamline (and therefore of the flow) indicates that there is a centrifugal acceleration acting on the particle as well. Hence, considering a coordinate n that is perpendicular to s, we can write the cross-streamline force balance as,

$$-\frac{\partial p}{\partial n} - \rho g \frac{\partial z}{\partial n} = \rho a_n = -\frac{\rho v^2}{R}. \qquad (8)$$

In equation (8), R is the local radius of curvature of the streamline. Clearly, this normal acceleration a_n is the sole effect of curvature of the streamline and not due to a change of magnitude of velocity.

With this lecture, we conclude dynamics of inviscid flows, where we have studied Euler's equation of motion in vector form and scalar form in Cartesian co-ordinate system, then in streamwise co-ordinate system, we have studied the unsteady and the more popular steady form of Bernoulli's equation, and we have used illustrative examples along the way to elucidate the various concepts.

In this chapter, we have often expressed the flow-behaviour in a Lagrangian framework, frequently using a control-mass approach. However, proceeding ahead in this course, we will be see that a key attribute of the discipline of in fluid mechanics is to translate this control mass based equations to control volume based equations, i.e. Lagrangian description to Eulerian description, and this needs to be done because all the parameters in fluid flow equations are essentially control volume based parameters. This is achieved by the renowned theorem called the Reynolds transport theorem, which we will take up in the next lecture.