**Advanced Concepts In Fluid Mechanics**
**Prof. Suman Chakraborty**
**Department of Mechanical Engineering**
**Indian Institute of Technology, Kharagpur**

**Lecture - 07**
# Bernoulli's Equation

## I. Recap of Last Lecture

We start this lecture by re-iterating the general form and the usually-employed form of Bernoulli's equation, equations (1) and (2) respectively.

$$\frac{\partial \vec{v}}{\partial t}\cdot d\vec{l}+d\left(\frac{p}{\rho}+\frac{v^2}{2}+gz\right)=\left(\vec{v}\times\vec{\Omega}\right)\cdot d\vec{l} \qquad (1)$$

$$p_1+\frac{\rho v_1^2}{2}+\rho g z_1=p_2+\frac{\rho v_2^2}{2}+\rho g z_2 \Rightarrow \frac{p_1}{\rho g}+\frac{v_1^2}{2g}+z_1=\frac{p_2}{\rho g}+\frac{v_2^2}{2g}+z_2 \qquad (2)$$

We list out the assumptions in deriving these equations as they sequentially emerge during the derivations.

1. In deriving equation (1),
   a. We started with the assumption of the flow being inviscid.
   b. Then we assumed that gravity is the only body force and is directed along the negative z direction
2. Subsequently, to obtain equation (2) from equation (1),
   a. We assumed steady flow, due to which, the first term of equation (1) vanished
   b. The we assumed constant density, as a consequence of which, $d\left(\frac{p}{\rho}\right)=\frac{dp}{\rho}$ holds true
   c. Lastly, the RHS of equation (1) becomes zero. This happens under three situations,
      i. When $d\vec{l}$ is along a streamline, i.e. points 1 and 2 are two points on the same streamline
      ii. When vorticity $\vec{\Omega}$ is the null vector, implying irrotational flow (for this situation, 1 and 2 can be any two points in the flow-field and are not restricted to lie on the same streamline line point 2.c.i above)
      iii. $\vec{v}\times\vec{\Omega}$ is perpendicular to $d\vec{l}$ - this situation implies a restriction on the choice of $d\vec{l}$ or equivalently the choice of points 1 and 2a

Focusing on equation (2), the terms suggest that it is an mechanical energy conservation equation. Further, dividing the equation on the left by $\rho g$ gives the equation on the right which is frequently employed by hydraulic engineers. For this equation on the right, all the terms have dimension of length. In hydraulics engineering, these terms are frequently termed as 'head'. Being more specific, the third term in RHS as well as LHS, $z_{1/2}$, is potential energy per unit weight, the second term in RHS as well as LHS, $\frac{v_{1/2}^2}{\rho g}$, is kinetic energy per unit weight, i.e. potential energy head and kinetic energy head respectively. The first term in LHS as well as RHS is termed as pressure energy in many texts. However, we require to elaborate some more on this term to understand its role. To understand it, we take the help of an example.

Let us assume flow through a pipe. The fluid at its inlet is ready to enter the pipe but is subjected to the pressure of the already flowing fluid inside the pipe. Considering a small length $\Delta x$ such that the pressure can be assumed to be constant along it, the work by the pressure for the incoming fluid to enter the pipe will be $p_{\text{inlet}}\Delta x A_{\text{inlet}}$, and dividing it by unit weight gives $\frac{p_{\text{inlet}}\Delta x A_{\text{inlet}}}{\rho \Delta x A_{\text{inlet}} g} = \frac{p_{\text{inlet}}}{\rho g}$. Therefore, the first term in RHS as well as LHS represents the additional energy the systems require to maintain flow in presence of pressure, and is hence termed as flow energy or flow work. This is a crucial difference between a stagnant system and a flowing system.

Summarily, subject to fulfillment of the assumptions listed out at the start of this lecture, the sum of kinetic energy, potential energy, and flow energy transferred between one point to the other remains conserved, as a result of there being a continuous flow process. It is important to understand that the energy in Bernoulli's equation is not being possessed by the fluid but instantaneously transferred from one point to the other. Therefore, what is conserved is not the sum of energy possessed by the fluid, but sum of energy transferred from one point in flow to another.

Having recapitulated Bernoulli's equation, and with the foundation for Bernoulli's equation and Euler equation thus being formed, we will work out a representative problem to illustrate some of the fundamentals of inviscid flows.

**Illustrative Example:**

Problem - Consider a flow-field $u = Ax, v = -Ay, w = 0$, $A$ where is a constant. Find pressure difference between any two points 1 and 2 in this flow field. Assume inviscid flow, constant $\rho$ and $g$ to be along negative $z$.

Solution - We first list down the Euler's equations for the flow (since the flow is inviscid):

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}$$

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y} \qquad (3)$$

$$0=-\frac{\partial p}{\partial z}-\rho g$$

Here, we have omitted the terms with $w$ beforehand as it is given zero.

With the given expression for flow-field, velocity is evidently not a function of time. Therefore, the flow is steady. Consequently, the equations simplify to,

$$\rho\left[u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right]=-\frac{\partial p}{\partial x}$$

$$\rho\left[u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y} \qquad (4)$$

$$0=-\frac{\partial p}{\partial z}-\rho g$$

Further, substituting the expression for $u$ and $v$, we can do partial-integration of the pressure term for each of these equations to get,

$$p=-\frac{\rho A^2 x^2}{2}+f_1(y,z)$$

$$p=-\frac{\rho A^2 y^2}{2}+f_1(x,z) \qquad (5)$$

$$\mathrm{p}=-\rho gz+f_3(x,y)$$

It should be noted that the third expression obtained in equation (5), particularly the $-\rho gz$ term, is the hydrostatic pressure variation that is recovered in fluid statics. Clearly, this hydrostatic pressure has been obtained here as a special case of the fluid dynamics when we set  to zero, i.e. the concept of fluid statics is recovered as a special case of fluid dynamics when flow is zero. It is for owing to this possibility that as fluid mechanics has evolved over the years, there is a pedagogical development of not teaching fluid statics dedicatedly in the interest of accommodating upcoming advancements like atmospheric flows and micro/nano-scale flows.

Combining the solutions in equation (5), the pressure in the flow-field as,

$$p=-\frac{\rho A^2 x^2}{2}-\frac{\rho A^2 y^2}{2}-\rho gz+c \Rightarrow p+\frac{1}{2}\left(u^2+v^2\right)+\rho gz=c \Rightarrow p+\frac{V^2}{2}+\rho gz=c \qquad (6)$$

where, $c$ is a constant and $V$ represents the velocity magnitude or speed.

The third form in equation (6) indicates that $p+\frac{V^2}{2}+\rho gz=c$ holds true over the entire flow-field, i.e. Bernoulli's equation applies between any two-points in the flow field. Therefore, we obtain the difference of pressure between any two points 1 and 2 in the flow-field as ,

$$p_1-p_2=-\left[\left(\frac{V_1^2}{2}+\rho gz_1\right)-\left(\frac{V_2^2}{2}+\rho gz_2\right)\right]. \tag{7}$$

While we have arrived at a solution, we can extract more insight from the fundamentals of this problem.

First, as we have seen, solving the equations of motion for inviscid flow (Euler's equation) for this problem has given us the same solution that we would arrive at by simply applying the Bernoulli's equation. This happens because in addition to being inviscid (as requirement for applying Euler's equation), this flow is also irrotational. This becomes clear by computing the angular velocity (rate of angular/shear deformation) of the flow, $\omega=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}=0+0=0$, which is seen to be zero. Hence, the alternative approach to solve this problem could have been to first investigate whether the flow is irrotational, and having confirmed that it is, simply use Bernoulli's equation between the two points in question, 1 and 2.

Second, the velocity of the flow-field also gives us $\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0-0=0$, which implies that not only is the flow irrotational, but there isn't any rigid-body rotation either.

Third, the velocity of the flow-field also satisfies $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=A-A=0$, implying that volume is conserved. However, the individual expressions are not zero. This means that there is stretching of fluid in one direction and compression in the perpendicular direction, such that the volume remains conserved. We obtain the equation of the streamlines,

$$\frac{du}{u}=\frac{dy}{v}\Rightarrow\frac{du}{Ax}=\frac{dy}{-Ay}\Rightarrow\ln(x)=-\ln(y)+\ln(k)\Rightarrow xy=k\ . \tag{8}$$

Hence, equation of the streamlines is a rectangular hyperbola with axes at $45^0$ to the x-y axes. Hence, a fluid element that is at a large y and small x flows to a point at a small y and large x, i.e. it compresses along the y-direction and stretches along the x-direction. An initially square fluid element deforms to a flattened and stretched rectangle as it flows, visualization presented in Fig 1. This fluid element continues to stay irrotational as it flows from ABCD to A'B'C'D' because the flow is initially irritation (i.e. when the element is at ABCD) as well inviscid.

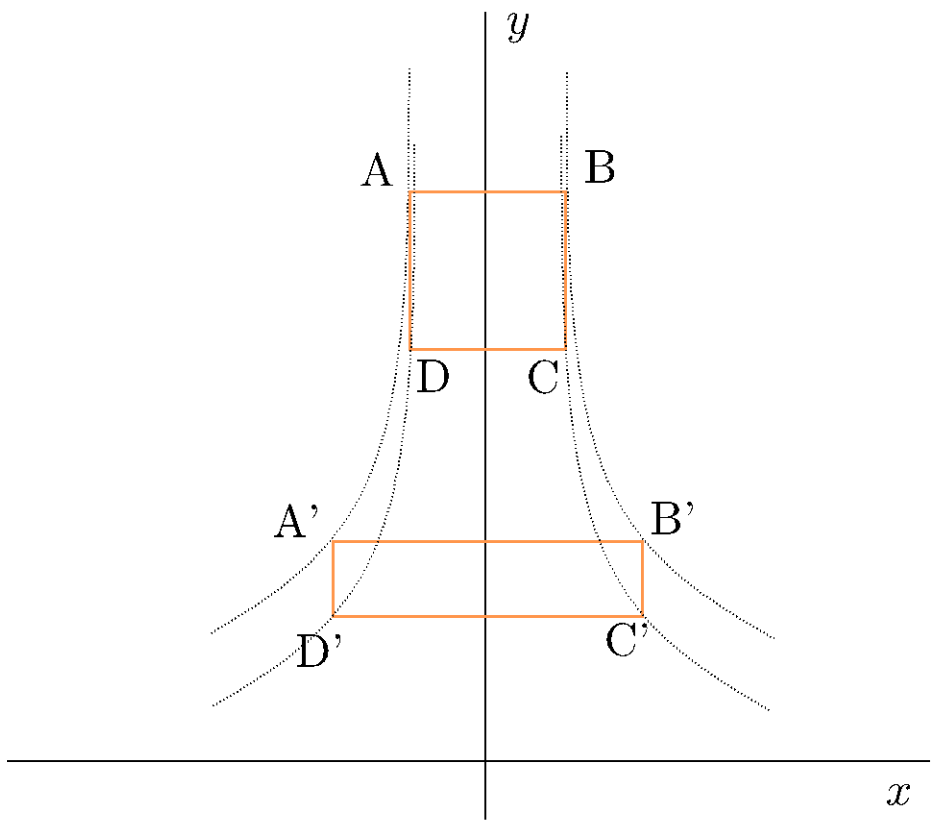


Figure 1: An initially square fluid element ABCD flows and deforms to A'B'C'D'

We now consider the question – "is it possible that a flow that does not rotate initially, but starts rotating after some time". The factors that can make an initially irrotational flow rotational are:

1. Viscosity
2. Thermal Stratification
3. Shock Waves
4. Coriolis Effect

The first factor, viscosity, is the most important. To elucidate this concept, we consider a person who steps out of a moving bus. This person would have to run a bit in the direction of the bus immediately after stepping down to avoid toppling. Why the person will topple if he/she didn't run a bit? This is because the body of the person had an inertia because of the movement of the bus but the ground causes his foot to stop, thus creating a veocity gradient over the body's length. This gradient occurs because of the friction of the ground. Similar to this, the viscosity of a fluid generates a disturbance of to the fluid's momentum. Thus disturbance has a rotational effect on the fluid elements, due to which an initially irrotational flow can become rotational. Summarily, because of viscosity there is a rotational effect.

We briefly and qualitatively elucidate the other three factors now. Thermal stratification means because of the temperature variation, there is a density variation and the lighter fluid flows to the top and denser fluid to the bottom. Shock waves are discontinuities that emerge in a domain where there is a very high speed flow. This happens the speed of the flow is greater than the sonic speed, which is the speed at which the disturbance propagates. As a result, the disturbances accumulate along a particular line which is called as a shockwave front and there is a release of this discontinuity called as a shockwave. Lastly, Coriolis effect is the effect on flow-field of a force due to the reference frame being a rotating one. This results in a side wise force. An example of this effect is that the ocean current in the northern hemisphere moves in a certain direction and in the southern hemisphere moves in another direction.