Advanced Concepts In Fluid Mechanics
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Lecture - 06
Euler Equation for Inviscid Flow

I. Inviscid Flows

In the former lectures, we have studied the kinematics of flow without getting into the details of what forces cause and affect the flow. We will now consider the forces involved in fluid flows.

The forces acting in a flow-field can be broadly categorized into two types – driving forces that drive the force, and resisting forces that oppose the flow. From our prior knowledge of physics, we know that resistive forces in flows are called as viscous forces. However, there can be situations where viscous forces, although present, do not affect the flow substantially. Such flows are termed as Inviscid Flows. One example is fluid flowing past a solid body at high velocity. The effect of the wall of the body on the flow, particularly the disturbance caused to the fluid momentum, occurs for only a small layer near the wall, beyond which there is an outer layer where this momentum disturbance does not propoagate. Therefore, in the region, although the viscosity is finite, the momentum disturbance is small. This implies the velocity gradient is small enough such that the viscous effect, which is the product of viscosity and velocity gradient, is negligible in comparison to inertial effects. This comparison is quantified in terms of Reynolds number, that we shall discuss at a later stage in the course.

With this qualitative example, we proceed with discussing inviscid flows. Mathematically, while viscous flows are represented by a second order differential equation, inviscid flows are represented by a first order differential equation.

Euler Equation of Motion: The governing differential equation for inviscid flows is called as Euler equation of motion. To derive this equation, we consider a fluid element (i.e. a control mass) as presented in Fig 1

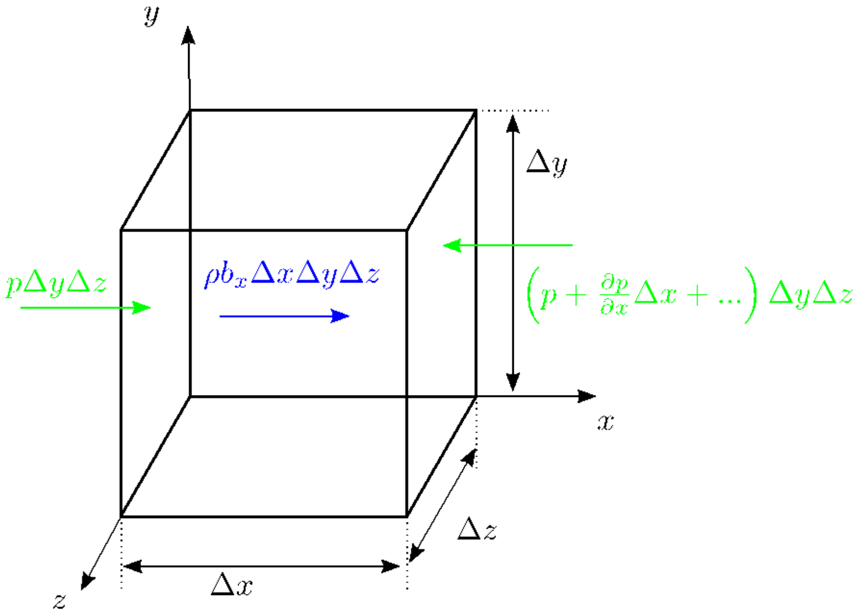

Fig 1: A fluid element for derivation of Euler equation of motion

We study the force balance along the x-direction, and therefore, the forces on the element along the x-direction are presented in Fig 1, where b_x is the body force on the fluid (which can be for example electrical force, magnetic force), and is therefore multiplied to the fluid element's volume, $\Delta x \Delta y \Delta z$. Expressing the Newton's second law for this fluid element, we have,

$$\sum F_x = (\Delta m) a_x$$

$$\Rightarrow p\Delta y \Delta z - \left(p + \frac{\partial p}{\partial x}\Delta x + ...\right)\Delta y \Delta z + \rho b_x \Delta x \Delta y \Delta z = \rho \Delta x \Delta y \Delta z a_x \qquad (1)$$

$$\Rightarrow -\frac{\partial p}{\partial x}\Delta x \Delta y \Delta z + \rho b_x \Delta x \Delta y \Delta z + \mathrm{H.O.T} = \rho \Delta x \Delta y \Delta z a_x$$

In equation (1), H.O.T. implies higher order terms and a_x is the fluid elements acceleration in the x-direction. Other notations have their usual meanings.

In order to obtain the differential form of Newton's second law from equation (1), we have to consider the fluid element as being vanishingly small. Therefore, $\Delta x, \Delta y, \Delta z \to 0$, giving us,

$$\rho a_x = -\frac{\partial p}{\partial x} + \rho b_x . \qquad (2)$$

Expression the acceleration, a_x, as per the Eulerian framework, we get,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho b_x . \qquad (3)$$

This is the x-component of the momentum balance equation for inviscid flows, i.e. the Euler equation of motion for inviscid flows. Similar equations are there for y and z components,

$$\begin{aligned} &\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho b_y \\ &\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho b_z \end{aligned} \tag{4}$$

Combined together, equations (3) and (4) give the Euler equation in vector form,

$$\rho\left[\frac{\partial \vec{v}}{\partial t}+(\vec{v}\cdot\nabla)\vec{v}\right]=-\nabla p+\rho\vec{b} \tag{5}$$

<u>Bernoulli's Equation Derivation:</u> We now systematically derive the Bernoulli's equation from the Euler equation of motion for invscid flow. For this, we first utilize a standard vector identity, $(\vec{v}\cdot\nabla)\vec{v}=\frac{1}{2}\nabla(\vec{v}\cdot\vec{v})-\vec{v}\times(\nabla\times\vec{v})$. With respect to fluid velocity, this vector identity signifies the relationship between the fluid's inertial effect $(\vec{v}\cdot\nabla)\vec{v}$ and gradient of its kinetic energy $\nabla(\vec{v}\cdot\vec{v})$ and dot product of velocity with vorticity $\vec{v}\times(\nabla\times\vec{v})$, where $\nabla\times\vec{v}=\vec{\Omega}$ is the vorticity. Utilizing this vector identity, we express equation (5) as,

$$\rho\frac{\partial \vec{v}}{\partial t}+\nabla p+\frac{1}{2}\rho\nabla(\vec{v}\cdot\vec{v})-\rho\vec{b}-\rho\vec{v}\times(\nabla\times\vec{v})=0 \tag{6}$$

We take the dot product of equation (6) with a line element in the flow-field $d\vec{l}$. The dot product of the second term with $d\vec{l}$ is,

$$\nabla p\cdot d\vec{l}=\left(\hat{i}\frac{\partial p}{\partial x}+\hat{j}\frac{\partial p}{\partial y}+\hat{k}\frac{\partial p}{\partial z}\right)\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})=\frac{\partial p}{\partial x}dx+\frac{\partial p}{\partial y}dy+\frac{\partial p}{\partial z}dz=dp\,. \tag{7}$$

Similarly, the dot product of the third term with $d\vec{l}$ is, $d\left(\frac{v^2}{2}\right)$.

Furthermore, is we assume that gravity is the only body force and is expressed as $\vec{b}=-g\hat{k}$, i.e. acts in the negative z-direction, $-\vec{b}\cdot d\vec{l}=gz$.

With these expressions at hand, the dot product of equation (6) with $d\vec{l}$ and then dividing by ρ gives,

$$\frac{\partial \vec{v}}{\partial t}\cdot d\vec{l}+\frac{dp}{\rho}+d\left(\frac{v^2}{2}\right)+bz=\left(\vec{v}\times\vec{\Omega}\right)\cdot d\vec{l}\,. \tag{8}$$

Equation 8 is Euler's equation motion in an alternate representation.

If we further assume ρ is constant and g is independent of z, equation (8) further transforms to,

$$d\left(\frac{dp}{\rho}+\frac{v^2}{2}+bz\right)+\frac{\partial\vec{v}}{\partial t}\cdot d\vec{l}=\left(\vec{v}\times\vec{\Omega}\right)\cdot d\vec{l}\ . \tag{9}$$

If we also assume that the flow is steady, equation (9) simplifies to,

$$d\left(\frac{dp}{\rho}+\frac{v^2}{2}+bz\right)=\left(\vec{v}\times\vec{\Omega}\right)\cdot d\vec{l}\ . \tag{10}$$

Now, $\left(\vec{v}\times\vec{\Omega}\right)\cdot d\vec{l}$ can be expressed as the determinant $\begin{vmatrix} u & v & w \\ \Omega_x & \Omega_y & \Omega_z \\ dx & dy & dz \end{vmatrix}$. This determinant vanishes if at least one of the following three conditions is satisfied:

1. $\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$, i.e. $d\vec{l}$ is directed along a streamline
2. $\vec{\Omega}=0$, i.e. the flow is irrotational
3. $\frac{u}{\Omega_x}=\frac{v}{\Omega_y}=\frac{w}{\Omega_z}$.

The third condition can only be co-incidentally be satisfied by some special flow-fields. However, condition 1 simply implies selection of the elemental length $d\vec{l}$ which is upto our choice, and condition 2 implies irrotational flow, which a family of flow-fields studied classically.

When the RHS of equation (10) vanishes as a consequence of one of the conditions above, it can be integrated along sequence of line elements to give,

$$\left(\frac{dp}{\rho}+\frac{v^2}{2}+bz\right)_1=\left(\frac{dp}{\rho}+\frac{v^2}{2}+bz\right)_2, \tag{11}$$

which is the Bernoulli's equation between points 1 and 2 in the flow-field.

In summary, for a steady inviscid flow, Bernoulli's equation, equation (11), can be applied between two points that are on the same streamline if the flow is rotational and between any two points in the flow-field if the flow irrotational.