Advanced Concepts in Fluid Mechanics
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Lecture – 59
Compressible Flow with Friction

One-dimensional compressible flow with friction:

In this lecture, we will study the effects of fluid friction in compressible flows. To keep the analysis simple, we will make some simplifications. We will not consider shock and will only study weak waves with differential changes in properties. We consider the flow to be a one-dimensional compressible flow in a constant area duct of area A and perimeter C. The setup is presented in figure 1.

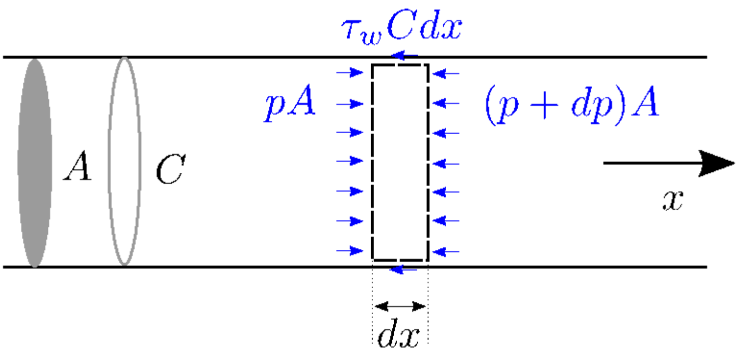

Figure 1: Setup being studied for one-dimensional compressible flow with friction

We first list out the basic equations for the system being studied. Here, we have a fluid flow, with thermal effects coming into the picture because of the compressible nature of the flow. For such a situation, the governing equations are:

1. Continuity equation or Mass conservation equation
2. Momentum conservation equation
3. Energy balance equation

Apart from these governing equations, we require two closure relations:

4. Equation of state
5. Definition of Mach number

Out of these five basic equations, only one will change explicitly due to fluid friction, the momentum conservation equation. The other equations will be implicitly affected as Mach number distribution (Mach number variation with axial length) will get altered due to the flow velocity getting affected by friction through the momentum conservation equation.

We first write the continuity equation,

$$\frac{d\rho}{\rho}+\frac{du}{u}=0. \tag{1}$$

To write the momentum balance, we now consider the dashed control volume presented in figure 1 at an arbitrary axial location. All the forces are depicted with blue colour. Note that in addition to the force due to pressure (pA and $(p+dp)A$ in the negative x direction) there is an additional force of $\tau_w Cdx$ in the negative x direction at the wall, arising out of the viscous effect of the fluid. Writing the momentum balance for this control volume, we have,

$$
\begin{aligned}
&\sum F_x = \dot{m}(u+du-u) \Rightarrow \\
&pA-(p+dp)A-\tau_w Cdx = \rho Audu \Rightarrow \\
&-dp-\tau_w \frac{C}{A}dx = \rho udu \Rightarrow \\
&\frac{dp}{\rho u^2}+\frac{\tau_w}{\rho u^2}\frac{C}{A}dx+\frac{du}{u}=0
\end{aligned}
\qquad (2)
$$

The last form of equation (2) is obtained such that the dimensions of equation (1) and (2) are the same, which is 1. Here, we specify that the wall shear stress τ_w is obtained only once we know the transverse velocity profile, as its expression as per Newton's law of viscosity is simply $\tau_w = \mu \left.\frac{\partial \bar{u}}{\partial n}\right|_w$, where n is the co-ordinate normal to the wall and $\bar{u}$ is the transverse velocity profile. Hence, even with one-dimensional treatment of flow (which essentially means we are dealing with the average velocity at any cross-section of the conduit, defined as $u = \frac{1}{A}\int_A \bar{u}dA$) as done here, the transverse velocity profile is important to incorporate the effects of friction in the fluid.

We now write the energy equation,

$$
\begin{aligned}
&h+\frac{u^2}{2} = constt \Rightarrow \\
&dh+udu=0
\end{aligned}
\qquad (3)
$$

In writing equation (3), we have made two assumptions. First, we have neglected any changes in potential energy. This is a suitable assumption as the high speed in compressible flows means that the changes in kinetic energy will be much higher than any potential energy changes. Second, the expression corresponding to the kinetic energy term in equation (3), $\frac{u^2}{2}$, is valid only when the flow velocity is uniform. In the case of variation of velocity along the transverse direction of the channel, as will be the case in the presence of fluid friction, we have to multiply this term with a kinetic energy correction factor α, defined as $\alpha = \frac{1}{A}\int_A \left(\frac{\bar{u}}{u}\right)^3 dA$. Hence, equation (3) gets modified to,

$$dh+\alpha udu=0. \qquad (4)$$

Assuming the gas to be ideal gas, we substitute $dh = c_p dT$ in equation (4) and divide by u^2 to get,

$$c_p \frac{dT}{u^2} + \alpha \frac{du}{u} = 0 . \qquad (5)$$

We now use the definitions of Mach number and sonic speed to substitute $u = M\sqrt{\gamma RT}$ into the first term of equation (5) to get,

$$\frac{c_p}{\alpha M^2 \gamma R} \frac{dT}{T} + \frac{du}{u} = 0 . \qquad (6)$$

Lastly, we write the equation of state,

$$p = \rho RT . \qquad (7)$$

Taking log and then expressing in differential form, equation (7) becomes,

$$\frac{dp}{p} = \frac{d\rho}{\rho} + \frac{dT}{T} . \qquad (8)$$

Lastly, the definition of Mach number gives us,

$$u^2 = M^2 \gamma RT . \qquad (9)$$

Again, taking log and then expressing in differential form, equation (9) becomes,

$$\frac{du}{u} = \frac{dM}{M} + \frac{1}{2}\frac{dT}{T} . \qquad (10)$$

Collected, equations (1), (2) (last form), (6), (8) and (10) are the five basic equations we listed out at the start.

We will now simplify these equations further such that we recover $\frac{du}{u}$, $\frac{dp}{p}$, $\frac{dT}{T}$, and $\frac{d\rho}{\rho}$ as functions of $\frac{dM}{M}$.

Substituting $\frac{du}{u}$ from equation (6) into equation (10), we get,

$$\frac{dT}{T} = -\left[\frac{c_p}{\alpha M^2 \gamma R} + \frac{1}{2}\right]^{-1} \frac{dM}{M} . \qquad (11)$$

Looking at equation (11), we obtain a physical insight. The factor $\left[\frac{c_p}{\alpha M^2 \gamma R} + \frac{1}{2}\right]^{-1}$ is always positive. Hence, the signs of dM and dT are opposite, i.e. if the Mach number is increasing, the temperature is decreasing and vice versa. Also note that we have not yet used the momentum conservation equation in arriving at relation (11). Hence, relation (11) is not dependent on whether there is friction in the fluid or not.

Substituting $\frac{dT}{T}$ from equation (11) into equation (6), we get,

$$\frac{du}{u}=\frac{c_p}{\alpha M^2\gamma R}\left[\frac{c_p}{\alpha M^2\gamma R}+\frac{1}{2}\right]^{-1}\frac{dM}{M}. \tag{12}$$

Substituting $\frac{du}{u}$ from equation (12) into equation (1), we get,

$$\frac{d\rho}{\rho}=-\frac{c_p}{\alpha M^2\gamma R}\left[\frac{c_p}{\alpha M^2\gamma R}+\frac{1}{2}\right]^{-1}\frac{dM}{M}. \tag{13}$$

Finally, we get $\frac{dp}{p}$ using equation (8) (which has been obtained from equation of state),

$$\frac{dp}{p}=-\left[\frac{c_p}{\alpha M^2\gamma R}+1\right]\left[\frac{c_p}{\alpha M^2\gamma R}+\frac{1}{2}\right]^{-1}\frac{dM}{M}. \tag{14}$$

We have now obtained $\frac{dT}{T}$, $\frac{du}{u}$, $\frac{d\rho}{\rho}$, and $\frac{dp}{p}$ as functions of $\frac{dM}{M}$ (equations (11), (12), (13), and (14)). We re-iterate here that as we shall see, friction in the fluid will affect the axial velocity profile by the momentum conservation equation (equation (2)), and will thus alter the variation of M and dM with x. But, equations (11), (12), (13), and (14) are independent of whether the fluid has friction or not, and hence, any changes to $\frac{dT}{T}$, $\frac{du}{u}$, $\frac{d\rho}{\rho}$, and $\frac{dp}{p}$ will occur only implicitly due to how the variation of M with x gets affected by fluid friction.

Now, to determine how variation of M with x gets affected by fluid friction, we will now look at the momentum conservation equation, equation (2),

$$\frac{dp}{\rho u^2}=-\left[\frac{\tau_w}{\rho u^2}\frac{C}{A}dx+\frac{du}{u}\right]. \tag{15}$$

Substituting $\frac{du}{u}$ from equation (12) and ρ from equation (13) into equation (15), we will have,

$$\frac{1}{u^2}\frac{dp}{d\rho}=\frac{\left[\frac{\tau_w}{\rho u^2}\frac{C}{A}dx+\frac{c_p}{\alpha M^2\gamma R}\left[\frac{c_p}{\alpha M^2\gamma R}+\frac{1}{2}\right]^{-1}\frac{dM}{M}\right]}{\frac{c_p}{\alpha M^2\gamma R}\left[\frac{c_p}{\alpha M^2\gamma R}+\frac{1}{2}\right]^{-1}\frac{dM}{M}}. \tag{16}$$

Now, as per definition of sonic speed, $\frac{dp}{d\rho}=c^2$. Substituting this in equation (16), and given the definition of Mach number M is $M=\frac{u}{c}$, we have,

$$\frac{1}{M^2}=\frac{\left[\frac{\tau_w}{\rho u^2}\frac{C}{A}dx+\frac{c_p}{\alpha M^2\gamma R}\left[\frac{c_p}{\alpha M^2\gamma R}+\frac{1}{2}\right]^{-1}\frac{dM}{M}\right]}{\frac{c_p}{\alpha M^2\gamma R}\left[\frac{c_p}{\alpha M^2\gamma R}+\frac{1}{2}\right]^{-1}\frac{dM}{M}}\Rightarrow \quad . \tag{17}$$

$$\frac{c_p}{\alpha\gamma R}\left(\frac{1-M^2}{M^2}\right)\frac{dM}{dx}=\frac{\tau_w}{\rho u^2}\frac{C}{A}\left[\frac{c_p M}{\alpha\gamma R}+\frac{M^3}{2}\right]$$

Solving equation (17) will give us the variation of M with x, which will evidently be affected by viscous effects due to the presence of the τ_w term that is predominantly dictated by the friction in the fluid.

Before proceeding to the next topic, we summarize that we have incorporated fluid friction into the formulation of one-dimensional compressible flow in the analysis above. Similar to incorporation of fluid friction by addition of the wall stress term in momentum conservation equation, any heat transfer occurring in the system can be incorporated by appropriately modifying the energy balance equation. And the analysis for such a case also remains similar to the one presented above.

Maximum Entropy and Change in Entropy:

Once we have obtained M as a function of x, with or without fluid friction, we will appeal to the second law of thermodynamics to see how the change of entropy occurs in a compressible flow.

First, we will observe what is the situation where maximum entropy occurs. As has been derived earlier in the course and is also an important equation in thermodynamics, we have,

$$Tds=dh-\frac{dp}{\rho}\,. \tag{18}$$

Any system approaches a state of highest disorder, i.e. maximum entropy. At a stage where maximum entropy has been achieved by the fluid, ds will be zero, and hence, equation (18) will become,

$$dh=\frac{dP}{\rho}. \tag{19}$$

From equation (3), we substitute dh into equation (19) to get,

$$-udu=\frac{dp}{\rho}\,. \tag{20}$$

Next, dividing equation (20) by u^2 and substituting $\frac{du}{u}$ from equation (1), we have,

$$\frac{d\rho}{\rho}=\frac{dp}{\rho u^2}\Rightarrow\frac{dp}{d\rho}=u^2\,. \tag{30}$$

Since the definition of sonic speed c is $c = \frac{dp}{d\rho}$, equation (30) becomes,

$$c^2 = u^2 \Rightarrow M = 1. \tag{31}$$

Hence, we can see that the state of maximum entropy corresponds to Mach number of 1, i.e. the sonic condition, a very important observation.

Now, we will obtain the change in entropy for situation where entropy is not maximized.

Continuing with equation (18) again, we divide the entire equation by T, then substitute $dh = c_p dT$ and $RT = \frac{p}{\rho}$ as the fluid being considered is an ideal gas. Hence, we have,

$$ds = c_p \frac{dT}{T} - R\frac{dp}{p}. \tag{32}$$

Now, substituting $\frac{dp}{p}$ in terms of $\frac{dM}{M}$ from equation (14) and then $\frac{dM}{M}$ in terms of $\frac{dT}{T}$ from equation (11), we have,

$$ds = \left[c_p - R\left(\frac{c_p}{\alpha M^2 \gamma R} + 1 \right) \right] \frac{dT}{T}. \tag{33}$$

Equation (33) implies that for the case where entropy is not maximum, the change in entropy will depend only on temperature. In figure 2, we present the entropy variation with temperature for a compressible flow. The maximum entropy corresponds to Mach number of 1, which corresponds to the point marked "*M* = 1" in figure 2. Furthermore, we have earlier deduced that the variation in Mach number and in temperature are opposite of each other, i.e. when Mach number decreases, temperature increases and vice versa. Hence, as we move leftward of "*M* = 1" along the curve, since the temperature is decreasing, Mach number will increase and we will have increasingly supersonic flow. Similarly, as we move rightward of "*M* = 1" along the curve, since the temperature is increasing, Mach number will decrease and we will have increasingly subsonic flow. We note here that figure 2 is equally applicable in presence and absence of fluid friction, as it has been obtained without appealing to equation (2), the momentum conservation equation, which brings fluid friction into the picture.

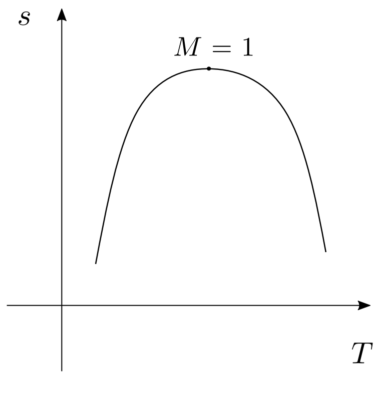

Figure 2: variation of entropy with temperature in a compressible flow