Advanced Concepts in Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 58
Compressible Flows (Converging Diverging Nozzle) (Contd.)

In this lecture, we will present another example of compressible flow through converging-diverging nozzle to further enhance our understanding of the topic.

Example: A converging-diverging nozzle is designed to expand air from a chamber in which p_0 is 800 kPa, T_0 is 40 ^{0}C to give M as 2.7 at the exit. Area of the throat is 0.08 m^2. We seek the answer to the following questions:

i. What is the area at exit, A_e?
ii. What is the design mass flow rate, $\dot{m}_{design}$?
iii. What is the lowest back pressure for which there is subsonic flow throughout the nozzle?
iv. What is the design back pressure?
v. What is the back pressure for which normal shock will occur at the exit plane?
vi. What is the range of back pressure for which there is no shock inside the nozzle?
vii. What is the range of back pressure for which there will be oblique shock at the exit?
viii. What is the range of back pressure for which there will be oblique expansion wave at the exit?

Solution:

This problem deals with different back pressure conditions for which different Mach number conditions occur at the exit. To aid us in obtaining the solution, we use the schematic as presented in figure 1.

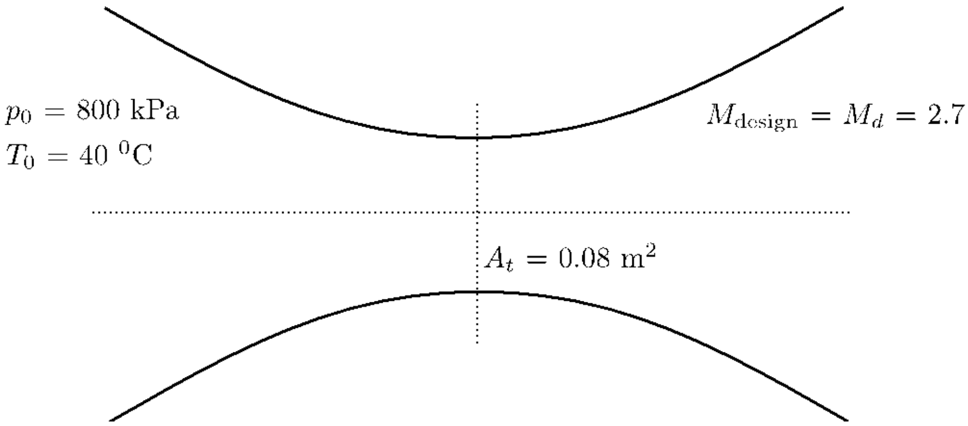

Figure 1: Schematic for problem

Before proceeding further, we re-iterate the point that generally, the area at the throat, A_t, is not necessarily the sonic condition area, A^*. It occurs in the particular solution when sonic condition exists in the converging-diverging nozzle – if sonic condition exists in the converging-diverging nozzle, it must exist at the throat and in such a scenario, $A_t = A^*$.

We now proceed with answering the queries in the problem.

First, we obtain the exit area A_e. For this, we start with the relation of $\frac{A}{A^*}$ as was derived earlier in the course,

$$\frac{A}{A^*} = \frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{\frac{1+\gamma}{2(1-\gamma)}}. \tag{1}$$

We apply equation (1) at the exit.

$$\frac{A_e}{A^*} = \frac{1}{M_e}\left[\frac{\gamma+1}{2+(\gamma-1)M_e^{\ 2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}}. \tag{2}$$

Note that equation (1) applies for a situation where there is no shock. Hence, we have been able to apply equation (1) at the exit for the design condition, which is characterized as having no shock. Furthermore, the sonic condition area A^* is the same throughout the nozzle. Also, since we are considering the nozzle to be operating at design condition (for obtaining the exit area as per design), sonic area matches the area at the throat and the outlet Mach number is same as the design Mach number, , i.e. $A^* = \mathrm{A}_t$ and $M_e = M_d$. Substituting these, equation (2) becomes,

$$A_e = \frac{A_t}{M_d}\left[\frac{\gamma+1}{2+(\gamma-1)M_d^{\ 2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}} \tag{3}$$

Knowing all the terms in the RHS of equation (3), and given γ = 1.4 for air, we obtain the value of A_e as <u>0.255 m^2</u>. *(answer to query i)*

Next, we obtain $\dot{m}_{design}$. We simply use the equations,

$$\dot{m}_{design} = \dot{m}^* = \rho^* A^* u^* = \frac{p^*}{RT^*} A^* M^* \sqrt{\gamma RT^*} = \frac{p^* A^* \sqrt{\gamma}}{\sqrt{RT^*}}. \tag{4}$$

In equation (4), * means the sonic condition, which exists at the throat. We have used the equation of state for an ideal gas to substitute $\rho^* = \frac{p^*}{RT^*}$ and the definition of sonic speed and Mach number to substitute $u^* = M^* \sqrt{\gamma RT^*}$. Lastly, since * corresponds to the sonic condition, M^* is 1. To get T^*, we recall the equation for $\frac{T_0}{T}$ and use it for the sonic condition (equation (5) of lecture 53),

$$\frac{T_0}{T^*} = \frac{\gamma + 1}{2}. \qquad (5)$$

Further, using the relation of pressure and temperature for adiabatic flow, we have,

$$\frac{P_0}{P^*} = \left(\frac{T_0}{T^*}\right)^{\frac{\gamma}{\gamma-1}} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}. \qquad (6)$$

where we have substituted the expression for $\frac{T_0}{T^*}$ from equation (5). Equation (5) gives T^* as -12.19 ^{0}C and equation (6) gives us p^* as 422.63 kPa. Finally, substituting these two into equation (4) gives us $\dot{m}_{design}$, which comes out as <u>146 kg/s</u>. *(answer to query ii)*

For the next part, i.e. query iii, we recall the variation of pressure inside the nozzle with decreasing value of back pressure, as was presented in the previous lecture and has been included in figure 2 below.

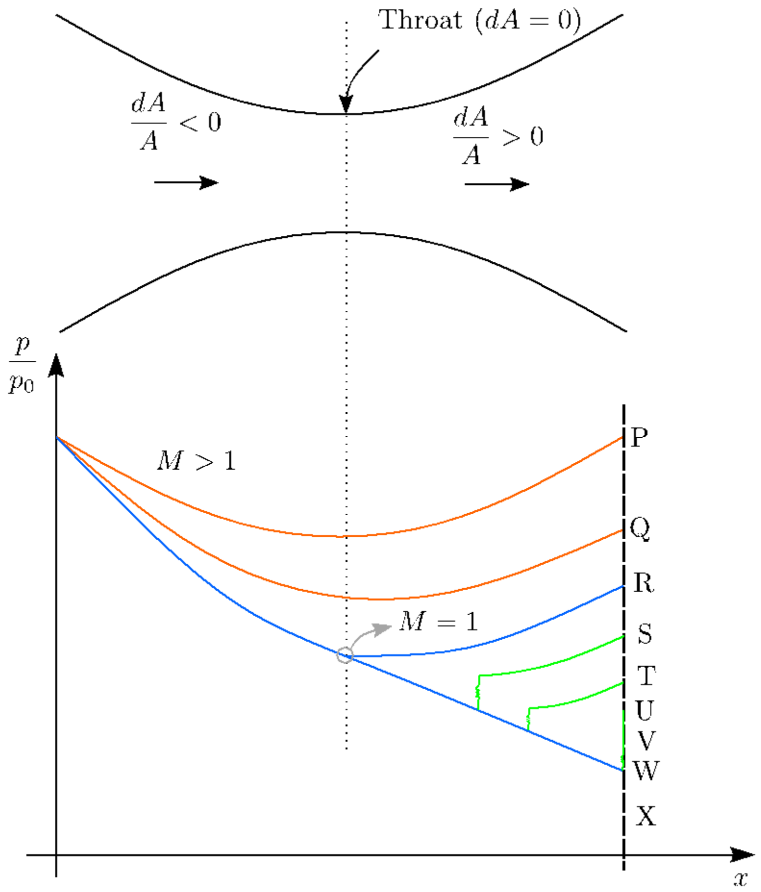

Figure 2: Variation of pressure inside the nozzle with decreasing value of back pressure

We know that as we decrease the back pressure, the critical condition of the throat occurs at the point R. For any back pressure lower than the point R, the flow becomes supersonic for part of the diverging section and undergoes a shock to become subsonic at nozzle exit. Hence, the minimum back pressure for which the flow is subsonic throughout the nozzle corresponds to R, denoted as P_R. To get this, we start with the equation (1) and apply it to the exit, giving us equation (2). However, now we have the exit area and sonic area (which is same as the throat area) known and we wish to know the other solution for M_e, i.e. the subsonic solution which corresponds to the back pressure at point R in figure 2 – we denote this Mach number as $M_{e(R)}$. Hence, we use,

$$\frac{A_e}{A_t} = \frac{1}{M_{e(R)}}\left[\frac{\gamma+1}{2+(\gamma-1)M_{e(R)}^{\ 2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}}, \quad (7)$$

to obtain $M_{e(R)}$, whose value is 0.185. To obtain the p_R, we recall the relation derived earlier in the course,

$$\frac{p_0}{p} = \left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}}, \quad (8)$$

and apply it to the exit with the subsonic Mach number,

$$\frac{p_0}{p_R} = \left[1+\frac{\gamma-1}{2}M_{e(R)}^2\right]^{\frac{\gamma}{\gamma-1}}. \quad (9)$$

Substituting the values, we get p_R as 780.5 kPa. *(answer to query iii)*

Next, we want to obtain the design back pressure $p_{e(W)}$, i.e. corresponding to point W in figure 2. Here again, we use equation (8) and apply it to the exit of the nozzle with the Mach number as the design Mach number (M_d),

$$\frac{p_0}{p_{e(W)}} = \left[1+\frac{\gamma-1}{2}M_d^{\ 2}\right]^{\frac{\gamma}{\gamma-1}}. \quad (10)$$

Substituting the numerical values, we get, $p_{e(W)}$ as 34.4 kPa. *(answer to query iv)*

Now, we want to ascertain the back pressure for which the normal shock will occur at the exit of the nozzle. This corresponds to the point U as we had deduced in the last lecture. For this situation, the flow is isentropic for the length of the nozzle upto just upstream of the exit plane and we also recover the design condition at just upstream of the exit plane.

Hence, we apply equation (1) at just the upstream of the exit, denoted by subscript ‘up’ (we denote the downstream of the exit with subscript ‘dw’), to get,

$$\frac{A_{up}}{A^*} = \frac{1}{M_d}\left[\frac{\gamma+1}{2+(\gamma-1)M_d^{\ 2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}}. \quad (11)$$

Equation (11) will give us the value for A_{up}, which will come out same as A_e i.e. 0.255 m^2. We have derived the relation between Mach numbers upstream and downstream of a normal shock (equation (4) of lecture 55), which we apply to the shock in this situation (substituting $\gamma = 1.4$ for air),

$$M_{dw}^2 = \frac{M_{up}^2 + 5}{7M_{up}^2 - 1} = \frac{M_d^2 + 5}{7M_d^2 - 1}. \qquad (12)$$

Upon substituting $M_d = 2.7$, equation (12) gives us M_{dw} as 0.4956. Similarly, we have the relation for pressure upstream and downstream of the normal shock (equation (7) of lecture 54) as,

$$p_{up}(1+\gamma M_{up}^2) = p_{dw}(1+\gamma M_{dw}^2). \qquad (13)$$

Now, p_{up} is simply $p_{e(W)}$, M_{up} is same as M_d which is 2.7 and M_{dw} has been obtained above as 0.4956. Substituting these in equation (13), we will get p_{dw} is simply the back pressure for which the normal shock will occur at the exit of the nozzle, i.e. $p_{e(U)}$ (corresponding to point U), whose numerical value is obtained as 286.5 kPa. *(answer to query v)*

Now, we want to ascertain the range of back pressure for which there is no shock inside the nozzle as well as the range of back pressure for which there is oblique shock at the exit. We know from the analysis of figure 2 in the last lecture that as the back pressure varies from the point P to X, there is no shock for back pressure from point P to point R. Thus, there is no shock for back pressure values higher than p_R , i.e. 780.5 kPa. However, this range is the trivial range of back pressure where there is no supersonic flow also. As we move to further lower back pressure (lower than point R), a normal shock occurs inside the nozzle up till the back pressure reaches the point U. At point U, there is normal shock at the nozzle exit (i.e. not inside the nozzle). For pressures between point U and the design point W, there is oblique shock outside the nozzle exit and for pressures lower than point W, there is oblique expansion wave outside the nozzle exit. Hence, for back pressure lower than $p_{e(U)}$ i.e. 286.5 kPa, there is no shock inside the nozzle. And the obtained flow is subsonic if back pressure is between $p_{e(U)}$ (286.5 kPa) and $p_{e(W)}$ (34.4 kPa), because the flow transitions to subsonic due to normal shock (if back pressure is $p_{e(U)}$ = 286.5 kPa) or oblique shock (if back pressure lies between $p_{e(U)}$ = 286.5 kPa and $p_{e(W)}$ = 34.4 kPa) at the nozzle exit. If the back pressure is $p_{e(W)}$ (34.4 kPa), the obtained flow is supersonic. *(answers to query vi and vii)*

Summarily, in the compressible flow component of the course, we have studied the fundamental of one-dimensional compressible flow with and without shock, with major focus on converging and converging-diverging nozzle. We have used illustrative examples at different points to elucidate the concepts, with the elaborate example problem studied in this lecture serving as a representative problem giving us the complete picture of the range of possibilities in a converging-diverging nozzle.

However, till now, we have made two important assumptions in the study of compressible flows. First, the flow is adiabatic and second, the flow is reversible. In particular, except for the location of the shock, the flow is assumed isentropic throughout. However, viscous effects (i.e. friction in the flow) can be non-negligible is some situations of compressible flows. In such a scenario, while the flow can still remain appreciable close to adiabatic (by virtue of rapidity of flow brought about by the high speeds in compressible flows), it will cease to be reversible. Thus, in the next lecture, we will see how considering friction in the flow alters the modelling and analysis of one-dimensional compressible flows.