Advanced Concepts in Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 57
Compressible Flows (Converging Diverging Nozzle)

In the previous lecture, we arrived at the important conclusion that a converging nozzle is limited to accelerate a flow only to Mach number 1 at its throat, and not to higher speeds. Hence, to accelerate the fluid to Mach numbers larger than 1, we need to design beyond a convergent nozzle and the standard approach is to design a converging-diverging nozzle.

Compressible flow through converging-diverging nozzle:

To understand how a converging-diverging nozzle will help us achieve super-sonic speed, we will first recall our analysis of compressible flow through a variable area conduit (lecture 52). We had arrived at the relation between change in area and change in Mach number (equation (13) of lecture 52) as,

$$\frac{dA}{A}=\frac{M^2-1}{1+\frac{\gamma-1}{2}M^2}\frac{dM}{M}. \qquad (1)$$

We point out here that we are considering converging-diverging nozzle as a variable area section attached ahead of the converging nozzle studied in the last lecture. Hence, the throat is now at an intermediate axial location and corresponds to the point of least nozzle area. Furthermore, at its maximum utilization, the converging section provides us with flow of Mach number as 1 at its throat, and we want to utilize the newly-attached variable area section ahead to further increase the Mach number along the rest of the nozzle length. In mathematical terms, it means that we want dM to be positive. Since $M>1 \Rightarrow M^2-1>0$ is expected for this additional length, we observe that all the terms on RHS of equation (1) are positive and this implies dA should be positive. Hence, this additionally attached variable-area section should be a diverging section. Resultantly, we have deduced that we require the commonly-studied converging-diverging nozzle to accelerate a compressible flow from sub-sonic to super-sonic speed. However, merely using a converging-diverging nozzle does not guaranteed that we will get Mach number greater than 1 in the diverging section and the outlet. To achieve this objective, we have to appropriately attune the back pressure, i.e. the pressure at the exit of the nozzle (exit of the diverging section for the converging-diverging nozzle being studied here).

For this, we will now look into the influence of back pressure on the Mach number. We present, in figure 1, the profile of the converging-diverging nozzle on top and variation of $\frac{p}{p_0}$ with the axial distance x along length of the nozzle on the bottom. Alongwith this, we present, in figure 2, the variation of $\frac{\dot{m}}{\dot{m}_{\max}}$ with $\frac{p_b}{p_0}$, where p_b is the back pressure.

Before proceeding, we had deduced (in lecture 52) that if sonic condition (i.e. $M=1$) exists in the nozzle, it will be at the throat (i.e. where area is minimum and thus dA is zero), but it is not assured that there will be sonic condition at the throat – the condition at the throat can also be subsonic, depending on the back pressure.

Hence, it is possible to have multiple scenarios. These scenarios are explored by keeping the stagnation pressure (i.e. the pressure in the reservoir that is connect to the inlet of the nozzle at the start of its converging section) fixed and varying the back pressure. The scenarios that we will get are bucketed into two cases:

Case 1 – The flow speed at the throat is less than the sonic speed. This is represented by the two orange plots in figure 1. The back pressure is lower for Q as compared to P.

Case 2 – The flow speed at the throat is equal to the sonic speed. This is represented by the two blue plots in figure 1. These plots are identical for the converging section up to the throat, i.e., the flow converges to a sonic condition at the throat for either scenarios. However, after this point, the two plots deviate from each other – one plot observes further decrease in pressure and culminates at S, leading to a supersonic flow, and, another plot observe increase in pressure after the throat and culminates at R, leading to a subsonic flow. To understand why these two possibilities arise, we recall that we had obtained the relation (equation (8) in lecture 53),

$$\frac{A}{A^*}=\frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{\frac{1+\gamma}{2(1-\gamma)}}. \tag{2}$$

In equation (2), A^* is the reference sonic condition area, which may or may not exist in the actual nozzle. However, for this particular scenario (i.e. the two blue plotlines in figure 1), flow at the throat is sonic and so, A^* is the area of the throat. Substituting the value of A as the area of the exit of the nozzle in equation (2), we can solve for the Mach number at the exit. However, the form of equation (2) indicates that it will admit two solutions for this Mach number. These two solutions actually correspond to the points R and S, i.e. the subsonic and supersonic conditions at outlet.

Here, we state a crucial point. The objective of attaching the additional divergent section was to obtain a supersonic flow at the outlet. This objective gets quashed if we allow the outlet condition to correspond to point R. Therefore, to recovers the maximum possible acceleration from the converging-diverging nozzle, one has to regulate the back pressure to the point W – hence, the point W is also called the design condition of the nozzle and specifies the designed back pressure for the nozzle that the user is expected to provide.

Now, if the back pressure is regulated to a value between R and W, we obtain a shock in the flow. We consider increasingly lower back pressure starting from R. We first consider the two points – S and T. For these back pressure values, we obtain a normal shock inside the diverging section of the nozzle, illustrated in the figure 1 as the green plots. The flow prior to the normal shock is the same as would be for design condition, but the flow after the normal shock is slowed to subsonic. Also, the location of the shock gets closer to the outlet as the back pressure gets smaller. This reaches a limiting case when the normal shock is at exactly the exit cross-section of the nozzle, corresponding to the point U. If the back pressure is regulated to any value between U and W, e.g. V, we obtain an oblique shock further outside the nozzle from the exit. Lastly, we note that it is possible to regulate the back pressure to be even smaller than W, i.e. X. In such a situation, we obtain an oblique expansion wave outside the nozzle.

Summarily, as we increase the back pressure from P to W,

P - the condition throughout the nozzle is subsonic

Q - the condition throughout the nozzle is subsonic, but the flow speed is higher than for P

R – the condition at the throat is sonic, but the back pressure provided is such that the flow at the exit has slowed down to subsonic condition

S – the flow becomes the supersonic for part of the divergent section of the nozzle after the throat but undergoes a normal shock within the divergent section and transitions to subsonic condition, and hence the exit flow is subsonic

T - the flow becomes the supersonic for part of the divergent section of the nozzle after the throat but undergoes a normal shock within the divergent section and transitions to subsonic condition, and hence the exit flow is subsonic; however, the shock front is further ahead than for S

U – the flow becomes the supersonic for part of the divergent section of the nozzle after the throat but undergoes a normal shock at the nozzle exit and transitions to subsonic condition, and hence the exit flow is subsonic

V – there is an oblique shock outside the nozzle

W – the provided back pressure matches the design condition and we obtain a supersonic flow with the maximum possible acceleration using the particular convergent-divergent nozzle being considered

X – we obtain an oblique expansion wave outside the nozzle

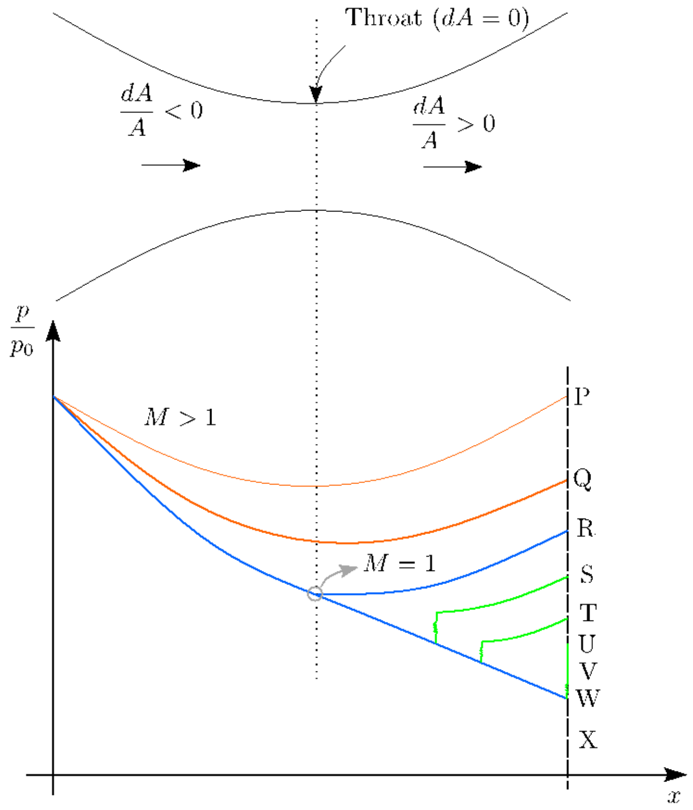

Figure 1: Variation of pressure inside the nozzle with decreasing value of back pressure

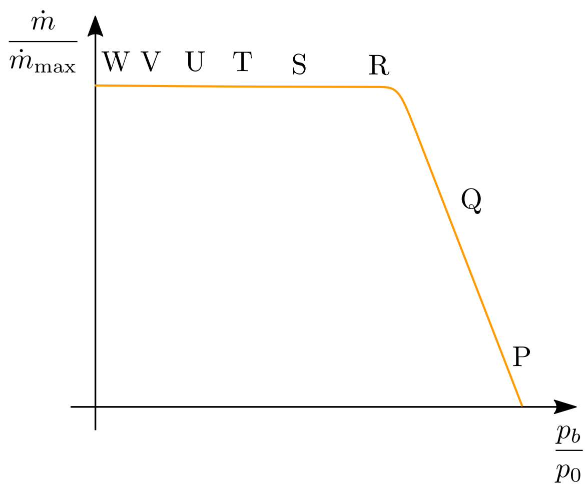

Figure 2: Variation of mass flow rate with back pressure

We now ascertain how the mass flow rate will vary with the back pressure, as presented in figure 2.

The mass flow rate increases as we decrease the back pressure starting from P, but it gets saturated once the back pressure reduces to R. Beyond this, any further reduction in back pressure leads to increase in Mach number, eventually transitioning to supersonic flow at exit when back pressure reduced to W, but it is not accompanied by any increase in the mass flow rate.

Hence, we have now formed a background that gives us the design specification for a nozzle based on the desired output – a converging nozzle can be used to accelerate a flow to subsonic or in the limiting case a sonic condition, whereas to obtain supersonic flow, we have to use a converging-diverging nozzle, the diverging section coming in use to accelerate the flow beyond sonic speed.

We will now work out an example problem to illustrate the concepts and develop quantitative understanding of compressible flow through a converging-diverging nozzle.

Example: Consider the converging nozzle, with the three sections 1, 2, and 3 with their areas given. Note that section 2 is any arbitrary section and not necessarily the throat. A normal shock occurs at the section 2. Also given is: M_1 = 2.5, p_1 = 40 kPa, T_1 = 30 ^{0}C. Find: (1) $\dot{m}_1$, (2) M_3, (3) stagnation pressure at section 3, $p_{0,3}$.

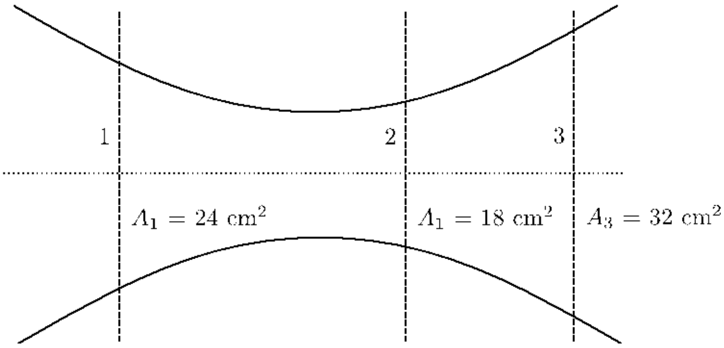

Figure 3: Schematic for example problem

Solution:

Before proceeding to obtain the solution, we highlight two crucial points.

First, in the converging section, we observe Mach number higher than 1 (M_1 = 2.5). While this might seem to contradict the deduction that 'we cannot obtain a Mach number higher than 1 in a converging nozzle or converging section of a converging-diverging nozzle', it is not so. Stated more clearly, the deduction actually says that a subsonic flow cannot be accelerated to a supersonic flow by passing it through a convergent nozzle (or convergent section of a convergent-divergent nozzle). However, if the inlet flow itself is supersonic, Mach number can be greater than 1 inside the converging nozzle – Mach number will decrease though, owing to the decreasing cross section area of the converging section along the axial length.

Second, because shock is occurring at the section 2, we expect the flow to be isentropic between section 1 and just before section 2 (we call this region 2-upstream and denote it with '2u'), and between just after section 2 (we call this region 2-downstream and denote it with '2d') and section 3. Between 2u and 2d, we have to use the relations for shock (as obtained in lectures 54 and 55) to relate the flow properties.

We first obtain $\dot{m}_1$, which is given as,

$$\dot{m}_1 = \rho_1 A_1 u_1 = \frac{p_1}{RT_1} A_1 M_1 \sqrt{\gamma R T_1} = \frac{\sqrt{\gamma_1}\, p_1 A_1 M_1}{\sqrt{RT_1}} \,. \tag{3}$$

Substituting the numerical values of all the parameters, we get $\dot{m}_1$ as 0.96 kg/s.

Next, we have to obtain M_3. To do this, we have to move along the flow and first obtain the M_{2u}. We start with equation (2) and apply it at section 1,

$$\frac{A_1}{A_1*} = \frac{1}{M_1}\left[\frac{\gamma+1}{2+(\gamma-1)M_1^{\,2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}} \Rightarrow A_1* = A_1 M_1 \left[\frac{\gamma+1}{2+(\gamma-1)M_1^{\,2}}\right]^{\frac{1+\gamma}{2(\gamma-1)}} . \tag{4}$$

Since A_1* is the only unknown in equation (4), we are able to obtain its value as 9.1022 cm^2. Now, A_1* is the same as $A_{2u}*$. Hence, applying equation (2) to section 2u now, we have,

$$\frac{A_{2u}}{A_{2u}*} = \frac{A_{2u}}{A_1*} = \frac{1}{M_{2u}}\left[\frac{\gamma+1}{2+(\gamma-1)M_{2u}^{\,2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}} . \tag{6}$$

Equation (6) has M_{2u} as the only unknown and upon solving, we will arrive at two solutions, one higher than 1 and one lower than 1. Since 2u is upstream of the shock and we have deduced earlier in the course that a shock transitions a supersonic flow to a subsonic flow, we admit the solution that is higher than 1, which is 2.1844. Next, we related M_{2u} and M_{2d} as per the relation between the Mach number at the two sides of a normal shock (equation (4) in lecture 55),

$$M_{2d}^2 = \frac{2+(\gamma-1)M_{2u}^2}{2\gamma M_{2u}^2-(\gamma-1)} . \tag{7}$$

This gives us M_{2d} as 0.5492.

Now, we again take the help of equation (2) to obtain $A_{2d}*$,

$$\frac{A_{2d}}{A_{2d}*} = \frac{1}{M_{2d}}\left[\frac{\gamma+1}{2+(\gamma-1)M_{2d}^{\,2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}} \Rightarrow A_{2d}* = A_{2d} M_{2d} \left[\frac{\gamma+1}{2+(\gamma-1)M_{2d}^{\,2}}\right]^{\frac{1+\gamma}{2(\gamma-1)}} , \tag{8}$$

which comes as 14.3295 cm^2. Note that due to the presence of shock at section 2, the $A*$ is different on either side and thus, we have to obtain $A_{2d}*$ separately from $A_{2u}*$. The obtained value of $A*$ at 2d remains the same at 3 due to the isentropic nature of flow between 2d and 3, i.e. $A_3* = A_{2d}*$. We again use equation (2) and apply it at section 3,

$$\frac{A_3}{A_3*} = \frac{A_3}{A_{2d}*} = \frac{1}{M_3}\left[\frac{\gamma+1}{2+(\gamma-1)M_3^{\,2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}} . \tag{9}$$

Equation (9) has M_3 as the only unknown, and upon solving we will obtain two roots, one larger than one and one smaller than 1. To determine which solution to admit, we examine the equation relating dA with dM,

$$\frac{dA}{A} = \frac{2(M^2 - 1)}{\gamma + 1} \frac{dM}{M}. \tag{10}$$

Since M_{2d} is less than 1, the term $\frac{2(M^2-1)}{\gamma+1}$ is negative at 2d. And dA is negative throughout the region between 2d and 3. Therefore, dM will start decreasing at 2d and will continue decreasing without $\frac{2(M^2-1)}{\gamma+1}$ switching sign anywhere. Therefore, M_3 is smaller than 1. Hence, the obtained value of M_3 is 0.27.

Lastly, to get the stagnation pressure $p_{0,3}$, we use a similar approach as is used to obtain M_3. That is, we start from section 1 and proceed with the flow, utilizing the $\frac{p_0}{p}$ relation (equation (9) of lecture 50),

$$\frac{p_0}{p} = \left[1 + \frac{\gamma - 1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}, \tag{11}$$

instead of equation (2). Applying equation (11) at section 1,

$$\frac{p_{0,1}}{p_1} = \left[1 + \frac{\gamma - 1}{2} M_1^2\right]^{\frac{\gamma}{\gamma-1}} \Rightarrow p_{0,1} = p_1 \left[1 + \frac{\gamma - 1}{2} M_1^2\right]^{\frac{\gamma}{\gamma-1}}, \tag{12}$$

we get $p_{0,1}$, which is the same as $p_{0,2u}$. Hence, we have,

$$p_{0,2u} = p_1 \left[1 + \frac{\gamma - 1}{2} M_1^2\right]^{\frac{\gamma}{\gamma-1}}. \tag{13}$$

As we had derived earlier (equation (7) of lecture 54), the pressure upstream and downstream of the shock at section 2 are related as,

$$p_{2u}(1 + \gamma M_{2u}^2) = p_{2d}(1 + \gamma M_{2d}^2), \tag{14}$$

and these pressures are related to the respective stagnation pressures (as per equation (11)) as,

$$\frac{p_{0,2u}}{p_{2u}} = \left[1 + \frac{\gamma - 1}{2} M_{2u}^2\right]^{\frac{\gamma}{\gamma-1}}, \frac{p_{0,2d}}{p_{2d}} = \left[1 + \frac{\gamma - 1}{2} M_{2d}^2\right]^{\frac{\gamma}{\gamma-1}} \tag{15}$$

Combining equation (14) and (15), and then equation (13), we have,

$$\frac{p_{0,2u}}{p_{0,2d}} = \left(\frac{1+\gamma M_{2d}^2}{1+\gamma M_{2u}^2}\right)\left[\frac{2+(\gamma-1)M_{2u}{}^2}{2+(\gamma-1)M_{2d}{}^2}\right]^{\frac{\gamma}{\gamma-1}} \Rightarrow$$

$$p_{0,2d} = \left(\frac{1+\gamma M_{2u}^2}{1+\gamma M_{2d}^2}\right)\left[\frac{2+(\gamma-1)M_{2d}{}^2}{2+(\gamma-1)M_{2u}{}^2}\right]^{\frac{\gamma}{\gamma-1}} p_{0,2u} \Rightarrow \tag{16}$$

$$p_{0,2d} = \left(\frac{1+\gamma M_{2u}^2}{1+\gamma M_{2d}^2}\right)\left[\left(\frac{2+(\gamma-1)M_{2d}{}^2}{2+(\gamma-1)M_{2u}{}^2}\right)\left(1+\frac{\gamma-1}{2}M_1{}^2\right)\right]^{\frac{\gamma}{\gamma-1}} p_1$$

Once we obtain $p_{0,2d}$ using equation (16), $p_{0,3}$ is the same, i.e. $p_{0,3} = p_{0,2d}$. Hence, we obtain $p_{0,3}$ as 435 kPa.