Advanced Concepts in Fluid Mechanics
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Lecture – 56
Compressible Flows (Converging Nozzle)

Compressible Flow through a Converging Nozzle:

Based on the formalism developed on supersonic flows and subsonic flows and sonic flows as special case, we will now analyse the very important engineering problem of compressible flow through nozzles.

To understand this topic, we need to first establish the purpose of a nozzle. A nozzle is primarily used to accelerate a flow and many nozzles are typical shaped as convergent (presented in figure 1). This shape is the appropriate design when compressibility effects are negligible (i.e. incompressible flow situation), as for such a case, the mass flow conservation translates directly in volume flow rate conservation, which establishes as inverse relation between change in area and average flow speed. However, for a compressible flow, the density, in particular its variation, becomes a crucial parameter and prevents the situation from being as trivial as for an incompressible flow situation.

To analyse how does compressibility exclusively affect the flow behaviour in a nozzle, we will first analyse the converging nozzle in the purview of a particular physical setup, which consists of a tank filled with gas in stagnation condition with pressure p_0 and temperature T_0. The tank is connected to a chamber (called as plenum chamber) by the convergent nozzle. The complete setup is presented in figure 1. The plenum chamber is a chamber such that the pressure inside it can be controlled by some experimental means (typically by regulation of a valve attached to the plenum chamber), and this pressure is commonly called as 'back pressure', p_b. Gas will flow out of the reservoir when the reservoir has higher pressure, i.e., $p_0 > p_b$. Now, let us consider the situation where an experimentalist attempts to indirectly control the mass flow rate of gas out of the reservoir (via the nozzle), $\dot{m}$, by varying the back pressure (p_b) while keeping the reservoir pressure fixed (p_0) and monitoring the mass flow rate through the nozzle. Intuitively, we expect a reduction in p_b will lead to higher pressure gradient between tank and plenum chamber, and hence will lead to higher flow rate. In this manner, we also anticipate that the mass flow rate can perpetually increase as well keep decreasing the back pressure so long the tank has enough gas to act like a reservoir at stagnant condition. However, we will demonstrate ahead that the increase of mass flow rate is not perpetual. Decreasing the back pressure p_b from a higher magnitude, mass flow rate initially increases but eventually saturates to a critical value $\dot{m}_{\max}$ when p_b reaches a threshold, the sonic pressure $p*$. At this condition, the nozzle is said to be 'chocked'. We have graphically depicted this phenomenon in figure 2, which presents the variation of mass flow rate through the nozzle with back pressure.

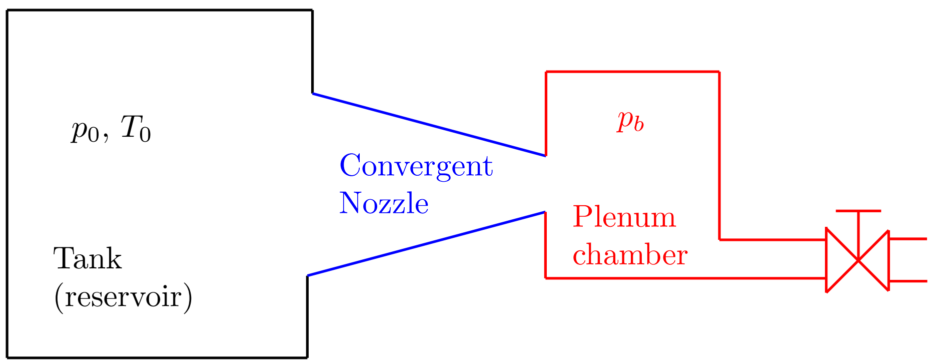

Figure 1: Physical setup to analyse effect of compressibility in flow through a convergent nozzle

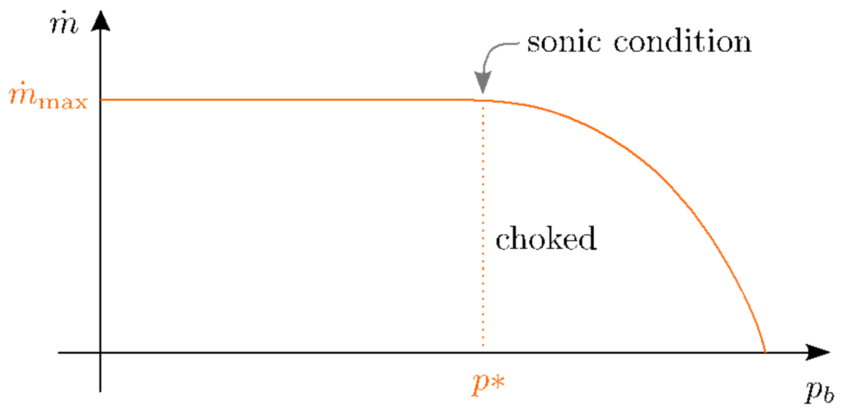

Figure 2: Variation of mass flow rate through the nozzle with back pressure, demonstrating the choking at the sonic condition

We will now analyse the physics behind this choking phenomenon. For this, we recall derivation of the relation between change in area and Mach number for flow through variable area ducts (as was derived in equation (5) of lecture 52). We had arrived at a crucial deduction in that derivation – 'while occurrence of sonic condition is not assured during flow through a variable area duct, if sonic condition does occur, it must occur at the throat i.e. the axial location where the duct area is the least'. The throat for the nozzle presented in figure 1 is the location where it is attached to the plenum chamber, which is essentially the exit of the nozzle as well. Hence, the state of gas at the throat of the nozzle is same as the state of gas inside the plenum chamber.

Now, as we decrease p_b, the pressure difference between the tank and plenum chamber decreases and mass flow rate increases. The mechanism by which this occurs is that the information of any decrease in p_b reaches the tank, and in response the mass flow rate out of the tank and through the nozzle increases. However, as the flow rate increases to the point where the condition in the plenum chamber (and hence the throat of the nozzle) becomes sonic, the speed of air flowing out (which equals the sonic speed) exactly cancels out the speed by which the information of decrease in p_b is transmitted to the tank (in the form of a pressure wave at sonic speed and in opposite direction). Hence, information of any further decrease in p_b does not get transmitted to the tank and therefore, the mass flow rate does not increase any further, even with decreasing p_b. This explains the variation of $\dot{m}$ with p_b as observed in figure 2. For a situation where the velocity of air flow would be say twice the sonic speed in the forward direction, this effect would be exaggerated any information moving in the backward direction at sonic speed relative to air will actually be moving in the forward direction at sonic speed. Summarily, if the flow of air out of the reservoir is at a speed higher than the sonic speed, no information about change in back pressure gets transmitted to the tank. This is the interpretation of the physical phenomena without having delved into the mathematical formulation.

In real-life scenarios, the purpose of increasing the mass flow rate can be an auxiliary effect, with the main objective being, for example, acceleration of flow. Consider the example of an aircraft. If it were possible to perpetually increase mass flow rate by decreasing pressure, it would be possible to accelerate the aircraft to extremely high values of Mach number. But the principle discussed above implies that an aircraft can be sped only upto Mach number of 1 by decreasing the back pressure (when using a convergent nozzle to do so).

Now, we look at the mathematical formulation, and obtain an expression for the maximum mass flow rate $\dot{m}_{\max}$ that occurs as the condition in plenum chamber (and at the exit of the nozzle) becomes sonic. Hence,

$$\dot{m}_{\max} = \rho^* A^* u^* = \frac{p^*}{RT^*} A_e \sqrt{\gamma RT^*} = \frac{p^* A_e \sqrt{\gamma}}{\sqrt{RT^*}}. \tag{1}$$

In equation (1), we have used the equation of state for an ideal gas to substitute $\rho^* = \frac{p^*}{RT^*}$ and the derived expression for sonic speed to substitute $u^* = \sqrt{\gamma RT^*}$. A_e is the area of the nozzle at its exit (i.e., its throat, i.e., the point where it is connected to the plenum chamber). We now recall the expressions $\frac{T_0}{T}$ (equation (8) of lecture 50),

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2 \tag{2}$$

and apply them for sonic condition, giving us,

$$\frac{T_0}{T^*} = 1 + \frac{\gamma - 1}{2} M^{*2} = 1 + \frac{\gamma - 1}{2} = \frac{\gamma + 1}{2}. \tag{3}$$

$\frac{p_0}{p}$ is related to $\frac{T_0}{T}$ simply as,

$$\frac{p_0}{p} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma - 1}}, \tag{4}$$

due to the process being adiabatic-reversible. Hence, we simply have,

$$\frac{P_0}{P^*} = \left(\frac{T_0}{T^*}\right)^{\frac{\gamma}{\gamma-1}}. \tag{5}$$

Equations (3) and (5) indicates that T^* and p^* are simply functions of T_0 and p_0. Resultantly, $\dot{m}_{\max}$, as per equation (1), will also be a function of T_0 and p_0. Hence, we arrive at the surprising and somewhat counter-intuitive deduction that the maximum mass flow rate in a converging nozzle in a setup like the one presented in figure 1 is dictated by the stagnation pressure and temperature in the tank and not by the back pressure in the plenum chamber.

We now work out an illustrative problem to understand the phenomenon in a more quantitative manner.

Problem: Air in a large tank at 100 ^{0}C and 150 kPa exhausts through a converging nozzle that has a throat area of 5 cm^2. Calculate $\dot{m}$ if (a) p_b is 100 kPa, (v) p_b is 60 kPa and (c) p_b is 30 kPa.

Solution:

We first obtain the value for sonic condition pressure, p^*, as we need to compare p_b with p^* to determine whether mass flow rate has saturated to maximum or not.

To get p^*, we recall the expression for $\frac{p_0}{p}$ (equation (9) of lecture 50),

$$\frac{p_0}{p} = \left[1 + \frac{\gamma-1}{2} M^2\right]^{\frac{\gamma}{\gamma-1}}, \tag{6}$$

And apply it for the sonic condition,

$$\frac{p_0}{p^*} = \left[1 + \frac{\gamma-1}{2} M^{*2}\right]^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{\gamma-1}{2}\right]^{\frac{\gamma}{\gamma-1}} = \left[\frac{\gamma+1}{2}\right]^{\frac{\gamma}{\gamma-1}}. \tag{7}$$

Substituting $\gamma = 1.4$ for air and knowing P_0 is simply the pressure in the tank, i.e. 150 kPa, we get P^* as 79 kPa.

Now, for case (a), P_b is 100 kPa which is greater than P^*. Hence, the condition at the throat is not sonic and we have to obtain the temperature and pressure at the throat of the nozzle to obtain the mass flow rate. The pressure at the throat is the given back pressure, i.e. P_e = 100 kPa. Substituting this in equation (6), the only unknown is M_e and we obtain its value as 0.784, which is smaller than 1. Substituting this Mach number in equation (2) and knowing that T_0 is simply the temperature in the tank, i.e. 100 ^{0}C, we get the throat temperature as T_e = 59.16 ^{0}C.

Now, we proceed with the expression for $\dot{m}$,

$$\dot{m} = \rho_e A_e u_e = \frac{P_e}{RT_e} A_e \sqrt{\gamma R T_e} \ . \qquad (8)$$

Since numerical values for all the terms in equation (8) are known, we get the mass flow rate as 0.15 kg/s.

For cases (b) and (c), the back pressure is smaller than the sonic pressure. Hence, for both these cases, the mass flow rate is the maximum possible mass flow rate (or the saturated mass flow rate), which as per equation (1) is,

$$\dot{m} = \dot{m}_{\max} = \frac{p^* A_e \sqrt{\gamma}}{\sqrt{RT^*}} \ .$$

p^* has been obtained as 79 kPa and T^* is obtained as 37.81 ^{0}C using equation (3) and the given data that T_0 is 100 ^{0}C. Substituting all the numerical values, we get this mass flow rate as 0.157 kg/s.

This numerical problem has helped us to improve our understanding of the physical basis of compressible flow through a converging nozzle. An important conclusion we have arrived at is that by attuning the back pressure, we are able to accelerate the flow to Mach number of 1 at the exit but not higher. Hence, a convergent nozzle is constrained to not provide us with supersonic flows, reaching sonic speed in the limiting case. In order to accelerate the flow to supersonic speed, we will have to use a converging-diverging nozzle, and we will discuss its working principles and design and off-design conditions in the next lecture.