Advanced Concepts in Fluid Mechanics
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Lecture – 55
Compressible Flows
(Normal Shock) (Contd.)

Continuing from the previous lecture, we will obtain an explicit expression for M_2 in terms of M_1, i.e. the downstream Mach number in terms of the upstream Mach number across a normal shock wave front. Hence, we proceed with the equation obtained in the previous lecture,

$$\frac{\left(1+\gamma M_2^2\right)^2}{\left(1+\gamma M_1^2\right)^2}=\frac{M_2^2}{M_1^2}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right). \tag{1}$$

To aid the ensuing algebra, let use denote $M_1^2 = x$ and $M_2^2 = y$. This, equation (1) is,

$$\frac{\left(1+\gamma y\right)^2}{\left(1+\gamma x\right)^2}=\frac{y}{x}\left(\frac{2+(\gamma-1)y}{2+(\gamma-1)x}\right)$$

$$\Rightarrow \frac{\left(1+\gamma y\right)^2}{\left(1+\gamma x\right)^2}=\frac{y}{x}\left(\frac{2+(\gamma-1)y}{2+(\gamma-1)x}\right)\Rightarrow \frac{\left(1+2\gamma y+\gamma^2 y^2\right)x}{\left(1+2\gamma x+\gamma^2 x^2\right)y}=\left(\frac{2+(\gamma-1)y}{2+(\gamma-1)x}\right)$$

$$\Rightarrow \left(x+2\gamma xy+\gamma^2 xy^2\right)(2+(\gamma-1)x)=\left(y+2\gamma xy+\gamma^2 x^2 y\right)(2+(\gamma-1)y)$$

$$\Rightarrow 2x+4\gamma xy+2\gamma^2 xy^2+(\gamma-1)\left(x^2+2\gamma x^2 y+\gamma^2 x^2 y^2\right)=2y+4\gamma xy+2\gamma^2 x^2 y+(\gamma-1)\left(y^2+2\gamma xy^2+\gamma^2 x^2 y^2\right)$$

$$\Rightarrow 2x-2y+4\gamma xy-4\gamma xy+2\gamma^2 xy^2-2\gamma^2 x^2 y+(\gamma-1)\left(x^2+2\gamma x^2 y+\gamma^2 x^2 y^2-y^2-2\gamma xy^2-\gamma^2 x^2 y^2\right)=0$$

$$\Rightarrow 2(x-y)-2\gamma^2 xy(x-y)+(\gamma-1)\left(x^2-y^2+2\gamma x^2 y-2\gamma xy^2+\gamma^2 x^2 y^2-\gamma^2 x^2 y^2\right)=0$$

$$\Rightarrow 2(x-y)-2\gamma^2 xy(x-y)+(\gamma-1)\left((x-y)(x+y)+2\gamma xy(x-y)\right)=0$$

$$\Rightarrow (x-y)\left(2-2\gamma^2 xy+(\gamma-1)(x+y+2\gamma xy)\right)=0$$

$$\Rightarrow (x-y)\left(2-2\gamma^2 xy+2\gamma^2 xy-2\gamma xy+(\gamma-1)(x+y)\right)=0$$

$$\Rightarrow (x-y)\left(2-2\gamma xy+(\gamma-1)(x+y)\right)=0$$

Hence, we finally get equation (1) as,

$$(x-y)\left(2-2\gamma xy+(\gamma-1)(x+y)\right)=0, \tag{2}$$

where x is M_1^2 and y is M_2^2.

Note that one solution of equation (2) is the trivial solution corresponding to $x = y$, i.e. $M_1 = M_2$. This trivial solution corresponds to the situation when the region being enclosed by control volume is not a normal shock wave front. Note that we recover this trivial solution because we had not explicitly restricted the control volume considered in the last lecture to enclose a normal shock wave front. Rather, it has the capability of accommodating a normal shock wave front and hence, equation (2) allows for two solutions – one corresponding to a normal shock wave front and another corresponding to a trivial solution which corresponds to any arbitrary region in the compressible flow field. Being interested in the normal shock wave front, we now further analyse the non-trivial solution. The non-trivial solution arises when,

$$
\begin{aligned}
&2-2\gamma xy+(\gamma-1)(x+y)=0 \\
&\Rightarrow -2\gamma xy+(\gamma-1)y=-2-(\gamma-1)x \\
&\Rightarrow \left[(\gamma-1)-2\gamma x\right]y=-\left[2+(\gamma-1)x\right], \\
&\Rightarrow y=\left[\frac{2+(\gamma-1)x}{2\gamma x-(\gamma-1)}\right]
\end{aligned}
\qquad (3)
$$

Substituting back $M_1^2 = x$ and $M_2^2 = y$, we get M_2 in terms of M_1 as,

$$M_2^2=\left[\frac{2+(\gamma-1)M_1^2}{2\gamma M_1^2-(\gamma-1)}\right]. \qquad (4)$$

For the case of air, $\gamma = 1.4$, and substituting this numerical value in equation (4), we get,

$$M_2^2=\frac{5+M_1^2}{7M_1^2-1}. \qquad (5)$$

While the algebra presented above is tedious, it is an important part of the analysis as it allowed us to observe that the trivial solution is also recovered from the mathematical formulation, enlightening us that we have not constrained the control volume being analysed (presented in the previous lecture) to enclose a normal shock wave front, rather it is capable of accommodating one.

We present below in figure 1, a graph of variation of M_2 with M_1, based on equation (4), i.e. the non-trivial solution of equation (2). While the complete trivial solution is not presented, its intersection with the non-trivial solution is presented as the point P, which corresponds to both M_2 and M_1 being 1.

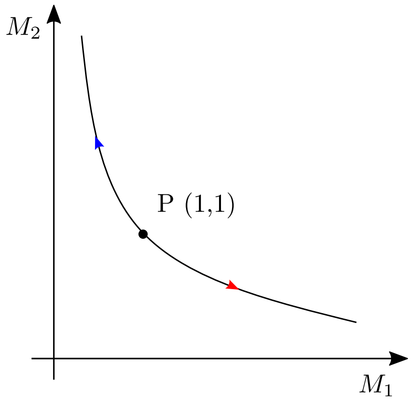

Figure 1: Variation of M_2 with M_1 obtained based on equation (4), the non-trivial solution of equation (2)

Observing the nature of the curve, we deduce that when M_1 is higher than 1, M_2 is constrained to be lower than 1, which corresponds to the section of the curve to the right of P (marked with the red arrowhead). On the other hand, when M_1 is lower than 1, M_2 is constrained to be higher than 1, which corresponds to the section of the curve to the left of P (marked with the blue arrowhead). The former situation represents a supersonic flow passing the normal shock wave front to become a subsonic flow and the latter situation represents a subsonic flow passing the normal shock wave front to become a supersonic flow.

Hence, the analysis of figure 1 done above has allowed us to deduce there are two possible scenarios at the normal shock wave front:

Possibility 1: $M_1 > 1$ and $M_2 < 1$

Possibility 2: $M_1 < 1$ and $M_2 > 1$

Note that the a flow that is close to the incompressible limit, which corresponds to both M_1 and M_2 less than 1, is not a possibility. Hence, we have mathematically deduced that a shock wave cannot occur close to the incompressible limit.

Returning to the topic of normal shock, we see that the analysis presented till here, which is based on principles of fluid mechanics and the first law of thermodynamics, is incapable of informing us which of the two possibilities can occur in a real-world situation. To ascertain this, we first observe that the question of 'which one of these possibilities is realizable' is related to the directionality of the system. And the directionality of a system is governed by the second law of thermodynamics. Therefore, we will now calculate the change in entropy across the normal shock wave front.

Change in entropy across the normal shock wave front:

To analyse the change in entropy across the normal shock wave front, we start with the first law of thermodynamics,

$$\delta q = di + \delta w. \qquad (6)$$

In writing equation (6), we have again neglected the changes in kinetic energy and potential energy. While this had been an appreciably valid assumption till now, its validity can be doubted in the case of a shock wave. However, for a process where thermal parameters are governed, the changes in kinetic energy get overpowered by the changes in thermal energy. Therefore, we continue with equation (6). We will now assume that the flow is quasi equilibrium.

It should be noted here that the reversible quasi-equilibrium process being considered here is only a hypothetical process that is being used to obtain an expression relating changes in thermodynamic properties. The real process will of course be different will include the irreversibility brought about by the shock wave. However, as iterated earlier, an expression relating changes in thermodynamic properties remains valid regardless of the process using which that they are derived and the process which they are undergoing. We are able to make this last statement because thermodynamic properties are point functions and hence path independent. In the current situation, the thermodynamic property is entropy.

Since we are choosing a reversible path to analyse the change in entropy, we write equation (6) as,

$$Tds = dh - vdp \,. \tag{7}$$

In obtaining equation (7) from equation (6), we have used the definitions $\delta q = Tds$ (which holds true for a reversible process) and $i = h - pv$. Further, considering that the fluid is an ideal gas, we use the equation of state and express equation (7) as,

$$\begin{aligned} Tds &= c_p dT - \frac{RT}{p} dp \\ \Rightarrow ds &= c_p \frac{dT}{T} - R\frac{dp}{p} \end{aligned} \tag{8}$$

Further assuming the gas to be calorically perfect (i.e. c_p is a constant), we are able to integrate equation (8) to get,

$$s_2 - s_1 = c_p \ln\left(\frac{T_2}{T_1}\right) - R\ln\left(\frac{p_2}{p_1}\right). \tag{9}$$

Now, the ratios $\frac{T_2}{T_1}$ and $\frac{p_2}{p_1}$ are functions of M_1 and M_2, as was derived in the previous lecture. Also, M_2 itself is a function M_1 as per equation (4). Knowing the expressions for $\frac{T_2}{T_1}$, $\frac{p_2}{p_1}$ and M_2 as functions of M_1 (equations (3) and (7) in last lecture and equation (4) in this lecture), we can evaluate $s_2 - s_1$.

We now write the second law of thermodynamics, which defines entropy generation s_{gen} as,

$$s_2 - s_1 = \frac{q_{12}}{T} + s_{gen} \,, \tag{10}$$

and states that s_{gen} should always be positive. Since the shock takes place adiabatically, q_{12} is zero and hence, s_{gen} is simply $s_2 - s_1$, which as stated above, is obtainable as a function of M_1. In figure 2, we present this functional variation of s_{gen} with M_1. As s_{gen} is constrained to be positive, only the section of the curve above the horizontal axis can be admitted as solution. Hence, M_1 is always greater than 1 with the limiting case of $M_1 = 1$ being the situation where there is no shock and no entropy generation. Hence, we have deduced that only 'Possibility 1' is realizable, i.e. "a shock always converts a flow from supersonic to subsonic and never from subsonic to supersonic, a restriction arrived at by the second law of thermodynamics." All the flow variables change abruptly across the shock wave front and temperature and pressure

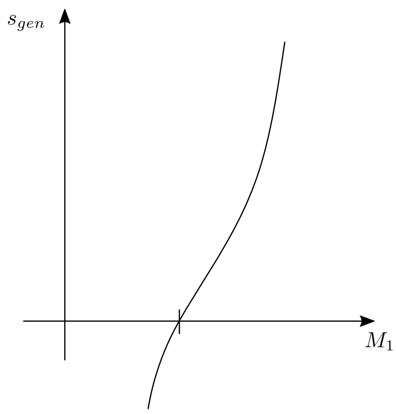

Figure 2: Variation of s_{gen} with M_1

undergo abrupt increase in magnitude.