Advanced Concepts In Fluid Mechanics
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Lecture – 54
Compressible Flow (Normal Shock)

In this lecture, we will discuss the concept of shock, more specifically normal shock, under the topic of compressible flows.

We briefly recapitulate some pertinent concepts. Let us consider a supersonic flow (a flow with Mach number greater than 1). A supersonic flow is one in which the disturbance speed (sonic speed) is smaller than the speed of the source of disturbance. Observed from the perspective of the source, this situation is equivalent to the entire fluid medium moving at the same speed but in opposite direction. Under such a situation, we have earlier discussed that there is formation of a Mach cone (also called as 'zone of influence') – the region within which the disturbance caused by the source is perceived. The region outside this Mach cone does not perceive the disturbance caused by the source, and is also referred to as the 'zone of silence'.

Consider the example of an aircraft moving at supersonic speed in the leftward direction. From the perspective of the aircraft, the air is moving in the rightward direction at the same speed and hence approaching it (and will eventually interface with it). Consequently, the air will eventually cross the border of the zone of silence and the zone of influence. As the air crosses this border, it is be subjected to a sudden discontinuity, which manifests itself as a discontinuity in various properties like pressure, density, temperature and so on. This discontinuity is what is called shock wave. Summarily, in a supersonic flow, the physical origin of the discontinuity in flow properties commonly called as shock is that the source of disturbance propagates at a speed which is faster than the disturbance speed itself, due to which, the disturbance accumulates at certain locations and the accumulation gets released as the shock wave propagates.

Shock wave can form in various ways. One of the commonly studied ways is when the direction of flow can be such that the front of the wave across which the above-discussed discontinuity exists is perpendicular to the flow direction. Such a shock is called normal shock. On the other hand, if the angle is different from 90^0, it is called as oblique shock. In this course, we will primarily study normal shock.

Theory of normal shock:

A shock wave front is a wave front where there is an abrupt change in properties of the flow, like pressure, density, velocity, temperature, etc. We recall that while considering the usual sound wave as well, we consider a front across which properties vary. However, these variations, for a usual sound wave, are smooth and infinitesimal, and hence, such a wave is called as a weak wave. In contrast, the variations are finite and abrupt for a shock wave, and hence, a shock wave is a strong wave. Furthermore, we encounter shock only under the purview of flows with Mach number close to 1, and shock does not occur for flows close to the incompressible limit. We will see why this is the case as we proceed in the theory of normal shock.

Our first objective in the theory of normal shock is to relate the properties downstream of the shock wave front with properties upstream of the shock wave front, and, to figure out under what circumstances (characterized by the Mach number) will the shock occur. To this end, we consider a control volume (dashed rectangle) around a normal shock wave front (solid line), as presented in figure 1. The control volume is thin but has a finite area, and it encloses the normal shock wave front. The properties upstream are denoted with the subscript 1 and the properties downstream are denoted with the subscript 2.

We will apply the mass balance, momentum balance and energy balance equations on this control volume.

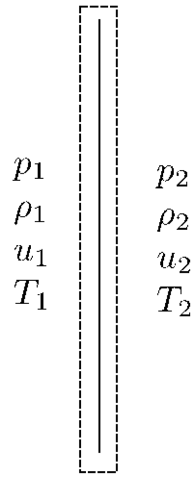

Figure 1: Normal shock wave front and the control volume enclosing it

We start with mass balance equation,

$$\rho_1 A u_1 = \rho_2 A u_2 \Rightarrow \rho_1 u_1 = \rho_2 u_2 . \tag{1}$$

Assuming the fluid to be ideal, we express density in terms of pressure and temperature, and thus, equation (1) becomes,

$$\frac{p_1}{RT_1} u_1 = \frac{p_2}{RT_2} u_2 . \tag{2}$$

Further, we know that $u = M\sqrt{\gamma RT}$. We recall that the expression for sonic speed as $\sqrt{\gamma RT}$ was derived for an ideal gas undergoing an isentropic process (the derivative of pressure with density, which is the sonic speed, comes out as $\sqrt{\gamma RT}$ when the process is adiabatic and reversible, i.e. isentropic) . However, the process by virtue of which shock occurs is by no means isentropic (as it includes a discontinuity). The reason we are still able to use the derived expression for sonic speed is because we are using sonic speed as a property. And while the expressions for thermodynamic properties can be obtained using a certain process, once derived, the expression remains valid regardless of what process the fluid is undergoing, i.e., expression for a thermodynamic property is not dependent on the process by which it is derived.

Hence, we continue with equation (2) as,

$$\frac{p_1 M_1 \sqrt{\gamma R T_1}}{R T_1} = \frac{p_2 M_2 \sqrt{\gamma R T_2}}{R T_2} \Rightarrow \frac{p_1 M_1}{\sqrt{T_1}} = \frac{p_2 M_2}{\sqrt{T_2}} . \quad (3)$$

After mass conservation equation, we will now consider momentum balance equation. The expression for momentum balance is the sum of forces on the control volume equals the net flux of momentum through its control surfaces, i.e.,

$$p_1 A - p_2 A = -\dot{m} u_1 + \dot{m} u_2 . \quad (4)$$

From mass conservation equation, we know,

$$\dot{m} = \rho_1 A u_1 = \rho_2 A u_2 \quad (5)$$

Hence, equation (4) becomes,

$$\begin{aligned} &-(p_2 - p_1) A = \rho_2 A u_2^2 - \rho_1 A u_1^2 \Rightarrow \\ &p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2 \end{aligned} \quad (6)$$

Note that in equation (6), we encounter terms of the form $p + \rho u^2$, as opposed to what we would get when using Bernoulli's principle, $p + \frac{1}{2}\rho u^2$. This excess $\frac{1}{2}\rho u^2$ is the outcome of the irreversibility in the system. We further substitute $\rho = \frac{p}{RT}$ and $u = M\sqrt{\gamma RT}$ to simplify equation (6) to,

$$p_1(1 + \gamma M_1^2) = p_2(1 + \gamma M_2^2) . \quad (7)$$

Lastly, we consider energy balance equation,

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2} . \quad (8)$$

This is essentially the first law of thermodynamics, with the assumptions of uniform properties at the boundaries of the control volume and negligible changes in potential energy.

We briefly remind ourselves the meaning of enthalpy h appearing in equation (8). Enthalpy is the sum of internal energy, i and flow energy, $\frac{p}{\rho}$. When a fluid is flowing with thermal effects, in addition to kinetic energy, potential energy and flow energy, (i.e. the three forms of energy we encounter in the context of Bernoulli's equation), the total energy of the fluid has another form of energy called the internal energy. This internal energy is the intrinsic thermal part of the fluid energy which is associated with the temperature of the fluid, and would have been the sole contributor to the fluid's total energy in absence of flow and potential energy effects. However, under flow, the total thermal energy, or flow, of the fluid is its internal energy plus its flow energy, together called as enthalpy.

Now, under the assumption of the fluid being a calorically perfect gas, equation (8) is expressed as,

$$c_p T_1 + \frac{M_1^2 \gamma R T_1}{2} = c_p T_2 + \frac{M_2^2 \gamma R T_2}{2} \Rightarrow$$
$$\frac{\gamma R}{\gamma - 1} T_1 + \frac{M_1^2 \gamma R T_1}{2} = \frac{\gamma R}{\gamma - 1} T_2 + \frac{M_2^2 \gamma R T_2}{2} \Rightarrow \qquad (9)$$
$$T_1 \left[1 + \frac{\gamma - 1}{2} M_1^2\right] = T_2 \left[1 + \frac{\gamma - 1}{2} M_2^2\right]$$

In writing the topmost version of equation (9), we have substituted $h = c_p T$ (obtained by integrating $dh = c_p dT$ thanks to the assumption of calorically perfect gas) and $u = M\sqrt{\gamma RT}$. In writing the middle version of equation (9), we have used the definitions $c_p - c_v = R$ and $\frac{c_p}{c_v} = \gamma$ to express c_p in terms of R and γ as $c_p = \frac{\gamma R}{\gamma - 1}$.

Summarily, equations (3), (7) and (9) are the equations of mass conservation, momentum conservation and energy balance for the control volume enclosing the normal shock front, as presented in figure 1.

Before proceeding with algebraic analysis, we discuss the interim objective. We know the upstream value of Mach number M_1. From this and the analysis above, we want to obtain the downstream Mach number, M_2. The reason we want to obtain the M_2 is that it will enable us to obtain all the downstream properties based on the available information of upstream properties, upstream Mach number and downstream Mach number. Also, we will be able to obtain the reference stagnation properties and the reference sonic condition properties on both sides of the normal shock wave front. Here, we emphasize that the flow is respectively isentropic on either side of the normal shock wave front, and it is adiabatic at the normal shock wave front due to the rapid nature of the process but it is not reversible. Hence, the reference stagnation properties and reference sonic condition properties are two different unequal sets of constants on either side of the normal shock wave front, and it is of interest to obtain these distinct sets of reference properties.

Hence, we want to obtain the missing information of the Mach number immediately downstream the normal shock wave front, and to do that, we use equations (3), (7) and (9). We eliminate p_1 and p_2 from equations (3) and (7) by expressing each in the form of LHS as $\frac{p_1}{p_2}$, and then equating the two. This way, we have,

$$\frac{1 + \gamma M_2^2}{1 + \gamma M_1^2} = \frac{M_2}{M_1}\sqrt{\frac{T_1}{T_2}} . \qquad (10)$$

We now substitute $\frac{T_1}{T_2}$ from equation (9) into equation (10) to get,

$$\frac{1+\gamma M_2^2}{1+\gamma M_1^2} = \frac{M_2}{M_1}\sqrt{\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}}\,, \tag{11}$$

Expressed alternatively by squaring both sides, we have,

$$\frac{\left(1+\gamma M_2^2\right)^2}{\left(1+\gamma M_1^2\right)^2} = \frac{M_2^2}{M_1^2}\left(\frac{1+\frac{\gamma-1}{2}M_2^2}{1+\frac{\gamma-1}{2}M_1^2}\right). \tag{12}$$

We want to algebraically manipulate equation (12) to obtain an explicit expression for M_2 in terms of M_1, which we will take up in the next lecture.