**Advanced Concepts In Fluid Mechanics**
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**Lecture – 53**
**Compressible Flows (Reference Sonic Properties)**

## Reference Sonic Conditions:

Reference sonic conditions were introduced in the previous lecture. These are hypothetical reference conditions which would exist in a given flow at a point if at that point, Mach number is one. While there is no necessity of a point with Mach number being 1 in a particular flow field, there is always such a hypothetical reference state where Mach number is unity, and the corresponding flow conditions are called "reference sonic conditions". These reference sonic conditions, for an isentropic flow, are demarcated using the superscript $*$.

Hence, the mass conservation relating any arbitrary cross section in a duct to the reference sonic condition in that duct is,

$$\rho A u = \rho^* A^* u^*. \quad (1)$$

Since $u^*$ is the sonic speed, it is simply $\sqrt{\gamma R T^*}$.

We re-emphasize that the flow being studied here is an isentropic flow, and therefore, the reference sonic conditions remain the same and is demarcated with the superscript $*$. For a change in the isentropic nature of the flow, the sonic conditions also change and hence, reference sonic conditions also change. An example where a change in isentropic nature of the flow occurs in the presence of a shock wave, as we will discuss ahead in the course.

Continuing with the described flow being isentropic, we have the expressions for $u$ and $u^*$ as,

$$\begin{aligned} u &= M\sqrt{\gamma R T} \\ u^* &= \sqrt{\gamma R T^*} \end{aligned}. \quad (2)$$

Dividing these terms, we have,

$$\frac{u}{u^*} = M\sqrt{\frac{T}{T^*}} \quad (3)$$

We have $\frac{T}{T^*}$ in the expression for $\frac{u}{u^*}$, and $\frac{\rho}{\rho^*}$ will also be a function of $\frac{T}{T^*}$. Now, to get $\frac{T}{T^*}$, we recall the expression for $\frac{T_0}{T}$ as obtained earlier in the course,

$$\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M^2. \quad (4)$$

From equation, we obtain the relation between $T_0$ and $T*$ as,

$$\frac{T_0}{T*}=1+\frac{\gamma-1}{2}=\frac{\gamma+1}{2}. \qquad (5)$$

In obtaining equation (5), we have substituted $M*=1$, which would be the case for the hypothetical reference sonic condition. Dividing equation (5) by equation (4) gives us the relation between $T$ and $T*$ as,

$$\frac{T}{T*}=\frac{\frac{\gamma+1}{2}}{1+\left(\frac{\gamma-1}{2}\right)M^2}=\frac{\gamma+1}{2+(\gamma-1)M^2}. \qquad (6)$$

Next, following the equation for adiabatic processes,

$$\frac{T}{\rho^{\gamma-1}}=c \Rightarrow \frac{T}{\rho^{\gamma-1}}=\frac{T*}{\rho*^{\gamma-1}} \Rightarrow \frac{\rho}{\rho*}=\left(\frac{T}{T*}\right)^{\frac{1}{\gamma-1}}=\left(\frac{\gamma+1}{2+(\gamma-1)M^2}\right)^{\frac{1}{\gamma-1}}, \qquad (7)$$

we have obtained the expression for $\frac{\rho}{\rho*}$. Next, we obtain the expression for $\frac{A}{A*}$ by using the equation (1), with $\frac{\rho}{\rho*}$ and $\frac{T}{T*}$ already known. Hence, using equations (1), (6) and (7), we obtain,

$$\frac{A}{A*}=\frac{1}{M}\left[\frac{\gamma+1}{2+(\gamma-1)M^2}\right]^{\frac{1+\gamma}{2(1-\gamma)}}. \qquad (8)$$

Hence, in equation (8), we have obtained an expression for area $A$ as an explicit function of Mach number $M$ and reference sonic area $A*$. This allows us to directly obtain the area at a location given the Mach number at that location. However, given area at a location, equation (8) admits two solutions for the Mach number - one corresponding to subsonic condition, another corresponding to supersonic condition. Another way to look at it is as follows. When M is 1, we can see from equation (8) that $A=A*$. And when $A$ varies from $A*$, it gives rise to one of two conditions - either M greater than 1 or M less than 1.

Summarily, we have obtained analytical expressions for terms like $\frac{p_0}{p}$, $\frac{T_0}{T}$, $\frac{\rho_0}{\rho}$ , $\frac{T}{T*}$, $\frac{\rho}{\rho*}$, $\frac{A}{A*}$ in terms of Mach number and $\gamma$. For a calorically perfect gas, it is easy to obtain a table for variation of these terms whereas for real gases like air, tabulated values for different combinations of Mach number and $\gamma$ are available in textbooks on compressible flows.

We have now learnt the analysis of compressible isentropic flow through a variable area duct. We will learn in coming lectures how the analysis gets altered when the condition of isentropic flow doesn't anymore hold true throughout the flow. For the rest of this lecture, we will consider a couple of illustrative problems.

Example 1: Consider isentropic flow in a channel of varying area between sections 1 and 2. Given $M_1 = 2$ and $\frac{v_2}{v_1} = 1.2$, estimate (1) $M_2 = ?$, (2) $\frac{A_2}{A_1} = ?$, (3) whether the duct is converging or diverging between sections 1 and 2.

Solution:

We begin with the given data, $\frac{v_2}{v_1} = 1.2$.

$$\frac{v_2}{v_1} = 1.2 \Rightarrow \frac{M_2\sqrt{\gamma RT_2}}{M_1\sqrt{\gamma RT_1}} = 1.2 \Rightarrow \frac{M_2\sqrt{\frac{T_2}{T_0}}}{M_1\sqrt{\frac{T_1}{T_0}}} = 1.2 \Rightarrow$$

$$\frac{M_2\left[1+\frac{\gamma-1}{2}M_1^2\right]^{\frac{1}{2}}}{M_1\left[1+\frac{\gamma-1}{2}M_2^2\right]^{\frac{1}{2}}} = 1.2 \qquad (9)$$

Knowing the value of $M_1$ and given $\gamma$ is 1.4 for air, we get $M_2$ as 2.98.

To get $\frac{A_2}{A_1}$, we make use of the fact that the flow is isentropic and thus $A^*$ is a constant. Thus,

$$\frac{A_2}{A_1} = \frac{\frac{A_2}{A^*}}{\frac{A_1}{A^*}} = \frac{f(M_1)}{f(M_2)} \qquad (10)$$

The function $f(M)$ is already known (equation (8) above) and substituting the numerical values, we get $\frac{A_2}{A_1}$ as 2.46.

Now the ratio of $\frac{A_2}{A_1}$ being 2.46 (higher than 1) does not tell us that the nozzle is necessarily diverging, it only tells the ratio of area between sections 1 and 2 (whose order along the axial length is not given). To get the shape of the nozzle, we have to use the $dA$ expression (equation (13) of last lecture) in terms of Mach number and its change, i.e.,

$$\frac{dA}{A} = \frac{dM}{M}\left(\frac{M^2 - 1}{1 + (\gamma - 1)\frac{M^2}{2}}\right). \quad (11)$$

Now, since both $M_1$ and $M_2$ are greater than 1, we know that the bracketed term on the RHS of equation (11) is positive throughout the region between sections 1 and 2. And $dM = M_2 - M_1 = 2.98 - 2.0 = 0.98 > 0$. Resultantly, $dA$ is also positive and hence the duct is diverging.

**Example 2:** Consider a converging-diverging nozzle as shown in figure 1. The channel diameters values are as depicted. Assume isentropic flow. Also given is the stagnation temperature $T_0$ = 300 K, velocity at section 1 is $u_1$ = 72 m/s and pressure at section 2 is $p_2$ = 124 kPa. Find: (1) pressure at section 1 ($p_1$), (2) Mach number at section 2 ($M_2$), and (3) mass flow rate of air ($\dot{m}_{air}$). Consider other properties as the standard ones for air.

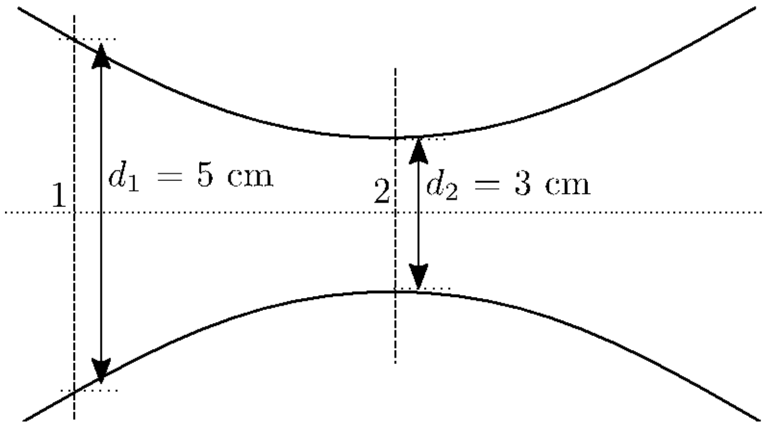


Figure 1: Figure for Example 2

Solution:

To get $M_1$, we use,

$$M_1 = \frac{u_1}{c_1} = \frac{u_1}{\sqrt{\gamma R T_1}}. \quad (12)$$

Here, $u_1$, $\gamma$ and $R$ are known. To obtain $T_1$, we use the energy conservation equation reference with stagnation condition,

$$c_p T_0 = c_p T_1 + \frac{u_1^2}{2} \Rightarrow T_1 = T_0 - \frac{u_1^2}{2c_p}. \quad (13)$$

Obtaining $T_1$ from equation (13) and substituting into equation (12), we get $M_1$ = 0.208.

Now, to evaluate $M_2$, we use equation (8). First, applying equation (8) for section 1,

$$\frac{A_1}{A^*} = \frac{1}{M_1}\left[\frac{\gamma+1}{2+(\gamma-1)M_1^{\,2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}}. \qquad (14)$$

In equation (14), $A^*$ is the only unknown and hence we obtain the value of $A^*$. Subsequently we apply equation (8) for section 2,

$$\frac{A_2}{A^*} = \frac{1}{M_2}\left[\frac{\gamma+1}{2+(\gamma-1)M_2^{\,2}}\right]^{\frac{1+\gamma}{2(1-\gamma)}}. \qquad (15)$$

Now, in equation (15), $M_2$ is the only unknown and hence we obtain $M_2$ = 0.831. Because the flow is subsonic at section 1 ($M_1$ = 0.208), it remains subsonic in the converging section and we have admitted the solution for $M_2$ which is less than 1 – we will see ahead in the course that a converging section cannot accelerate a subsonic flow to supersonic (and can only maximize it to sonic) at its throat.

Next, to obtain $p_1$, we first obtain $p_0$, for which we use the equation,

$$\frac{p_0}{p} = \left[1+\frac{\gamma-1}{2}M^2\right]^{\frac{\gamma}{\gamma-1}}, \qquad (16)$$

which has been derived in Lecture 50. We apply this equation at the section 2,

$$\frac{p_0}{p_2} = \left[1+\frac{\gamma-1}{2}M_2^{\,2}\right]^{\frac{\gamma}{\gamma-1}}, \qquad (17)$$

where only $p_0$ is unknown and hence we obtain its value. Subsequently, applying equation (15) at section 1,

$$\frac{p_0}{p_1} = \left[1+\frac{\gamma-1}{2}M_1^{\,2}\right]^{\frac{\gamma}{\gamma-1}}, \qquad (18)$$

we have $p_1$ as the only unknown and we get its value as 189 kPa.

In obtaining $M_2$ and $p_1$ above, we have demonstrated the utility of the expression relating fluid properties at given point in flow to their reference properties, i.e. stagnation properties and reference sonic properties.

Lastly, to obtain $\dot{m}_{air}$, we use,

$$\dot{m}_{air} = \rho_1 A_1 u_1 = \frac{p_1}{RT_1}A_1 u_1 = 0.313 \text{ kg/s}.$$

To summarize, in this lecture, we have seen how to handle problems related to variable area ducts in isentropic compressible flow. The interesting situation is when the flow continues to be adiabatic but is no longer reversible (and hence deviated from being isentropic), a situation that occurs very prominently during a shock wave in the flow. In the next lecture, we will take up a special case of shock wave in the flow, i.e. when the front across which there is an abrupt discontinuity in fluid properties because of the shock is perpendicular to the flow direction, commonly called as a normal shock.