Advanced Concepts In Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 52
Compressible Flows (Variable Area)

Isentropic compressible flow through variable area duct (continued):

Continuing from the last lecture, we proceed with the analysis of compressible flow through a variable area duct, presented in figure 1. Here, we study a special class of compressible flows, those which are also isentropic.

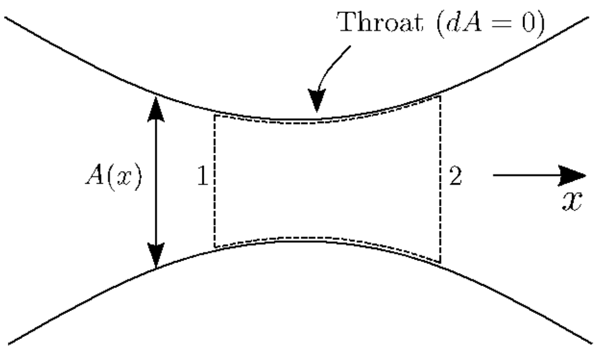

Figure 1: Variable area duct

To analyse the flow behaviour in this variable area duct, we begin with the basic conservation equations, mass conservation equation (or continuity) equation and momentum conservation equation.

Importing from the previous lecture, the mass conservation equation is,

$$\rho_1 A_1 u_1 = \rho_2 A_2 u_2 \Rightarrow \rho A u = c \Rightarrow \ln\rho + \ln A + \ln u = \ln c\,. \qquad (1)$$

where, ρ is density, A is cross-sectional area and u is average axial velocity, and the subscripts 1 and 2 imply two arbitrary cross sections along the axial length of the duct. Equation (1) is simply obtained by applying the Reynolds transport theorem on a control volume over a given section of the length of the duct, as illustrated in figure 1. Please note that equation (1) holds true even when v is non-zero, i.e. there is a vertical component to the velocity field, which is a realistic possibility with a variable-area duct. This is because the boundaries 1 and 2 are vertical and hence, the vertical velocity (whose dot product with the normal to the boundaries becomes zero) does not contribute to the mass flux across boundaries (the mass flux across the other two boundaries is already zero as these boundaries are coincident with the duct walls). Also, please note that u in equation (1) is the average axial velocity average over the cross-section.

Expressing in differential form, equation (1) is,

$$\frac{d\rho}{\rho} + \frac{dA}{A} + \frac{du}{u} = 0 \Rightarrow d\rho = -\rho\left[\frac{dA}{A} + \frac{du}{u}\right] \Rightarrow \rho = -d\rho\left[\frac{dA}{A} + \frac{du}{u}\right]^{-1}. \qquad (2)$$

Next, we consider the momentum conservation equation. For the situation presented here, as we had discussed in the last lecture, the momentum conservation does not boil down to Bernoulli's equation due variations in density. Here, we consider the flow to be inviscid. We observe that the consideration of inviscid flow is not an independently new assumption but actually follows from the assumption of isentropic flow– this is because any friction present in the flow, which is bound to occur for a flow with viscosity, will disturb the reversibility of the flow and hence prevent the flow from being isentropic (as it will prevent the flow from being reversible). Therefore, when considering isentropic flow, we cannot consider the flow to have viscosity. With simplification of inviscid flow, we know that the momentum conservation equation reduced to Euler equation.

Proceeding with the Euler equation for momentum conservation, we have the one-dimensional equation (imported equation (5) from lecture 48),

$$\frac{dp}{\rho} + udu = 0 \Rightarrow dp = -\rho udu \tag{3}$$

Absence of a body force term in equation (3) indicates that we have neglected the changes in potential energy of the fluid. Because of the high magnitudes of kinetic energy and its change (as compressible flows are very high-speed flows), potential energy changes will indeed be negligible.

Substituting the expression for ρ from equation (2), we get,

$$\frac{dp}{d\rho} = \frac{udu}{\frac{dA}{A} + \frac{du}{u}} \tag{4}$$

Since $\frac{dp}{d\rho}$ is c^2 as was derived in earlier lectures (c is sonic speed), equation (4) further transforms to,

$$\frac{c^2}{u^2} = \frac{\frac{du}{u}}{\frac{dA}{A} + \frac{du}{u}} \Rightarrow \frac{1}{M^2} = \frac{\frac{du}{u}}{\frac{dA}{A} + \frac{du}{u}} \Rightarrow \frac{dA}{A} + \frac{du}{u} = M^2 \frac{du}{u} \Rightarrow \frac{dA}{A} = \left(M^2 - 1\right)\frac{du}{u}\,. \tag{5}$$

The final form of equation (5) indicates that whether an increase in area will lead to an increase or decrease in flow speed is determined by the Mach number. If $M < 1$, the changes in A and u are conversely related, and hence increase in area will lead to decrease in flow speed and vice versa. This is the case for subsonic flows (i.e. flows with flow speed smaller than sonic speed) and the inference above continues to hold in the incompressible limit as well. On the other hand, if $M > 1$, the changes in A and u are directly related, and hence increase in area will lead to increase in flow speed and vice versa. This the case for supersonic flows (i.e. flows with velocity larger than sonic speed). A design principle which one can recover from the analysis above is that if one wants to achieve acceleration in a supersonic flow, the duct has to be designed with a diverging profile (opposite is the case for subsonic flows).

A word of caution here is that here we are considering isentropic flow, and so, the inferences above cannot be generalized to any flow – a situation where the flow will cease to be isentropic is when viscous effects become important.

The duct presented in figure 1 is a converging-diverging passage and is a crucial one in the context of compressible flow.

Based on the two regimes of relation between change in duct area and change in flow speed as obtained above, we realize that if a fluid has to be accelerated from subsonic speed to supersonic speed, one needs to pass it through a converging-diverging duct. The is a prime reason for the importance of converging-diverging ducts in compressible flows. Furthermore, if we want to maximize the mass flow rate in this mode of acceleration (from subsonic to supersonic), we need to take a keener look at what happens at the throat – the throat is defined as the axial location where the duct area is least and thus, $\frac{dA}{A}=0$. Hence, at the throat, equation (5) becomes,

$$0=\left(M^2-1\right)\frac{du}{u}. \qquad (6)$$

For equation (6) to be satisfied, either Mach number has to be 1, in which case, change in velocity is not restricted to any particular value, or, change in velocity is zero and hence velocity is either maximum or minimum (depending on whether Mach number is larger than 1 on smaller than 1). Furthermore, looking at equation (5), we observe that when Mach number is 1, dA will necessarily be zero. Summarily, if Mach number is equal to 1 at an axial location, that axial location must be at the throat, but the converse is not true (i.e. if an axial location is the throat, Mach number at that axial location is not constrained to be 1). Indeed, the situation when Mach number at throat is not 1 is when the velocity is not changing at the throat, and is either maximum or minimum.

An important point about equation (5) is that while it expresses $\frac{dA}{A}$ as a function of $\frac{du}{u}$, it does not present any dependence of $\frac{dA}{A}$ on dM or $\frac{dM}{M}$. Since the characteristics of flow are uniquely dictated by Mach number, finding such a dependence is of interest. Also, the variation in Mach number with the axial distance will dictate the relation between $\frac{dA}{A}$ and $\frac{du}{u}$. To arrive at the dependence of $\frac{dA}{A}$ on $\frac{dM}{M}$, we will assume some additional constraints and utilize the energy conservation equation and equation of state.

Using the equation of state for an ideal gas and under the consideration of the flow being a reversible adiabatic process, we have c as $\sqrt{\gamma RT}$. Also, the definition of Mach number M is $M=\frac{u}{c}$. Hence,

$$u=cM \Rightarrow \ln u=\ln c+\ln M \Rightarrow \frac{du}{u}=\frac{dc}{c}+\frac{dM}{M}. \qquad (7)$$

Hence, we have arrived at a relation of $\frac{du}{u}$ with $\frac{dM}{M}$. Since $\frac{dc}{c}$ is not going to be handy for algebraic analyses, we use,

$$c=\sqrt{\gamma RT}\Rightarrow \ln c=\frac{1}{2}\ln\left(\gamma R\right)+\frac{1}{2}\ln T\Rightarrow \frac{dc}{c}=\frac{1}{2}\frac{dT}{T}. \qquad (8)$$

Hence, equation (7) becomes,

$$\frac{du}{u}=\frac{1}{2}\frac{dT}{T}+\frac{dM}{M}. \qquad (9)$$

To obtain $\frac{du}{u}$ exclusively in terms of $\frac{dM}{M}$, we have to eliminate $\frac{dT}{T}$ from equation (9), for which we use the energy conservation equation (i.e. the first law of thermodynamics),

$$h+\frac{u^2}{2}=c. \qquad (10)$$

We express (10) in differential form,

$$\begin{aligned}
& dh+udu=0\Rightarrow c_p dT+udu=0\Rightarrow \\
& \frac{c_p dT}{u^2}+\frac{udu}{u^2}=0\Rightarrow \frac{c_p dT}{M^2c^2}+\frac{du}{u}=0\Rightarrow \frac{c_p dT}{M^2\gamma RT}+\frac{du}{u}=0 \\
& \Rightarrow \frac{dT}{(\gamma-1)M^2T}+\frac{du}{u}=0 \\
& \Rightarrow \frac{dT}{T}=-(\gamma-1)M^2\frac{du}{u}
\end{aligned}. \qquad (11)$$

Above, we have use the relations $c_p-c_v=R$ and $\frac{c_p}{c_v}=\gamma$ to replace $\frac{c_p}{\gamma R}$ with $\frac{1}{\gamma-1}$. Substituting $\frac{dT}{T}$ from equation (11) into equation (9), we get,

$$\left[1+\frac{1}{2}(\gamma-1)M^2\right]\frac{du}{u}=\frac{dM}{M}. \qquad (12)$$

Further substituting $\frac{du}{u}$ from equation (12) into equation (5), we get,

$$\frac{dA}{A}=\frac{\left(M^2-1\right)\frac{dM}{M}}{\left[1+\frac{1}{2}(\gamma-1)M^2\right]}. \qquad (13)$$

Equation (13) gives us the relation between the change in area and the change in Mach number. Based on the obtained expressions, we will consider the two cases of Mach number greater than 1 and Mach number less than 1 for a variable area duct.

In figure 2, we have presented the variation of Mach number with axial distance, x, along the converging-diverging duct, with the profile of the duct presented above for reference. Looking at equation (12), we observe that $dA = 0$ implies $dM = 0$ unless $M = 1$. Hence, for non-sonic ($M \neq 1$) velocity at the throat (throat has $dA = 0$ by definition), Mach number at the throat will also reach an extremum. We also observe that the denominator of the RHS is always positive (because $\gamma = \frac{c_p}{c_v}$ is always greater than 1 as c_p is always greater than c_v). Hence, the extremum of M at the throat is a minimum if $M > 1$ and a maximum if $M < 1$.

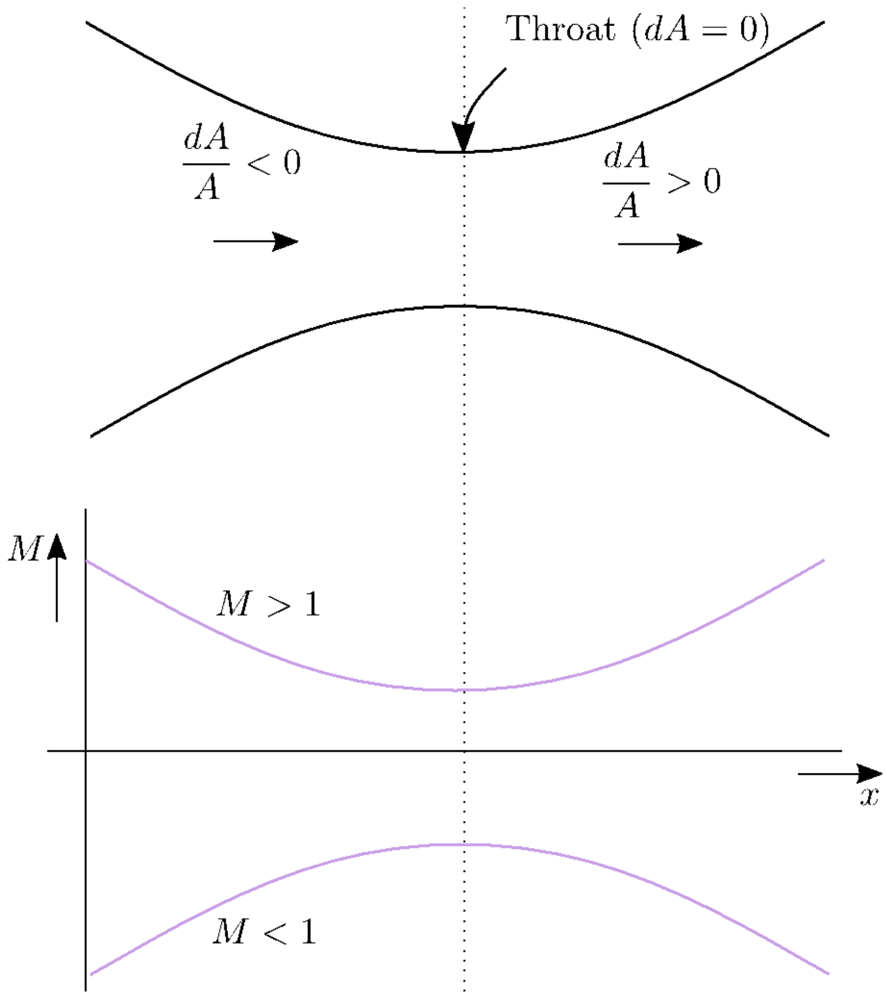

Figure 2: Variation of Mach number with axial length in a converging-diverging duct

We deduced above in the lecture that if sonic condition exists (i.e. Mach number is unity), then it must be at the throat. Hence, if Mach number is 1, it means that A is A^*. Every compressible flow corresponds to a hypothetical reference sonic state which is designated using the superscript * .

If we are solving $\frac{dA}{A}$ as a function of $\frac{dM}{M}$, there are parametric solutions based on two Mach numbers. This concept will be clearer if we think of an algebraic equation (rather than the differential equation, equation (13) above) having A referenced by a hypothetical sonic area A^*, and see how it varies with Mach number.