Advanced Concepts In Fluid Mechanics
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Lecture – 51
Compressible Flows (Stagnation Properties, Variable Area)

We briefly recap 'Stagnation Properties', discussed in the previous lecture. While the expressions for stagnation properties (pressure, temperature, density) were derived based on certain processes, these properties now correspond to the particular thermodynamic state of the gas and are independent of the process by which this state is attained. Furthermore, these stagnation properties are defined as the thermodynamic properties, corresponding to a thermodynamic state, that the gas would have if they are **hypothetically** brought to stagnation from the given thermodynamic state, and such a stagnation doesn't have to be brought about in reality. Furthermore, the stagnation properties change as the gas moves from one state to another.

As derived in the last lecture, the stagnation properties p_0, T_0, ρ_0 are related to the current thermodynamic properties p, T, ρ , as,

$$\frac{p_0}{p} = f_1(M), \frac{T_0}{T} = f_2(M), \frac{\rho_0}{\rho} = f_3(M), \quad (1)$$

where, M is the Mach number and f_i are unique functions of Mach number.

Now, to elucidate why the stagnation properties are being strongly emphasized, we consider stagnation enthalpy. If the flow is adiabatic and the change in potential energy and work done is negligible, the stagnation enthalpy for all points in the flow is the same, i.e.,

$$h_0 = h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_3^2}{2} = ... = h_k + \frac{u_k^2}{2}. \quad (2)$$

Equivalent statement holds true for stagnation temperature.

A subtle distinction occurs when extending the above discussion to stagnation pressure and stagnation density. Stagnation temperature is physically the temperature which can be attained by a fluid which is adiabatically brought to rest under no work transfer and with negligible changes in potential energy. On the other hand, the expressions for stagnation pressure and stagnation density were derived using the equation of state for an ideal gas undergoing reversible adiabatic process. Hence, the extra requirement of process being reversible comes into picture.

Having recapitulated the concepts of stagnation properties, we now discuss an illustrative example problem:

Problem:

Consider a large tank (figure 1) which is filled with air at 30 ^{0}C. There is an exit tube attached to the tank from which air is exiting at 235 m/s. The tank is also equipped with a mercury manometer and the manometric height h is 30 m (take density of mercury as 13550 kg/m^3). Find the temperature, pressure, and Mach number at the exit and the pressure in the tank.

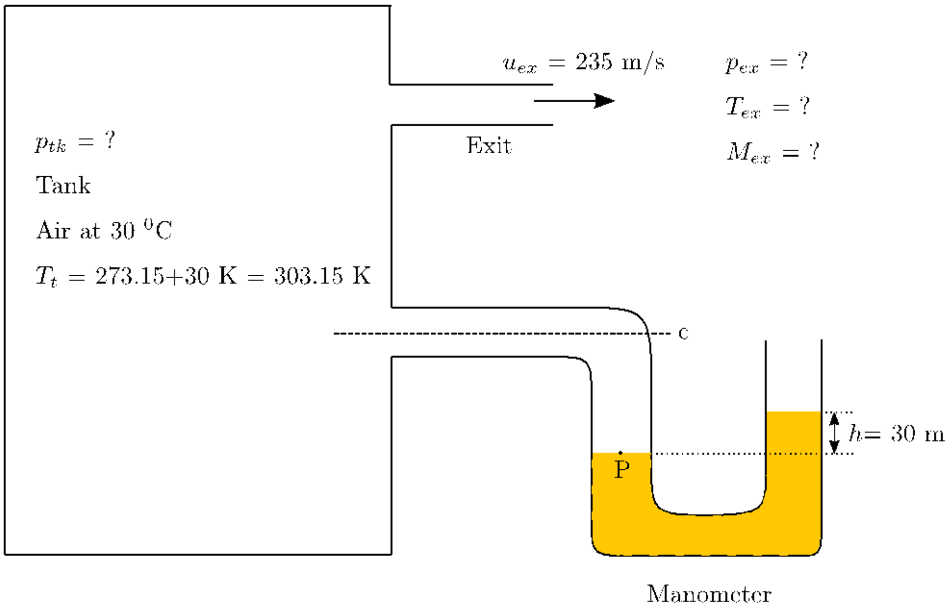

Figure 1: Figure for illustrative problem

We first assess the physical scenario at hand before attempting a solution. The illustrated manometer indicates that the pressure is higher in the tank compared to atmosphere. The pressure at point P is the same as the pressure on the centerline c. And the thermodynamic state on centerline c is same as the thermodynamic state in the tank. Hence, the pressure at point P is the pressure in the tank, given as,

$$p_{tk} = p_P = p_{atm} + \rho_{Hg} gh \,. \tag{3}$$

Since the gas well inside the tank from the exit, and hence along the line c as well, is stationary, p_{tk} is same as the stagnation pressure p_0. Here, we have made the additional assumptions that the tank is perfectly insulated from the surrounding, i.e. there is no heat exchange between tank and surrounding.

Now if you apply the energy equation (essentially the first law of thermodynamics) between the tank and the exit,

$$h_{tk} = h_{tk} = h_{ex} + \frac{u_{ex}^2}{2} \Rightarrow c_p T_k = c_p T_0 = c_p T_{ex} + \frac{u_{ex}^2}{2} \,. \tag{4}$$

Since T_0 and u_{ex} are known and c_P is available to us in literature as 1 kJ/kgK, we obtain T_{ex} as 276 K. Now, to obtain Mach number at the exit, we have the value of u_{ex} and the sonic speed c is given as $\sqrt{\gamma R T_{ex}}$, with $\gamma = 1.4$ for air. Substituting all the values, $M_{ex} = \frac{u_{ex}}{c} = 0.706$.

Now, to obtain p_{ex}, we use the equations,

$$\frac{p_0}{p} = \frac{p_{tk}}{p_{ex}} = \left[1 + \frac{\gamma - 1}{2} M_{ex}^2\right]^{\frac{\gamma}{\gamma-1}}, \qquad (5)$$

$$p_{tk} + \rho_{air} hg = p_{ex} + \rho_{Hg} hg\ . \qquad (6)$$

Equation (5) is the expression derived in the previous lecture relating stagnant pressure to pressure at a point in the flow. Equation (6) is simply the expression for pressure at point P and its counterpart on the right arm of the manometer. We note here that though $\rho_{air} hg$ has been included explicitly in equation (5), it will be negligible compared to $\rho_{Hg} hg$. p_{tk} can be eliminated between equations (5) and (6) and then we are left with an equation with only p_{ex} unknown,

$$\frac{p_{ex} + \rho_{Hg} hg - \rho_{air} hg}{p_{ex}} = \left[1 + \frac{\gamma - 1}{2} M_{ex}^2\right]^{\frac{\gamma}{\gamma-1}}. \qquad (7)$$

Substituting all the values, we obtain p_{ex} as 101 kPa, which is sufficiently close to atmospheric pressure.

Similarly, p_{ex} can be eliminated between equations (5) and (6) and then we are left with an equation with only p_{tk} unknown,

$$\frac{p_{tk}}{p_{tk} - \rho_{Hg} hg + \rho_{air} hg} = \left[1 + \frac{\gamma - 1}{2} M_{ex}^2\right]^{\frac{\gamma}{\gamma-1}}.$$

Substituting all the values, we obtain p_{tk} as 140.8 kPa.

This example problem has illustrated the relations between stagnation properties and current properties and their utility in the evaluation for an example problem .

Compressible flow through a variable area duct:

We will now study compressible flow through a variable area duct under one-dimensional analysis. This topic constitutes a fundamental cornerstone of the formalism on compressible flow.

Before proceeding further, we assess the situation using the often-used equations for flow continuity,

$$A_1 \bar{u}_1 = A_2 \bar{u}_2 \qquad (1a)$$

$$\frac{\partial u}{dx} + \frac{\partial v}{dy} = 0 \qquad (1b)$$

Equation (1a) relates the average axial velocity at two cross-sections 1 and 2 whose areas are A_1 and A_2 respectively. Equation (1b) applies to any particular point in the flow field. If the area of the duct is varying, we expect $\frac{\partial u}{dx}$ to be non-zero and resultantly $\frac{\partial v}{dy}$ to be non-zero as well. This indicates that solving the flow in a variable area duct using a one-dimensional formulation is a gross simplification and equation (1b) is more suitable. However, we observe here that since the flow we want to study is a compressible flow (where density varies), neither of equations (1a) and (1b) (which are both defined for constant density flows) is valid. While this does not imply that a one-dimensional analysis for compressible flow through a variable area duct does not amount to any simplifications of its own, it does mean that we cannot derive an understanding of these simplifications from equations (1a) and (1b).

Here, we also highlight an intuitive error that many students are prone to make regarding flow through a variable area duct. When asked whether the average flow velocity increases or decreases as the duct area decreases, the common answer, based firmly on intuition as well as high school physics knowledge, is that flow velocity increases. This intuition is mathematically validated by equation (1a). However, a subtle point comes into picture here. While equations (1a) and (1b) are statements of volume conservation, these are simplified versions of the mass conservation equation (which is the physical basis) under constant density assumption,

$$\rho_1 A_1 \bar{u}_1 = \rho_2 A_2 \bar{u}_2 \Rightarrow \rho A_1 \bar{u}_1 = \rho A_2 \bar{u}_2 \Rightarrow A_1 \bar{u}_1 = A_2 \bar{u}_2 \tag{2a}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u}{dx} + \frac{\partial v}{dy} = 0 \Rightarrow \left.\frac{\partial u}{dx} + \frac{\partial v}{dy} = 0\right|_{d\rho=0} \tag{2b}$$

On the other hand, for a compressible flow (where density changes), all is controlled by the band master, density. Mathematically, this control is dictated by Mach number. If the Mach number is small, the intuitive answer of increasing average velocity with decreasing duct area is reasonably correct. But if the Mach number is not small, there will be a physical correction. In addition to the mathematical understanding, the physical understanding is that a situation where area as well as velocity is decreasing is characterised by a tremendously increasing density.

We now highlight another common error. We conventionally expect an increase in flow speed to be associated with a decrease in flow pressure. This expectation arises out of our elementary mathematical study of fluid mechanics, where Bernoulli’s equation is used vehemently. However, Bernoulli’s equation is derived under multiple restrictions, two crucial ones (that are pertinent here) being flow is frictionless and density is constant. While many compressible flows can be appreciably close to frictionless (where we use the Euler equation), considering density to be constant in compressible flow is incorrect in all scenarios. Hence, we cannot apply Bernoulli’s equation to a compressible flow, and therefore, this commonly-used equation relating pressure and flow velocity is not at our disposal.

So, to summarize flow through variable area duct offers us a very interesting proposition of fine tuning the velocity with the density variation in compressible flow. This paradigm of varying the flow features via density variation is not available with us for incompressible flows.