**Advanced Concepts In Fluid Mechanics**
**Prof. Suman Chakraborty**
**Department of Mechanical Engineering**
**Indian Institute of Technology, Kharagpur**

**Lecture – 50**
**Compressible Flows (Stagnation Properties)**

In this lecture, we discuss some illustrative examples to better understand the concept of Mach number.

**Example 1:** A weak pressure wave with pressure change of 40 Pa propagates through still air at 20 $^0$C and 1 atm. Estimate:

(1) density change across the wave

(2) velocity change across the wave

Solution:

$\Delta p$ = 40 Pa, $T$ = 20 $^0$C, $p$ = 1 atm

We know that velocity change is related to the pressure change through the momentum conservation equation, as was derived in the first lecture on compressible flows (lecture 48). We import the corresponding expression,

$$\Delta p = \rho c \Delta u \,. \tag{1}$$

We get $\rho$ from the equation of state, $\frac{p}{\rho} = RT$ as $\rho = \frac{p}{RT}$ and we have $c = \sqrt{\gamma RT}$ as derived in the previous lecture. Hence, we now obtain $\Delta u$ by substituting these expressions in equation (1) and putting in the numerical values, giving us $\Delta u$ as 0.097 m/s.

Change in density, $\Delta \rho$, has similarly been derived using the mass conservation equation, as,

$$\Delta \rho = \frac{\rho}{c} \Delta u \,, \tag{2}$$

The numerical value obtained is 0.00034 kg/m$^3$. This simple illustrative example requires the appropriate application of mass and momentum balance equations to analyze the movement of the pressure wave.

**Example 2:** A supersonic aircraft flies horizontally at 1500 m altitude with a constant speed of 750 m/s. The aircraft passes directly over a stationary ground observer. How much time elapses after it has passed over the observer before the observer hears the aircraft? Assume: sonic speed is 335 m/s and the aircraft creates a small disturbance that may be treated as a sound wave.

Solution:

Importing the concept of Mach cone from the previous lecture, we see that speed of the source (750 m/s) is higher than speed of the disturbance (335 m/s). Thus, there exists a Mach cone having vertex as the aircraft and the observer will hear the aircraft, i.e. will be subject to the disturbance caused by the aircraft, when he/she enters the Mach cone. The observer will enter the Mach cone at the instance illustrated in figure 1. Examining the figure, we have,

$$\tan\theta = \tan\left(\sin^{-1}\left(\frac{1}{M}\right)\right) = \tan\left(\sin^{-1}\left(\frac{c}{u}\right)\right) = \frac{h}{u\Delta t}$$

Knowing $c$, $u$, and $h$, we obtain $\Delta t$ as 4.006 s.

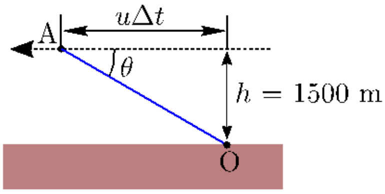


Figure 1: Instance when the observer (O) hears the aircraft (A) for the first time, i.e. enters the Mach cone

**Stagnation Properties:**

Consider the flow of fluid through a conduit as presented in figure 2. We assume fluid properties are respectively uniform at the cross-sections 1 and 2. Stagnation properties are defined as the properties of the fluid at cross-section 2 if the fluid is hypothetically brought to rest at cross-section 2 in an adiabatic process. Stagnation properties serve as reference properties, and hence, other properties in flow can be obtained with respect to these stagnation properties. This helps simplify the analysis of a compressible flow problem in many situations.


Figure 2: Section of a conduit with fluid flowing from left to right, with the control volume in consideration outlined using dotted line

To ascertain the stagnation properties, we start with the mathematical expression for the first law of thermodynamics for a flow process through a control volume,

$$\dot{Q}_{CV} + \dot{m}\left(h_1 + \frac{u_1^2}{2} + gz_1\right) = \dot{W}_{CV} + \dot{m}\left(h_2 + \frac{u_2^2}{2} + gz_2\right). \quad (3)$$

Note that in contrast to the expression for a control mass that has specific internal energy $u_1$ and $u_2$, here we have specific enthalpy $h_1$ and $h_2$. This is the case because of the additional flow work that gets added when switching from a control mass approach to a control volume approach.

We now assume any work done on the control volume to be negligible, i.e. $\dot{W}_{CV} \approx 0$. And since the process is adiabatic, $\dot{Q}_{CV} = 0$. Furthermore, if we consider the conduit to be horizontal, $z_1 = z_2$. Thus, equation (3) simplifies to,

$$h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}. \tag{4}$$

Invoking the stagnation condition at cross-section 2, $u_2 = u_0 = 0$ and $h_2 = h_0$, equation (4) becomes,

$$h_1 + \frac{u_1^2}{2} = h_0. \tag{5}$$

Written generally, i.e. in terms of any arbitrary point in flow field and not necessarily cross-section 1, equation (5) is,

$$h + \frac{u^2}{2} = h_0. \tag{6}$$

Assuming the fluid as an ideal gas, $dh = c_p dT$. Further assuming that the gas is calorically perfect, i.e. $c_p$ remains constant and does not change with temperature, this expression can be integrated to get,

$$h_0 - h = c_p(T_0 - T). \tag{7}$$

Note that for an ideal gas, $c_p$ is a function of only temperature. If $c_p$ is further constrained to be a constant, the ideal gas qualifies as a calorically perfect gas.

Substituting equation (7) into equation (6), we have,

$$\begin{aligned} &c_p(T_0 - T) = \frac{u^2}{2} = \frac{M^2c^2}{2} = \frac{M^2\gamma RT}{2} \Rightarrow \\ &T_0 = T\left(1 + \frac{M^2\gamma R}{2c_p}\right) = T\left(1 + \frac{M^2\gamma R}{2\dfrac{R\gamma}{\gamma-1}}\right) = T\left(1 + \frac{(\gamma-1)M^2}{2}\right) \Rightarrow \\ &\frac{T_0}{T} = 1 + \frac{(\gamma-1)M^2}{2}. \end{aligned} \tag{8}$$

Here, we have used the definitions $c_p = c_v + R$ and $\dfrac{c_p}{c_v} = \gamma$ to express $c_p$ in terms of $\gamma$ and $R$ as $c_p = \dfrac{R\gamma}{\gamma - 1}$.

Similar expressions for $\dfrac{p_0}{p}$ and $\dfrac{\rho_0}{\rho}$ exist.

To obtain these, we utilize the equation for a reversible adiabatic process and the equation of state for ideal fluid, $\frac{p}{\rho^{\gamma}} = c$ (constant) and $\frac{p}{\rho} = RT$ respectively. Combining these and equation (8), and eliminating $\rho$, we get,

$$\frac{p_0}{p} = \left[1 + \frac{(\gamma-1)M^2}{2}\right]^{\frac{\gamma}{\gamma-1}}. \tag{9}$$

Similarly, combining these and equation (8), and eliminating $p$, we get,

$$\frac{\rho_0}{\rho} = \left[1 + \frac{(\gamma-1)M^2}{2}\right]^{\frac{1}{\gamma-1}}. \tag{10}$$

Hence, given a Mach number and given pressure, temperature and density of the fluid, the stagnation properties (as reference properties) can be obtained. Furthermore, if the flow remains isentropic, the stagnation properties do not change as other properties of the fluid change, meaning that the stagnation properties for a particular Mach number remain constant, irrespective of the flow process.

Incompressible Flow Limit:

We now see the stagnation properties in the incompressible flow limit. Considering the process by which the fluid is brought to stagnation is reversible and adiabatic, the irreversible effect of viscosity is not present.

We now additionally apply the restriction of reversibility and consider the Bernoulli's equation in the incompressible flow,

$$p + \frac{\rho u^2}{2} = p_0. \tag{11}$$

to get an incompressible flow stagnation pressure, (we have substituted $u_0 = 0$ in the RHS). Proceeding further,

$$\begin{aligned} p_0 &= p + \frac{\rho M^2 c^2}{2} = p + \frac{\rho M^2 \gamma RT}{2} = p + \frac{M^2 \gamma p}{2} \Rightarrow \\ p_0 &= p\left(1 + \frac{M^2 \gamma p}{2}\right). \end{aligned} \tag{12}$$

Here, we have used the equation of state for ideal gas $p = \rho RT$, as well as the expression for sonic speed, i.e. $c = \gamma RT$.

We now expect that the incompressible flow stagnation pressure $p_0$ obtained as per the Bernoulli's equation, as presented in equation (12) above, should match with the incompressible limit $p_0$ from equation (9). Towards this end, we express the RHS of equation (9) using binomial expansion and neglect higher order terms of $M^2$ as we are considering the incompressible limit, for which $M << 1$. This yields the same expression as presented in equation (12). Here, we emphasize the caveat that the stagnation pressure derived in equations (9) and (12) are for a reversible and adiabatic process. This is in contrast to the expression for stagnation temperature (equation (8)) which required the process to only be adiabatic and not necessarily reversible. Lastly, a crucial distinction between equations (9) and (12) is that the ideal gas equation of state is used in deriving the former whereas derivation of the latter used an explicit relation between pressure and flow velocity, the Bernoulli's equation.