Advanced Concepts In Fluid Mechanics
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Lecture - 05
Continuity Equation in Integral Form: Stream Function and Velocity Potential

I. The Popular Form of Continuity Equation

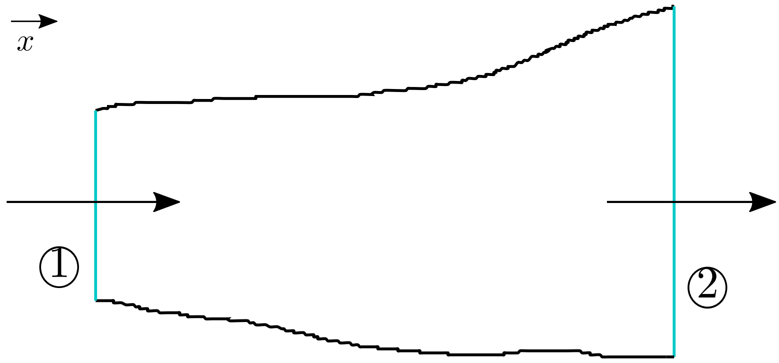

Fig 1: A arbitrary chamber through which fluid is flowing. Fluid enters at 1 and leaves at 2. The other two walls (i.e. the black walls) are impermeable (i.e. fluid doesn't flow through them).

An expression for continuity equation that is popularly used is

$$A_1 v_1 = A_2 v_2 , \tag{1}$$

where A_1 and A_2 are the areas of and v_1 and v_2 are the velocities at the inlet and outlet of a chamber. For illustration, one such chamber is presented in Fig 1 with the inlet and outlet marked. However, we derive this popular expression by using the standard form of continuity equation (that was derived in the last lecture),

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 . \tag{2}$$

First, we assume the flow is steady. This simplifies the equation (2) to,

$$\nabla \cdot (\rho \vec{v}) = 0 . \tag{3}$$

Integrating equation (3) over the volume of the chamber and then applying divergence theorem,

$$\int_V \nabla \cdot (\rho \vec{v}) dV = 0 \Rightarrow \int_A (\rho \vec{v}) \cdot \hat{\eta} dA = 0 . \tag{4}$$

We assume the flow is uni-directional in the horizontal i.e. x-direction at the inlet and outlet i.e. walls 1 and 2. Further, given the walls 1 and 2 are vertical i.e. perpendicular to x and the black colored walls are impermeable (i.e. there is no liquid flux through them), equation (4) becomes,

$$-\int_{1}\rho u dA+\int_{2}\rho u dA=0\,. \tag{5}$$

Further considering constant fluid density,

$$\int_{1} udA=\int_{2} udA\Rightarrow \bar{u}_1 A_1=\bar{u}_2 A_2\,. \tag{6}$$

Therefore, we have obtained the popular expression for continuity in equation (6), where the velocities are actually the average velocities at the inlet and outlet and not necessarily the velocities at say the midpoints on the inlet and outlet (a common mistake).

II. Stream Function and Velocity Potential

We conclude the module on fluid kinematics by considering two important classical functions in fluid kinematics, the stream function and the velocity potential.

<u>Stream Function:</u> We consider a two-dimensional incompressible flow. Therefore, following the continuity equation (which is equivalent to a kinematic constraint for compressible flow),

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0\,. \tag{7}$$

Since this equation has two unknowns, u and v, it is useful to use a parametric form of these unknows that could aid further analysis. Therefore, we consider the function ψ such that,

$$u=\frac{\partial \psi}{\partial y},\; v=-\frac{\partial \psi}{\partial x}\,. \tag{8}$$

Substituting these in equation (7), we see that equation (7) gets trivially satisfied (i.e. it gets converted to 0=0).

There are two important concerns often raised by students in the context of stream function. First, is there is any sanctity to the signage in the expressions in equation (8). The answer to this question is that there is no sanctity to the signage so far as the sign for expressions for the two components are opposite (as equation (7) would not be trivially satisfied otherwise). Therefore, stream function could just as well be defined as,

$$u=-\frac{\partial \psi}{\partial y},\; v=\frac{\partial \psi}{\partial x}\,. \tag{9}$$

Second, why is this function called a stream function. To address this, we use,

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy\,, \tag{10}$$

which is a standard identity in differential calculus. Substituting the expressions in equation (8), equation (10) becomes,

$$d\psi = -vdx + udy\,. \tag{11}$$

This implies that for $d\psi = 0$, $\frac{dx}{u} = \frac{dy}{v}$, which is the equation of a streamline. In other words, stream function is constant along a steamline, hence the term stream function. However, while stream function is constant along a streamline, all streamlines don't necessary have a constant stream-function value. The reason is that while streamlines are present in three-dimensional flows and compressible flows as well, stream function isn't defined for these flows. Therefore, streamlines have constant stream function values only for flows for which both are defined.

<u>Velocity potential:</u> We now consider an irrotational flow. For such a flow,

$$\nabla \times \vec{v} = 0\,. \tag{12}$$

With this equation at hand, we use the vector identity $\nabla \times (\nabla \phi) = 0$ to define a velocity potential ϕ such that $\vec{v} = \nabla \phi$.

In field theory, a field can be expressed as a gradient of a scalar potential if it is a conservative field. Hence, in an irrotational flow, if there is no viscous effect, the flow will remain conservative and there will be no viscous dissipation, which is why we are able to expression velocity as the gradient of the scalar, velocity potential (or simply potential).

<u>Constant Stream Function and Constant Velocity Potential Lines:</u> Having defined stream function and velocity potential, we consider a 2-D (two-dimensional) incompressible irrotational flow. For this flow, both the stream function and the velocity potential are defined. In the discussion for streamfuntion, we have already obtained $d\psi = 0 \Rightarrow \frac{dx}{u} = \frac{dy}{v} \Rightarrow \left.\frac{dy}{dx}\right|_{\psi=\text{constant}} = \frac{v}{u}$, utilizing equation (10). Using similar approach for velocity potential, we have $d\phi = 0 \Rightarrow \frac{dx}{v} = -\frac{dy}{u} \Rightarrow \left.\frac{dy}{dx}\right|_{\phi=\text{constant}} = -\frac{u}{v}$. Combining these two, we have,

$$\left.\frac{dy}{dx}\right|_{\phi=\text{constant}} \times \left.\frac{dy}{dx}\right|_{\phi=\text{constant}} = \frac{v}{u}\cdot\left(-\frac{u}{v}\right). \tag{13}$$

While one would be tempted to multiply the terms in RHS of equation (13) to get -1, we should be cautious of the case where u or v is zero, in which case we are not allows to do the multiplication. Once special case is when both u and v are zero, which is termed as a stagnation point. However, if both u and v are non-zero we can perform this multiplication to get,

$$\left.\frac{dy}{dx}\right|_{\phi=\text{constant}} \times \left.\frac{dy}{dx}\right|_{\phi=\text{constant}} = -1. \tag{13}$$

This implies that constant stream function lines and constant velocity potential lines for a 2-D incompressible irritation flow are perpendicular to each other, i.e. they for a set of orthogonal curves. In computational fluid mechanics, one can generate gridlines using constant stream function and constant velocity potential lines to get an orthogonal grid form, which is also called as a flow net.

Lastly, for this flow, substituting the expression for velocity components in terms of stream function (equation (9)) into the condition for irrotationality, i.e. $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$, gives,

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{14}$$

Similarly, substituting the expression for velocity components in terms of stream function (i.e. $\vec{v} = \nabla \phi$) into the continuity equation for incompressible flow, equation (7), gives,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0. \tag{15}$$

Equations (14) and (15) are 2-D Laplace equations for ψ and ϕ respectively. Hence, both stream function and velocity potential satisfy the 2-D Laplace equation in the flow domain. However, their solutions are not the same but a set of orthogonal lines. This occurs because the boundary conditions for the two are different.

Illustrative Example: Before concluding this lecture, we consider an illustrative example to improve our understanding of stream function and velocity potential.

Consider a flow field,

$$u = a(x+y)$$
$$v = a(x-y)$$
$$w = b$$

For this flow field, (i) determine the velocity potential should it exist, and (ii) determine the projection of the streamline on the $x-y$ plane.

Solution:

(i) First, to determine whether potential function exists or not, we check whether the flow is irrotational or not. Since $\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=a-a=0$, the flow is irrotational. Therefore, potential function, ϕ, exists. To determine ϕ, we use the expressions for velocity components as derivatives to ϕ,

$$u=\frac{\partial \phi}{\partial x}=a(x+y)\Rightarrow \phi=\frac{1}{2}ax^2+axy+f(y,z)$$
$$v=\frac{\partial \phi}{\partial y}=a(x-y)\Rightarrow \phi=-\frac{1}{2}ay^2+axy+g(x,z)$$
$$w=\frac{\partial \phi}{\partial z}=b\Rightarrow \phi=bz+h(x,y)$$

The integrations done above are called partial integrations, as a consequence of which, the constants of integration are function of the other co-ordinates (for example, partial integration of the topmost expression is done w.r.t. x and therefore, the constant of integration, $f(y,z)$, is a function y and z). It is a common mistake to consider the constant of integrations for partial integrals as absolute constants rather than function of the other co-ordinates.

Combining all these expressions, the velocity potential is, $\phi=\frac{1}{2}ax^2-\frac{1}{2}ax^2+axy+bz+c$, where c is a constant. Lastly, to get the projection of streamlines on the $x-y$ plane, consider the sum of $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$. This sum is $a-a$ which is zero. Therefore, even though the flow is three-dimensional, we see that the equation $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$, which is the 2-D incompressible flow continuity equation, is satisfied. Since this equation is satisfied, it becomes possible to use the stream function, and therefore, the projection of streamlines on the $x-y$ would simply be the lines on constant stream function. The stream function is obtained as $\psi=\frac{a}{2}(x^2+y^2)+axy++k$ where k is a constant.