Advanced Concepts In Fluid Mechanics
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Lecture – 49
Compressible Flow (Contd.)

Sonic Speed:

The effect of compressibility is quantified using the Mach number M, as discussed in the previous lecture. We also determined that Mach number is dependent on 'sonic speed', i.e. the speed of sound in a medium.

We now assess the physical meaning and associations of sonic speed. Sonic speed in a medium was obtained in the last lecture as,

$$c = \sqrt{\frac{dp}{d\rho}}, \quad (1)$$

where c is sonic speed, p is pressure in the medium and ρ is density of the medium. More formally, this variation of pressure with density should be derived for a specified process. Hence, we typically define sonic speed c as the isentropic sonic speed c_s, which is defined as,

$$c_s = \sqrt{\left(\frac{dp}{d\rho}\right)_s}, \quad (2)$$

where the subscript s implies the process is isentropic. 'Isentropic' process means the entropy of the system remains constant as the process occurs. From our understanding of thermodynamics, we know that a process which is reversible and adiabatic is isentropic.

We briefly recap the definition of reversibility using an example. Consider a piston cylinder setup where the piston is being pulled outward. If the piston is being pulled outward at a sufficiently low speed such that the gas inside the cylinder is at equilibrium at any intermediate instant, then the process is called reversible. The slowness of pulling the piston is the interpretation of reversibility particular to this case, whereas, the equilibrium state of the gas in the cylinder at any intermediate instant is the principle of reversibility. On similar lines, if the gas in the cylinder is being heated across a finite temperature difference, the reversibility of the process gets violated and hence, the process becomes irreversible. Here, we recall that often in elementary physics, we identified a process that is fast enough to prohibit any heat transfer to/from the system as adiabatic. While such a process is adiabatic, it is not reversible. Hence, we summarily deduce that a system insulated from the surroundings undergoing a process that is sufficiently slow qualified for a ‘reversible adiabatic’ process and by extension an isentropic process.

Delving into the mathematical formulation for such a process, we write the first law of thermodynamics,

$$\delta Q = dI + \delta W \Rightarrow \delta q = di + \delta w, \tag{3}$$

where, δQ is the heat entering the system, dI is change is internal energy, and δW is work being derived from the system. It is customary to express change in internal energy with dU and change in total energy (which includes kinetic, potential and internal energy) with dI. To avoid confusion with the flow velocity which is denoted by u and given that we are considering a thermodynamic process which involves change in only internal energy with negligible change in kinetic and potential energy, here we use dI to represent change in internal energy of the system. δq, di, and δw are the per unit mass, i.e. ‘specific’ counterparts of δQ, dI and δW respectively.

δq is zero for an adiabatic process, $\delta w = pdv$ for a reversible process due to the qausi-static nature of system volume change, and $di = c_v dT$ for an ideal gas. Hence,

$$0 = c_v dT + pdv, \tag{4}$$

for the process we study here, a reversible adiabatic process.

Here, we also observe that the definition of change in (specific) entropy ds in a reversible process is $\delta q = Tds$, and δq being zero (as is for an adiabatic process) implies ds is also zero given the process is reversible as well – this is the mathematical basis for the statement ‘a reversible adiabatic process is isentropic’. Now, we observe that for an ideal gas,

$$pv = RT. \tag{5}$$

This implies,

$$pdv + vdp = RdT. \tag{6}$$

Equation (6) enables us to eliminate dT in equation (4), giving us,

$$pdv\left(\frac{c_v}{R}+1\right) + vdp\frac{c_v}{R} = 0 \Rightarrow c_p pdv + c_v vdp = 0 \Rightarrow \frac{dp}{p} + \frac{c_p}{c_v}\frac{dv}{v} = 0. \tag{7}$$

In equation (7), we have made the substitution $c_p = c_v + R$, which holds true for an ideal gas. Integrating, we get,

$$pv^{\gamma} = k = \text{constant}, \tag{8}$$

where, γ is the ratio c_p / c_v. Equation (8) derived above is the commonly-known expression for a reversible adiabatic process.

Since density ρ and specific volume v are related as $\rho v = 1$, equation (8) can also be expressed as,

$$\frac{p}{\rho^{\gamma}} = k. \tag{9}$$

Proceeding ahead with equation (9),

$$\ln p - \gamma \ln \rho = \ln k \Rightarrow \frac{\partial p}{p} - \gamma \frac{\partial \rho}{\rho} = 0 \Rightarrow$$
$$\frac{\partial p}{\partial \rho} = \frac{\gamma p}{\rho} = \gamma pv = \gamma RT. \tag{10}$$

In equation (7), we have made the substitution $pv = RT$, which holds true for an ideal gas. Hence, we have obtained the expression for c_s as,

$$c_s = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} = \sqrt{\gamma RT}. \tag{11}$$

It should be noted that this speed c_s is a reference sonic speed derived for a perceived isentropic process.

Physical Implications of Mach Number M:

To illustrate the physical implications of Mach number, we consider three illustrative scenarios, corresponding to $M > 1$, $M = 1$ and $M < 1$. Let us consider a disturbance propagating in a stationary medium in which the source of the disturbance is moving from right to left at speed u.

We first consider the scenario of $M < 1$, and take $M = 0.5$. This scenario is presented pictorially in figure 1. The source speed u is half of the disturbance propagation speed c for this scenario. In figure 1, we present the situation at time $t = 3\Delta t$. The three fronts (from left to right) presented as dotted black lines correspond to the location of the pressure wave front of disturbance generated by the source at times $t = 0$, $t = \Delta t$, and $t = 2\Delta t$. Even the disturbance produced the smallest duration ago in history, i.e. at $t = 2\Delta t$ (just Δt ago than the current instance $t = 3\Delta t$) has proceeded beyond the current location of the source. Hence, the disturbance reaches any location on the path of the source prior to the source reaching that location.

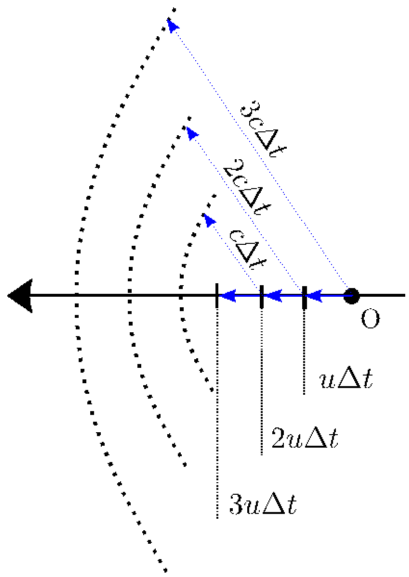

Figure 1: Scenario with $M = 0.5 < 1$.

We next consider the scenario of $M = 1$. This scenario is presented pictorially in figure 2. The source speed u is equal to the disturbance propagation speed c for this scenario. Similar to figure 1, in figure 2, we present the situation at time $t = 3\Delta t$. The three fronts presented as dotted black lines correspond to the location of the pressure wave front of disturbance generated by the source at times $t = 0$, $t = \Delta t$, and $t = 2\Delta t$. The disturbance produced at each of the three instances in history have fronts co-incident with the current location of the source. Hence, the disturbance reaches any location on the path of the source at the same instance as the source reaches that location.

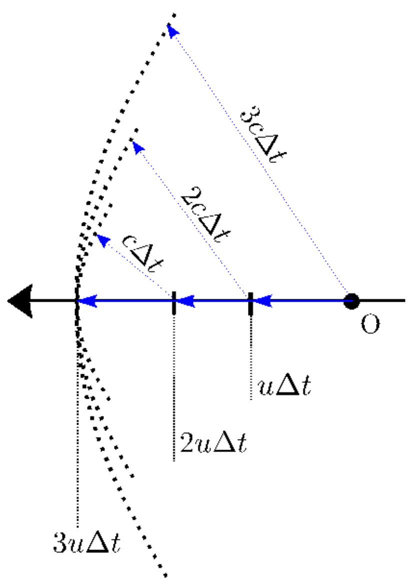

Figure 2: Scenario with $M = 1$.

We lastly consider the scenario of $M > 1$, and take $M = 2$. This scenario is presented pictorially in figure 3. The source speed u is double the disturbance propagation speed c for this scenario. Similar to figures 1 and 2, in figure 3, we present the situation at time $t = 3\Delta t$. The three fronts presented as dotted black lines correspond to the location of the pressure wave front of disturbance generated by the source at times $t = 0$, $t = \Delta t$, and $t = 2\Delta t$. The source has crossed through the region of disturbance propagation corresponding to each of the three instances in history. Hence, the disturbance reaches any location on the path of the source after the source has passed through that location. Furthermore, all the fronts are such that they are tangent to the two dashed lines shown, which passes through the current location of the source as well. In three-dimensional space, these lines are actually edges of a cone called the ‘Mach cone’. If we denote the angle either of these dashed lines makes with the horizontal by θ, simple examination of figure 3 indicates that,

$$\sin\theta = \frac{c\Delta t}{u\Delta t} = \frac{c}{u} = \frac{1}{M} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{M}\right). \quad (12)$$

The region outside the cone is not subject to any disturbance whereas the region within the cone is. Hence, surface of this cone represents a ‘discontinuity’ of disturbance in the medium, and it constitutes the characteristics of the mathematical equations of flows that exhibit such discontinuity in the disturbance of the medium. We note here that the discontinuity arises because the speed of the source is comparable to the sonic speed, meaning that the disturbance propagates through the medium at a finite speed which gets surmounted by the speed of source. This becomes possible because – sonic speed is finite, which means the density changes as the medium undergoes some process, which can occur only when the flow is compressible.

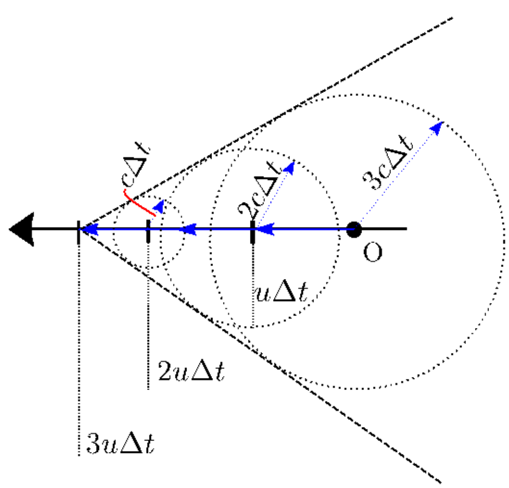

Figure 2: Scenario with $M = 2 > 1$.