Advanced Concepts In Fluid Mechanics
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Lecture – 48
Compressible Flow

Compressible Flows:

In this lecture, we start the topic of compressible flows. While all flows exhibit some measure of compressibility, the term 'compressible flows' refers to a special type of flows having some common qualities.

While not a part of the formal definition of compressibility, it is intuitively expected that compressibility is related to change in density. In other words, a strong change in density is one important effect under which a fluid flow becomes compressible. However, the formal definition of an incompressible flow is a flow characterized with no change in volume of any arbitrary fluid element, i.e. $\nabla \cdot \vec{v} = 0$ – a purely kinematic definition. Thus, any flow that deviates from this behaviour is strictly-speaking compressible.

Speaking in terms of one-dimensional flow, a compressible flow satisfies,

$$\frac{\partial u}{\partial x} = 0 . \qquad (1)$$

In contrast a 'general' one-dimensional steady flow (the term 'general' implying that the effect of compressibility is accommodated in the analysis) satisfies,

$$\frac{\partial (\rho u)}{\partial x} = 0 . \qquad (2)$$

While equation (1) is a pure kinematic condition, equation (2) is the true expression for conservation of mass and is not a pure kinematic constraint. Expanding equation (2), we have,

$$u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 . \qquad (3)$$

Recall that for one-dimensional steady incompressible flow, $\frac{\partial u}{\partial x}$ is zero (as per equation (1)) and this implies that as per equation (3) (which applies to any general one-dimensional steady flow, compressible or otherwise), $\frac{\partial \rho}{\partial x}$ is also zero. Similarly, we also arrive at the deduction that if $\frac{\partial u}{\partial x}$ is not zero, as occurs for a one-dimensional steady compressible flow, equation

(3) dictates that $\frac{\partial \rho}{\partial x}$ is constrained to be not zero as well. Furthermore, the order of the two terms of equation (3) should be the same, i.e.,

$$\rho \Delta u \sim u \Delta \rho \text{ or } \Delta \rho \sim \frac{\rho \Delta u}{u}. \tag{4}$$

$\Delta \rho$ indicates the degree of incompressibility.

We now write the momentum conservation equation for the one-dimensional steady inviscid flow, i.e. the one-dimensional steady Euler equation,

$$\rho u \frac{du}{dx} = -\frac{dp}{dx} \Rightarrow \frac{dp}{\rho} + u du = 0. \tag{5}$$

From equation (5), we obtain the order relation,

$$\Delta p \sim \rho u \Delta u. \tag{6}$$

Dividing equation (6) by equation (4), we have,

$$\frac{\Delta p}{\Delta \rho} \sim u^2. \tag{7}$$

From equation (7), we deduce that when the u^2 is of the same order as $\frac{\Delta p}{\Delta \rho}$, then compressibility effects in the flow are strong. On the other hand, when the order of u^2 is much smaller than the order of $\frac{\Delta p}{\Delta \rho}$, the flow is appreciably close to incompressible.

Now, we analyse the relevance of the term $\frac{\Delta p}{\Delta \rho}$ in the propagation of compressibility effect through a medium. For this, we consider a simple experiment of a piston in a cylinder with the piston being pushed to the right (i.e. towards decreasing the cylinder volume). The region enclosed by the piston and cylinder is filled with a fluid. This setup is presented in figure 1.

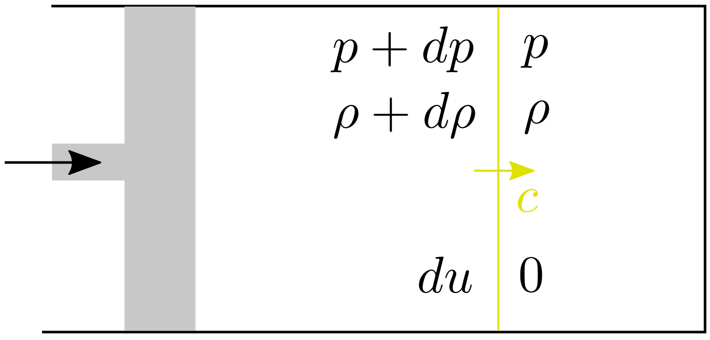

Figure 1: A piston fitted on a cylinder moving towards the right and compressing the fluid inside

As the piston is pushed, the fluid just adjacent to the piston are subjected to a compression but fluid that is significantly inside the cylinder does not get the information of this compression immediately. Rather, information of the compression gets propagated through the fluid by a compression-rarefaction wave originating at the piston location that propagates at a finite speed. This propagation of the information of compression by means of a compression-rarefaction wave of finite speed is similar to a sound wave and is commonly called as ‘disturbance propagation in the form of a pressure wave’.

Now, we consider the wave front presented in figure 1 by the yellow line. It propagates towards the right with a speed of c. There is an infinitesimal discontinuity at this wave front (this discontinuity is in contrast to a finite discontinuity, which is obtained in a shock wave in the context of compressible flows, as we will learn ahead in the course). The pressure, density and flow velocity on either side of the wave front (upstream, i.e. left side or the side on the opposite direction of propagation of the wave front, and, downstream, i.e. right side or the side on the same direction of propagation of the wave front) are depicted. We note here that the wave front is located at the leftmost part of the undisturbed region of the fluid, as the downstream region of the wave front has flow velocity of zero. The wave front is essentially what carries the information about the piston being pressed and hence induces compression in the flow as it propagates.

We now want to find out how the changes in pressure and density are related to c, the speed of propagation of the disturbance, i.e. speed of the wave front, commonly called ‘sonic speed’. To understand this, we consider the wave front to have an area A and we now consider a frame moving with the wave front. In transitioning from a stationary frame to this frame that is attached to the wave front, a left to right velocity of magnitude c gets added to entire fluid. Hence, the velocity downstream of the wave front is c right-to-left whereas the velocity upstream of the wave front is $c-du$ right-to-left.

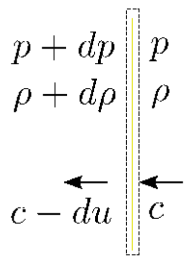

Figure 2: Wave front in a frame attached to the wave front, and, control volume (dashed enclosure) enclosing the wave front

We now consider a control volume enclosing the wave front, depicted in figure 2. We write the mass conservation and momentum conservation equations for this control volume,

Mass conservation: $\rho Ac = (\rho + d\rho)A(c - du)$. (1)

Momentum conservation: $pA-(p+dp)A=\dot{m}\left[(c-du)-c\right]$. (2)

We expand equation (1),

$$\rho Ac = A(\rho c-\rho du+d\rho c-d\rho du). \quad (3)$$

In equation (3), the term $d\rho du$ is negligible, hence we drop this term. Equation (3) now becomes,

$$\rho du = cd\rho \Rightarrow d\rho = \frac{\rho du}{c}. \quad (4)$$

We now simplifying equation (2),

$$-Adp=\rho Ac(-du) \Rightarrow dp=\rho cdu, \quad (5)$$

where we have made the substitution $\dot{m}=\rho Ac$. Dividing equation (5) by equation (6) (our objective is to get a term similar to $\frac{\Delta p}{\Delta \rho}$), we get,

$$\frac{dp}{d\rho}=\frac{\rho cdu}{\frac{\rho du}{c}}=c^2. \quad (6)$$

Hence, we can say, $\frac{\Delta p}{\Delta \rho} \sim c^2$ for the wave front, and therefore, to determine if compressibility is significant or not, we should compare u^2 and c^2, i.e. flow speed and sonic speed. Thus, we define a quantity, Mach number M as,

$$M=\frac{u}{c}, \quad (7)$$

and, the benchmark for onset of compressibility is $M^2 \sim 1$ (note that it is $M^2 \sim 1$ and not $M \sim 1$).

Compressibility is negligible when $M^2 << 1$, i.e., the compressibility wave does not induce a significant velocity by virtue of compression and rarefaction of the fluid. For instance, if $M^2 = 0.1$, i.e. $M = 0.3$, compressibility is negligible.

Conventionally, based on the sonic speed c, we obtain the Mach number M, and,

if $M < 1$, the flow is called subsonic

if $M = 1$, the flow is called sonic

If $M > 1$, the flow is called supersonic

The compressibility effect gets more important as the Mach number increases (i.e. as the flow goes from subsonic to supersonic).

It should be noted here that Mach number is not a constant in a flow region/device, and it transitions as one considers different locations in the flow region/device. A common example is compressible flow through a variable-area nozzle – for example a flow can transition from supersonic to subsonic in a variable-area nozzle. We will look into how the Mach number varies, and what are the parameters on which Mach number depends, in the next lecture.