Advanced Concepts In Fluid Mechanics
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Lecture – 47
Thin Film Dynamics (Contd.)

In the previous chapter we have discussed about the procedure to derive the thin film equation from all the boundary conditions and the governing equations under the thin film assumption. Now the equation that we have derived is quite an involved one and we need to actually appreciate the method of solving this problem. To do that we choose a very simple physical problem and we try to analyze the method using that. We consider the example of the gravitational spreading of a cylindrical drop.

Gravitational spreading of a cylindrical drop:

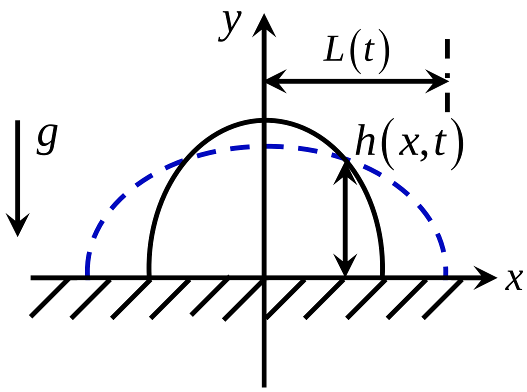

Figure 1. Gravitational spreading of a cylindrical drop.

As shown in figure 1, we have a drop (shown by the semi-circle drawn using black-colored solid line) which has a large length perpendicular to the plane. So it is basically a two-dimensional description which gives a complete picture of its spreading phenomenon. If the drop is placed on a surface, eventually the drop will spread (the final shape of the drop is shown by the semi-circle drawn using blue-colored dotted line) because of the gravity which is acting in the downward direction. At some point of time the local height $h(x,t)$ is a function of position *x* and time *t*. After some time, we have a situation when the entire droplet is spread on the surface so that the maximum height is zero. When the maximum height is equal to zero we can say that the spreading process is complete. Here one of our interest is to determine this spreading time or in general the variation of *h* as a function of position *x* and time *t*.

The first question arises in this context is whether this problem can be considered as a thin film problem or not. In the thin film problem, the basic premise is the existence of a small parameter $\varepsilon = \frac{h_0}{l_c}$. h_0 is the characteristic thickness of the film or the maximum thickness of the film and l_c is the length scale along the axial (*x*) direction and the ratio ε has to be small. In the present problem, when the droplet starts spreading, the maximum height h_0 and the axial length scale l_c are comparable. Only when the spreading is towards the end of its journey, around that time we will have h_0 much less as compared to l_c. So we can apply the thin film theory for solving the film thickness only during that time. At the early stage we cannot use this thin film theory because at the early stage, the height of the droplet is significant as compared to its footprint. Its thickness becomes negligible as compared to its footprint on the surface only towards the end of the spreading and then only the thin film theory can be applied.

Now we will solve the present problem from the first principles by using the set of equations which was reported in the earlier chapters. For the ease of understanding, these set of governing equations and boundary conditions are rewritten below:

Governing equations:

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

x-momentum equation:

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x' \qquad (2)$$

y-momentum equation:

$$0 = -\frac{\partial p}{\partial y} + \frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y' \qquad (3)$$

where f_x' and f_y' are the dimensional body forces per unit volume.

Boundary conditions:

Tangential force balance (at the interface):

$$\frac{\partial u}{\partial y} = \frac{\varepsilon}{Ca} \tilde{\nabla} \sigma \qquad (4)$$

Normal force balance (at the interface): $p_s - p_0 = -\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2} + p_{ex}$ (5)

Kinematic boundary condition: $v_i = \left(\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x}\right)_i$ ('*i*' indicates interface) (6)

Along with that there is additional no-slip boundary condition, i.e. at $y = 0$, $u = 0$.

Now we have to think of the simplifications that can be done for our present problem. Here we have the knowledge of the procedure to derive the thin film equation from the first principle. But one thing is missing, i.e. the knowledge of the characteristic velocity scale u_c. As we have already discussed, the mathematics of the thin film dynamics problem is very structured. But to identify the correct physics is not so structured based on which only one can determine the characteristic velocity scale u_c. However, in the present problem, the identification of the involving physics is quite obvious.

First of all, in the present problem, there is no body force in the *x*-momentum equation because the entire body force is acting vertically in the downward direction because of the gravity. So we set the term $\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x'$ of the *x*-momentum equation (2) equal to zero. Now, in the *y*-momentum equation (3), the body force term f_y' will be equal to $-\rho g \sin\theta$. Since in the present problem θ is equal to 90^0, $-\rho g \sin\theta$ will be equal to $-\rho g$, i.e. $f_y' = -\rho g$. In the previous chapters we have discussed about the various possibilities of physics which can define the characteristic velocity scale u_c. If the body force in the *x*-momentum equation $\left(\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x'\right)$ is important then it will define the velocity scale. Similarly, if the body force in the *y*-momentum equation $\left(\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y'\right)$ is important then it will define the velocity scale. If the tangential gradient of the surface tension $\left(\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma\right)$ is important then it will define the velocity scale. If the Laplace pressure is important then the coefficient of the curvature term $\left(\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}\right)$ will decide the velocity scale. If the disjoining pressure $\left(p_{ex}\right)$ is important then it will define the velocity scale.

In our present problem, the *y*-momentum equation is important which is given by

$0=-\frac{\partial p}{\partial y}-\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}\rho g$. So, $\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}\rho g$ is the term which will dictate the characteristic velocity scale u_c. The term $\frac{\partial p}{\partial y}$ is of the order of 1 because it is the non-dimensionalized pressure gradient which is made dimensionless by using proper scales. So the term $\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}\rho g$ also must be of the order of 1 such that terms $\frac{\partial p}{\partial y}$ and $\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}\rho g$ become competitive. So, the term $\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}\rho g$ becomes of the order of 1 from which we get $u_c \sim \frac{\varepsilon^3 l_c^{\,2}}{\mu}\rho g$. This is the first step as well as the most important step because the remaining part is mathematics (which contains algebra, calculus etc.). But the first step involves the physics of the problem and we have to be very careful about it.

Based on this velocity scale $u_c \sim \frac{\varepsilon^3 l_c^{\,2}}{\mu}\rho g$, one can estimate the Capillary number $Ca=\frac{\mu u_c}{\sigma}=\frac{\varepsilon^3 l_c^{\,2}\rho g}{\sigma}$. If we divide both the numerator and the denominator by l_c, we get $Ca=\varepsilon^3\frac{l_c\,\rho g}{\sigma/l_c}$. Now σ/l_c is nothing but the capillary pressure and $l_c\,\rho g$ is the pressure due to the hydrostatic effect or the gravity. So the ratio $\frac{l_c\,\rho g}{\sigma/l_c}$ is essentially the ratio of the gravity force and the surface tension force which is called as the Bond number (*Bo*), i.e. $Bo=\frac{l_c\,\rho g}{\sigma/l_c}$. Since our present problem is the gravitational spreading of drop, gravity is the dominating force. When gravity is the dominating force we can assume that the Bond number (*Bo*) is very large. In this context one can argue that the curvature effect is also important. The curvature effect is important but not at the very end stage of spreading. At the very beginning of spreading, the effect of curvature is very critical which is equally important with the gravity force; eventually the droplet flattens down and the curvature effect becomes less. Towards the end the curvature effect $\left(\sigma/l_c\right)$ becomes very less as compared to the gravity effect $\left(l_c\, \rho\, g\right)$. So the regime of the problem where we are working, the Bond number (*Bo*) is essentially much larger than 1,

i.e. *Bo* >> 1. It means that $\frac{\varepsilon^3}{Ca}$ is much less than 1, i.e. $\frac{\varepsilon^3}{Ca} = \frac{1}{Bo} \ll 1$. Now we look into the boundary condition where the term $\frac{\varepsilon^3}{Ca}$ appears in the normal force balance boundary condition (5) which reads as $p_s - p_0 = -\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2} + p_{ex}$. Here, the term $\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}$ represents the effect of the curvature in which $\frac{\partial^2 h}{\partial x^2}$ is non-dimensionalized second-order derivative and hence, it is of the order of 1. σ is the non-dimensional surface tension which is also of the order of 1. So if we consider the condition $\frac{\varepsilon^3}{Ca} \ll 1$, it means the entire $\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}$ term (or the effect of curvature) can be neglected from the normal force balance boundary condition (5). Then the simplified form of the normal force balance boundary condition becomes $p_s - p_0 = p_{ex}$ (after neglecting the curvature effect). Now we neglect an additional term which is the disjoining pressure term p_{ex}. One need to keep in mind that here we are forcefully neglecting it to simplify the algebra with an understanding that this is negligible to a large extent. However, this may not be valid under certain research problems. At the finishing stage of the droplet spreading process when the droplet has its footprint on the surface, then the film thickness becomes so thin that the Van der Waals forces can actually become very important. So, not at the end stage but almost at the ending of the spreading process the effect of disjoining pressure is important. In the present problem we are neglecting the effect of disjoining pressure and the normal force balance boundary condition becomes $p_s = p_0$. This is the first condition to be taken into account. The second condition is that in our present problem there is no tangential gradient of the surface tension which means the term $\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma$ in the tangential force balance equation (4) becomes equal to zero. With these two considerations we will start with the *y*-momentum equation and integrate it over the film thickness. The y-momentum equation is given by $\frac{\partial p}{\partial y} = -\frac{\rho\, g\, \varepsilon^3 l_c^{\,2}}{\mu\, u_c}$. As already discussed, the term on the right hand side $\frac{\rho\, g\, \varepsilon^3 l_c^{\,2}}{\mu\, u_c}$ is of the order of 1 and we can write $\frac{\partial p}{\partial y} = -1$. If we integrate it with respect to y, then we get $p = -y + c_1(x,t)$. Now we apply the boundary condition at $y = h$, $p = p_s$. Also p_s is equal to p_0 from the normal force balance boundary condition. So we get $p_s = p_0 = -h + c_1$, or, $c_1 = p_0 + h$ and $p = p_0 + (h - y)$. Now we differentiate this expression of pressure with respect to x to get $\frac{\partial p}{\partial x} = \frac{\partial h}{\partial x}$ (since p_0 is a constant, its derivative with respect to x will be equal to zero). The expression of $\frac{\partial p}{\partial x}$ is required in the x-momentum equation (2) which reads as $0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}$ or, $\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x} = \frac{\partial h}{\partial x}$. Now we integrate this x-momentum equation twice with respect to y to get the expressions of $\frac{\partial u}{\partial y}$ and u which are given by $\frac{\partial u}{\partial y} = \frac{\partial h}{\partial x} y + c_2(x,t)$ and $u = \frac{\partial h}{\partial x}\frac{y^2}{2} + c_2\, y + c_3$ respectively. Regarding the boundary conditions, we have used the considerations of the body force in the momentum equations as well as the tangential force balance and the normal force balance boundary condition. But we have not used the no-slip condition. Here, in the x-momentum equation, we use the two following boundary conditions: at $y = 0$, $u = 0$ and at $y = h$, $\frac{\partial u}{\partial y} = 0$. Using the boundary condition at $y = 0$, $u = 0$ we get $c_3 = 0$. Using the boundary condition at $y = h$, $\frac{\partial u}{\partial y} = 0$ we get $c_2 = -\frac{\partial h}{\partial x} h$. So the expression of the velocity distribution is given by $u = \frac{\partial h}{\partial x}\left(\frac{y^2}{2} - y\, h\right)$ (this is the velocity profile for the present problem). Our next step is to integrate the continuity equation. We integrate the continuity equation and then we use the Leibnitz rule. We are not going into the detail of the Leibnitz rule since it was already discussed earlier. Using the Leibnitz rule along with the kinematic boundary condition we get $\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left[\int_0^h u\,dy\right]=0$. We derived this expression in previous chapters. Substituting the expression of the velocity $u=\frac{\partial h}{\partial x}\left(\frac{y^2}{2}-y\,h\right)$ in the equation $\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left[\int_0^h u\,dy\right]=0$ we get

$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left[\int_0^h \frac{\partial h}{\partial x}\left(\frac{y^2}{2}-y\,h\right)dy\right]=0$. The integral $\int_0^h \frac{\partial h}{\partial x}\left(\frac{y^2}{2}-y\,h\right)dy$ becomes $\int_0^h \frac{\partial h}{\partial x}\left(\frac{y^2}{2}-y\,h\right)dy=\frac{\partial h}{\partial x}\left(\frac{h^3}{6}-\frac{h^3}{2}\right)=-\frac{\partial h}{\partial x}\frac{h^3}{3}$. After this substitution, the governing equation $\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left[\int_0^h u\,dy\right]=0$ becomes $\frac{\partial h}{\partial t}-\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right)=0$.

The equation $\frac{\partial h}{\partial t}-\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right)=0$ is a partial differential equation (PDE) but it can be transformed into an ordinary differential equation (ODE) by virtue of a transformation called as stretching transformation. Such a stretching transformation is available or is permissible if the method or the physics of the problem is self-similar with respect to position *x* and time *t*. In our present problem we define $h=t^{\alpha} f(\eta)$ where η is defined as $\eta=\frac{a\,x}{t^{\beta}}$. So we define the similarity variable η by combining *x* and *t* because at a given value of *x*, the height *h* is dependent on *t*. So if we freeze *x* and *t* together, we can come up with a unique similarity variable that can include the solution. The next part is the algebraic part where the objective of the aforesaid transformation is to convert the PDE into an ODE. If it cannot convert, then similarity solution does not exist. Now using the expression $h=t^{\alpha} f(\eta)$ we substitute it in the equation $\frac{\partial h}{\partial t}-\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right)=0$. The first term $\frac{\partial h}{\partial t}$ becomes $\frac{\partial h}{\partial t}=\alpha\,t^{\alpha-1} f+t^{\alpha} f'\frac{a\,x(-\beta)}{t^{\beta+1}}$ where t^{α} is a direct function of *t* and $f(\eta)$ is a function of *t* implicitly because η is a function of *t*. Here, the derivative of $f(\eta)$ with respect to *t* is chosen as $\frac{\partial}{\partial t}\left[f(\eta)\right]=\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial t}$. In the expression

$\frac{\partial h}{\partial t} = \alpha t^{\alpha-1} f + t^{\alpha} f' \frac{a\,x(-\beta)}{t^{\beta+1}}$, one can use the definition of $\eta = \frac{a\,x}{t^{\beta}}$ in the last term and the expression of $\frac{\partial h}{\partial t}$ can be rewritten as $\frac{\partial h}{\partial t} = t^{\alpha-1}\left[\alpha\, f - \beta\eta\, f'\right]$. So we have calculated the first term of the equation $\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right) = 0$. Now we will calculate the second term of this equation which is $\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right)$. This calculation is more straightforward as compared to the previous one because $f(\eta)$ is only function of *x*. So, $\frac{\partial h}{\partial x} = t^{\alpha} f' \frac{a}{t^{\beta}} = t^{\alpha-\beta} a\, f'$, $\frac{h^3}{3}\frac{\partial h}{\partial x} = \frac{t^{3\alpha}}{3} f^3 t^{\alpha-\beta} a\, f'$ and $\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right) = \frac{d}{d\eta}\left[f^3\, f'\right]\frac{a^2}{3} t^{4\alpha-2\beta}$ (here $\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right)$ is rewritten as $\frac{d}{d\eta}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right)\frac{d\eta}{dx}$ where $\frac{d\eta}{dx} = \frac{d}{dx}\left(\frac{a\,x}{t^{\beta}}\right) = a\,t^{-\beta}$ is substituted). Substituting the expressions of $\frac{\partial h}{\partial t}$ and $\frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right)$, the equation $\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right) = 0$ boils down to the form $t^{\alpha-1}\left[\alpha\, f - \beta\eta\, f'\right] = t^{4\alpha-2\beta}\frac{a^2}{3}\frac{d}{d\eta}\left[f^3\, f'\right]$.

Now the most important mathematical step to convert this equation into an ODE is that we must have only the dependent variable f as a function of η. There should not be any presence of *t*. So, if there is no presence of *t*, $\alpha - 1$ must be equal to $4\alpha - 2\beta$ but it does not tell us any information about the parameters α and β. We have not used another very important physical constraint; whenever the droplet is spreading in the absence of evaporation the volume of the droplet is conserved. So the constraint is the volume conservation. If we are modeling droplet evaporation then we can put a modified volume constraint by taking into account the evaporation flux otherwise the volume conservation can be considered. The volume conservation constraint is given by $\int_0^{L(t)} h(x,t)\,dx = \text{constant} = c$ where $L(t)$ is the length of the footprint of the droplet which is a function of time *t*. We can scale the constant c arbitrarily. We substitute the expression of $h = t^{\alpha} f(\eta)$ in the integral and we get $\int_0^{L(t)} t^{\alpha} f(\eta)\, dx = c$. Also from the expression $\eta = \frac{a\,x}{t^{\beta}}$ we get $dx = \frac{1}{a} t^{\beta} d\eta$ and the integral becomes $\int_0^{\eta_0} t^{\alpha} f(\eta) \frac{1}{a} t^{\beta} d\eta = c$ (here we have changed the limit from $L(t)$ to η_0 because at $x = L(t)$, the corresponding value of η is equal to η_0). Now the integral $\int_0^{\eta_0} t^{\alpha+\beta} f(\eta) \frac{1}{a} d\eta = c$ has to be independent of *t* because the volume has to be independent of *t*. It means $\alpha + \beta$ must be equal to zero. As a consequence, the integral becomes $\int_0^{\eta_0} f\, d\eta = a\,c$. Since these are scaled similarity variables we can normalize this and choose c to be equal to 1. $\alpha + \beta = 0$ means we get $\alpha = -\beta$. We have also found earlier that $\alpha - 1 = 4\alpha - 2\beta$. Now substituting $\alpha = -\beta$ here we get $\alpha - 1 = 4\alpha + 2\alpha$, or, $\alpha = -\frac{1}{5}$. If α is equal to $-\frac{1}{5}$, from the expression $h = t^{\alpha} f(\eta)$ we can clearly see the dependence of h as a function of time *t*. Although we have got an information of one dependence we have not yet obtained the solution of the equation $\frac{\partial h}{\partial t} - \frac{\partial}{\partial x}\left(\frac{h^3}{3}\frac{\partial h}{\partial x}\right) = 0$. Substituting $\alpha = -\frac{1}{5}$ in the equation $t^{\alpha-1}\left[\alpha\, f - \beta \eta\, f'\right] = t^{4\alpha-2\beta} \frac{a^2}{3} \frac{d}{d\eta}\left[f^3\, f'\right]$ we get $-\frac{1}{5}\left[\, f + \eta\, f'\right] = \frac{a^2}{3}\frac{d}{d\eta}\left[f^3\, f'\right]$. Then we can choose $a^2 = \frac{3}{5}$ and the resulting equation gets simplified significantly in the form $-\left[\, f + \eta\, f'\right] = \frac{d}{d\eta}\left[f^3\, f'\right]$. All these are scaling coefficients and therefore, the values of these coefficients can be chosen arbitrarily. The left hand side of this equation $\left[\, f + \eta\, f'\right]$ can be rewritten as $\left[\, f + \eta\, f'\right] = \frac{d}{d\eta}\left[f\,\eta\right]$ and the equation becomes $\frac{d}{d\eta}\left[f\,\eta\right] = -\frac{d}{d\eta}\left[f^3\, f'\right]$. Now if we integrate both sides with respect to η, we get $f\,\eta = -f^3\, f' + k$ where *k* is an integration constant. We need to apply the boundary condition to determine the constant *k*. At $x = 0$ we have $\frac{\partial h}{\partial x} = 0$ because the droplet is symmetric with respect to the *y*-axis. It means that at $\eta = 0$,

$f' = 0$ which means that k is equal to zero and we get $f^3 \frac{df}{d\eta} = -f\,\eta$, or, $f^2\, df = -\eta\, d\eta$.

Integrating both sides we get $\frac{f^3}{3} = -\frac{\eta^2}{2} + k_2$. So there is a requirement of the next boundary condition for the evaluation of k_2. At $\eta = \eta_0$ which is the footprint, the height is equal to zero which also means that f is equal to zero. Using the condition $f = 0$ at $\eta = \eta_0$ we get $k_2 = \frac{\eta_0^2}{2}$. So we get the solution in the form $\frac{f^3}{3} = \frac{\eta_0^{\,2} - \eta^2}{2}$. But still we do not know about η_0 which now can be obtained using the constraint $\int_0^{\eta_0} f\, d\eta = a$.

From the expression $\frac{f^3}{3} = \frac{\eta_0^{\,2} - \eta^2}{2}$ we get $f = \left[\frac{\eta_0^{\,2} - \eta^2}{2} \cdot 3 \right]^{1/3}$ and we substitute it in the integral $\int_0^{\eta_0} f\, d\eta = a$, i.e. $\int_0^{\eta_0} \left[\frac{\eta_0^{\,2} - \eta^2}{2} \cdot 3 \right]^{1/3} d\eta = \sqrt{\frac{3}{5}}$. The integral on the left hand side cannot be evaluated analytically but we can evaluate it numerically. Here we have to apply a trick; η can be replaced as $\eta = \eta_0 \sin\theta$ and then we can get the value of η_0 upon the numerical integration since all other remaining parameters are known. From the numerical integration, we get the value of η_0 to be equal to 2.23, i.e. $\eta_0 = 2.23$. Using the expression $\eta = \frac{a\,x}{t^\beta}$, we get $\eta_0 = \frac{a\,L(t)}{t^\beta}$ where $\beta = \frac{1}{5}$ (since η is equal to η_0 at $x = L(t)$). So one we know the value of η_0, we also know the length of the droplet $L(t)$ from the expression $\eta = \frac{a\,x}{t^\beta}$ because the value of the β is already known and we have chosen the value of a as $a = \sqrt{\frac{3}{5}}$. So this gives the final solution of how the length of the droplet will evolve as a function of time t which was given an initial length.

In this way we have come to an end of the topic thin film dynamics. The discussion of the last example on the thin film dynamics is quite comprehensive which gives us a general idea of how to solve the most general problems. All additional field effects like electric field, magnetic field, changing boundary condition from no-slip to slip boundary condition (these scenarios typically appear in the research problems) can be incorporated using the present common theoretical framework.