Advanced Concepts In Fluid Mechanics
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Lecture - 45
Thin Film Dynamics (Contd.)

In this chapter we will continue with the dimensionless forms of the governing equations and the boundary conditions with which we concluded in the previous chapter. The summary of these governing equations and the boundary conditions are given below for ease of understanding.

Governing equations:

Continuity equation: $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (1)$$

x-momentum equation: $$0=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x' \qquad (2)$$

y-momentum equation: $$0=\frac{\partial p}{\partial y}-\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y' \qquad (3)$$

Boundary conditions:

Kinematic boundary condition: $$v_i=\left(\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}\right)_i \quad \text{('}i\text{' indicates interface)} \qquad (4)$$

Tangential force balance: $$\frac{\partial u}{\partial y}=\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma \qquad (5)$$

Normal force balance: $$p_s-p_0=-\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}+p_{ex} \qquad (6)$$

There are certain important facts which need to be highlighted first. One important fact is that we do not have any idea about the characteristic velocity scale u_c. If we do not have any idea about u_c, then the reason is the ignorance about the physics of the problem. Since we have all the governing equations, if we know what is governing the physics then we can clearly pinpoint the characteristic velocity scale u_c. The procedure to indentify the characteristic velocity scale u_c is illustrated below.

When we look into the equations (1)-(6), there are certain special features in the governing equations and the boundary conditions. There can be several possibilities depending on which the characteristic velocity scale u_c can be decided. One possibility is the body force f_x' dominating in the *x*-momentum equation; other possibility is the body force f_y' dominating in the *y*-momentum equation. Besides there can be other possibilities; the surface tension gradient $\tilde{\nabla}\sigma$ can dominate the physics while the Laplace pressure (denoted by the term $\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}$) can also dominate the physics. Additionally, there is another possibility in which the Van der Waals force r the disjoining pressure (denoted by the term p_{ex}) is the dominating physics. The characteristic velocity scale u_c depends on what is the dominating physics. Now we write all these possibilities in a very structured way. We will not get a possibility outside these even in a research problem.

Possibilities:

1) *x*-momentum body force dominating:

For this case we look into the *x*-momentum equation which is given by $0=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x'$. The term $\frac{\partial p}{\partial x}$ is of the order of 1 because that is how we have chosen our scales. The term $\frac{\partial^2 u}{\partial y^2}$ is also of the order of 1. So, if the body force in the *x*-momentum equation is dominating, then, the term $\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x'$ is also of the order of 1. This means $\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x' \sim 1$, or, $u_c \sim \frac{\varepsilon^2 l_c^{\,2}}{\mu}f_x'$. So, if the body force in the *x*-momentum equation is dominating, the characteristic velocity scale u_c is of the order of $\frac{\varepsilon^2 l_c^{\,2}}{\mu}f_x'$. This is our decision about the physics of the problem which mathematics cannot tell. Out of all the forcing parameters, we have to make a judgment from the physics of the problem that what is the dominating physics. So, for the body force dominating in the *x*-direction, $u_c \sim \frac{\varepsilon^2 l_c^2}{\mu} f_x'$.

2) *y*-momentum body force dominating:

If the body force in the *y*-momentum equation is dominating, by looking into the *y*-momentum equation $0 = \frac{\partial p}{\partial y} - \frac{\varepsilon^3 l_c^2}{\mu u_c} f_y'$, we can see that the term $\frac{\partial p}{\partial y}$ is of the order of 1. Therefore, the term $\frac{\varepsilon^3 l_c^2}{\mu u_c} f_y'$ has to be of the order of 1 from which we get $\frac{\varepsilon^3 l_c^2}{\mu u_c} f_y' \sim 1$, or, $u_c \sim \frac{\varepsilon^3 l_c^2}{\mu} f_y'$. So, for the body force dominating in the *y*-direction, $u_c \sim \frac{\varepsilon^3 l_c^2}{\mu} f_y'$.

One should not get disturbed by looking into the order ε^2 and ε^3 terms and presuming that the value of the characteristic velocity scale u_c is less because of these terms. There are other multipliers like l_c, μ which we do not know how small or how big they are. We will be absolutely confident about neglecting the order ε^2 and ε^3 terms if the multiplier is of the order of 1 or smaller. But here multiplier contains property (μ), length scale (l_c) and we do not have any idea about these parameters. Therefore it is not legitimate to trivially rule out order ε^2 and order ε^3 terms.

3) Surface tension gradient dominating:

The case when the gradient of the surface tension is dominating is a typical situation where the physics of the flow is governed by the surface tension gradient. For this case, we need to look into the tangential force balance boundary condition which reads as $\frac{\partial u}{\partial y} = \frac{\varepsilon}{Ca} \tilde{\nabla} \sigma$. The term $\frac{\partial u}{\partial y}$ is of the order of 1 while the term $\tilde{\nabla} \sigma$ is also of the order of 1. To make the gradient of surface tension as the dominating physics, the term $\frac{\varepsilon}{Ca}$ has to be of the order of 1, i.e. $\frac{\varepsilon}{Ca} \sim 1$. Using the definition of Capillary number (*Ca*)

$Ca = \frac{\mu u_c}{\sigma_0}$, we get $\frac{\varepsilon \sigma_0}{\mu u_c} \sim 1$, or, $u_c \sim \frac{\varepsilon \sigma_0}{\mu}$. We have two more cases left which are discussed in the following.

4) Laplace pressure dominating:

For this case one need to look into the normal force balance boundary condition which reads as $p_s - p_0 = -\frac{\varepsilon^3}{Ca}\sigma \frac{\partial^2 h}{\partial x^2} + p_{ex}$. The Laplace pressure is the pressure difference due to the curvature of the interface. If the Laplace pressure is the dominating factor, then, of course, the disjoining pressure (p_{ex}) is not dominating. In the expression $p_s - p_0 = -\frac{\varepsilon^3}{Ca}\sigma \frac{\partial^2 h}{\partial x^2} + p_{ex}$, the term $(p_s - p_0)$ is of the order of 1 and the term $\sigma \frac{\partial^2 h}{\partial x^2}$ is also of the order of 1. For Laplace pressure to be the dominating factor, the term $\frac{\varepsilon^3}{Ca}\sigma \frac{\partial^2 h}{\partial x^2}$ has to be of the order of 1 which means $\frac{\varepsilon^3}{Ca}$ will be of the order of 1, i.e. $\frac{\varepsilon^3}{Ca} \sim 1$. Using $Ca = \frac{\mu u_c}{\sigma_0}$, we get, $\frac{\varepsilon^3 \sigma_0}{\mu u_c} \sim 1$ or, $u_c \sim \frac{\varepsilon^3 \sigma_0}{\mu}$.

5) Disjoining pressure dominating:

The final possibility is the scenario when the disjoining pressure becomes the dominating factor. If the disjoining pressure (p_{ex}) in the normal force balance boundary condition $p_s - p_0 = -\frac{\varepsilon^3}{Ca}\sigma \frac{\partial^2 h}{\partial x^2} + p_{ex}$ is dominating, then p_{ex} is of the order of 1. We can also write p_{ex} in terms of its dimensional form as $p_{ex} = \frac{p_{ex}'}{p_c}$ where p_{ex}' is the dimensional form of the disjoining pressure and p_c is the characteristic scale of pressure. So, the term $\frac{p_{ex}'}{p_c}$ will be of the order of 1. We use the expression of $p_c = \frac{\mu\, u_c}{\varepsilon^2\, l_c}$ which means that $\frac{p_{ex}' \varepsilon^2\, l_c}{\mu\, u_c}$ will be of the order of 1, i.e. $\frac{p_{ex}' \varepsilon^2\, l_c}{\mu\, u_c} \sim 1$, or, $u_c \sim \frac{p_{ex}' \varepsilon^2\, l_c}{\mu}$.

Very interestingly, for all these possibilities, irrespective of whichever is dominating the physics one can see that in the expression of the characteristic velocity scale u_c there is always either order ε, order ε^2 or order ε^3 terms. So, for an untrained eye, it makes an elusive appearance that these terms are negligible since ε is a small quantity. However in reality it is not true because, although ε is small, we do not have any control over the

terms like l_c, μ etc. These multipliers can make the terms effective despite the presence of ε (or ε^2, ε^3).

Therefore, to solve a problem, we have to decide that out of these five cases which one will be relevant for our specific problem. We cannot just train this thing by mathematics but it should come from our physical understanding of the problem. Of course we will work out a problem later on but at this stage we should keep in mind that we cannot really go ahead in solving a problem until and unless we pinpoint the dominating force that governs the physics of the problem. Now there can be a critical situation when more than one force or even all five forces are of equal strengths. Under such circumstances, if more than one force is of equal strength, we can choose any one of them to ascertain the characteristic velocity scale. If two or more forces are of the same order, we can decide the velocity scale based on one of the forces and it will automatically take care of the other forces as well.

Overall, we have not really advanced in the context of solving the thin film problem. We do not want to just represent the governing equations (1)-(6). We need to remember the physical problem with which we started. This problem consists of a surface on which there is an undulated thin film of liquid. Our objective is to see that how the film thickness $h(x,t)$ evolves as a function of *x* and *t*. In order to know the film thickness $h(x,t)$ as a function of *x* and *t*, we have to solve the governing equations along with the boundary conditions described by equations (1)-(6). Now we will proceed towards the solution strategy for the thin film problem.

Solution strategy for the thin film problem:

We will start with the *y*-momentum equation which reads as $0=\frac{\partial p}{\partial y}-\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y{}'$. Since it is the simplest governing equation, it is easy to deal with. Integrating this *y*-momentum equation $0=\frac{\partial p}{\partial y}-\frac{\varepsilon^3 l_c^{\ 2}}{\mu u_c} f_y'$ with respect to *y*, we get $p=\frac{\varepsilon^3 l_c^{\ 2}}{\mu u_c} f_y' y+\text{constant}$. Since we have the partial derivative $\frac{\partial p}{\partial y}$ with respect to *y*, the integration constant will be a function of *x* and *t*, we define this constant as $C_1(x,t)$. So, the pressure distribution (*p*) is given by $p=\frac{\varepsilon^3 l_c^{\ 2}}{\mu u_c} f_y' y+C_1(x,t)$. So we have three variables *x*, *y* and *t* respectively. To evaluate $C_1(x,t)$ we need to apply the boundary condition. At $y=h(x,t)$, the pressure p is equal to p_s which is the pressure in the liquid side of the free surface. Using this boundary condition we get $p_s=\frac{\varepsilon^3 l_c^{\ 2}}{\mu u_c} f_y' h+C_1(x,t)$ or, $C_1(x,t)=p_s-\frac{\varepsilon^3 l_c^{\ 2}}{\mu u_c} f_y' h$. Substituting this expression of $C_1(x,t)$ in the pressure distribution we get the final form of the pressure distribution

$$p=\frac{\varepsilon^3 l_c^{\ 2}}{\mu u_c} f_y'(y-h)+p_s \tag{7}$$

So we have taken care of the *y*-momentum equation part through finding the pressure distribution *p*. Before looking into the *x*-momentum equation, we use the normal force balance boundary condition $p_s-p_0=-\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}+p_{ex}$ to get an expression for p_s (the pressure at the surface) which is given by $p_s=p_0-\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}+p_{ex}$. Substituting this expression of p_s in equation (7), we get $p=\frac{\varepsilon^3 l_c^{\ 2}}{\mu u_c} f_y'(y-h)+p_0-\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}+p_{ex}$ which is the final expression of the pressure distribution.

Now we will discuss something which is very interesting. From the beginning of this topic we are trying to draw an analogy between the thin film theory and the boundary layer theory. We have seen that there is a remarkable difference in the physics of these two theories. In case of boundary layer theory, we are normally working with high Reynolds number flow. But in case of thin film theory, we are normally working with low Reynolds number flow. However, there is a remarkable similarity which is the separation of the length scales. There is a length scale in the transverse direction which is very less as compared to the length scale in the longitudinal direction and it is the scale ratio (the small quantity) which matters in our calculations. Now the question arises that if the two theories are so similar then whether there is any other striking dissimilarity between the two theories or not. The striking dissimilarity is the implication of the *y*-momentum equation. In the boundary layer theory, the *y*-momentum equation has its implication that the pressure gradient along the *y*-direction is equal to zero, i.e. $\frac{\partial p}{\partial y}=0$. It means that within the boundary layer, there is no pressure gradient across the boundary layer and all the pressure gradient is imposed from the free stream outside the boundary layer. But in the thin film theory, the pressure gradient within the film is very important. The reason is the pressure variation $p=\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y{}'\left(y-h\right)+p_0-\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}+p_{ex}$ within the film is a strong function of *h* and that dictates the evolution of the thin film. This expression of the pressure distribution is now required in the *x*-momentum equation (2) which reads as $0=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x{}'$. So we need to evaluate $\frac{\partial p}{\partial x}$ which is given by $\frac{\partial p}{\partial x}=-\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y{}'\frac{\partial h}{\partial x}+\frac{\partial p_s}{\partial x}$. We define a function $A\left(x,t\right)$ (i.e. function of both *x* and *t*) to be equal to $-\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y{}'\frac{\partial h}{\partial x}+\frac{\partial p_s}{\partial x}$, so, $\frac{\partial p}{\partial x}=-\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y{}'\frac{\partial h}{\partial x}+\frac{\partial p_s}{\partial x}=A\left(x,t\right)$. The reason of writing A as a function of *x* and *t* is that with respect to the *y*-derivative it can be treated as a constant. Using this form of $A\left(x,t\right)$ in the *x*-momentum equation we get $0=-A\left(x,t\right)+\frac{\partial^2 u}{\partial y^2}+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x{}'$, or, $\frac{\partial^2 u}{\partial y^2}=A\left(x,t\right)+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x{}'$. If we integrate the equation $\frac{\partial^2 u}{\partial y^2}=A\left(x,t\right)+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x{}'$ with respect to *y*, we get $\frac{\partial u}{\partial y}=A\left(x,t\right)y+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x{}' y+C_2\left(x,t\right)$. To evaluate $C_2\left(x,t\right)$, we will use the tangential force balance boundary condition (5) which reads as $\frac{\partial u}{\partial y}=\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma$. We have to keep in mind that all the three boundary conditions, namely tangential force balance; normal force balance and kinematic boundary condition are applicable at the interface, i.e. at $y=h\left(x,t\right)$. Since we are doing a general formulation big expressions/terms are appearing but if we solve a specific problem then many of these terms will not be there. So, using the tangential force balance boundary condition $\frac{\partial u}{\partial y}=\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma$ at $y=h(x,t)$ we get $\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma=Ah+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x'h+C_2$ from which an expression of C_2 can be obtained as

$C_2=\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma-Ah-\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x'h$. We can now use this expression of C_2 in the equation $\frac{\partial u}{\partial y}=Ay+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x'y+C_2$. Now, integrating this equation with respect to y we get the velocity distribution $u=\left(A+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x'\right)\frac{y^2}{2}+C_2y+C_3(x,t)$ where $C_3(x,t)$ is the integration constant. Now $C_3(x,t)$ can be easily calculated by using the no-slip boundary condition at $y=0$. If we have a slip boundary condition (in this context one can think of a microfluidic or a nanofluidic problem with slip), then we have to simply introduce the slip boundary condition at $y=0$ to accommodate the slip at the wall.

We have not written this no-slip boundary condition at $y=0$ earlier. So, this boundary condition is written as at $y=0$, $u=0$. Additionally, at $y=0$, we also have $v=0$ which is known as the no-penetration boundary condition. The earlier boundary conditions (i.e. the tangential force balance; normal force balance and kinematic boundary condition) are at the fluid-gas interface, so there should also be boundary condition at the wall. The no-slip and no-penetration boundary conditions are applicable at the wall. Since these boundary conditions are very obvious we have not written these conditions earlier. Despite being very obvious, sometimes the no-slip boundary condition is violated. However, the no-penetration boundary condition will never be violated.

So, the boundary conditions at $y=0$, $u=0$ and $v=0$ should be included in the set of boundary conditions that was written earlier (such that it will become a complete set of boundary conditions). Using the condition $u=0$ at $y=0$ we get $C_3=0$. We have got the final form of the velocity distribution $u=\left(A+\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x'\right)\frac{y^2}{2}+C_2y$ with $C_2=\frac{\varepsilon}{Ca}\tilde{\nabla}\sigma-Ah-\frac{\varepsilon^2 l_c^{\,2}}{\mu u_c}f_x'h$ and $A(x,t)=-\frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}f_y'\frac{\partial h}{\partial x}+\frac{\partial p_s}{\partial x}$. So we have overcome one of the major challenges, i.e. we have got the velocity profile in the film. We have to keep a note about the expressions of the two terms A and C_2 since they are required in the final expression of the velocity distribution. In the expression of A, there is another parameter p_s which is defined as $p_s = p_0 - \frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2} + p_{ex}$. This completes the description of u as a function of x and t. But it does not tell us about how h evolves as a function of x and t. So we have to somehow use the remaining resources. Therefore question arises about the remaining resources. We have not used the continuity equation and we have not used the kinematic boundary condition. Kinematic boundary condition needs to be applied at the interface as well as at the solid boundary which is known as the no-penetration boundary condition. So we have not used these boundary conditions and the continuity equation till now. In the next chapter we will use these conditions along with the derived velocity profile to get the governing equation for the film thickness.