Advanced Concepts In Fluid Mechanics
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Lecture – 44
Thin Film Dynamics (Contd.)

In the previous chapter, we have discussed about the tangential force balance. In the present chapter, we will focus on the normal force balance. We have already discussed about the normal force balance but here our objective is to write the normal force balance in a dimensionless form.

To do this, we first write the normal force balance in the dimensional form which is given by

$$p_s' - p_0' - \left(\vec{\mathrm{T}}' \cdot \hat{n}\right) = \frac{\sigma'}{R'} + p_{ex}' \tag{1}$$

First we explain the physical meanings of individual terms of this equation. Here, p_s' is the pressure at the interface, p_0' is the outside pressure; $\vec{\mathrm{T}}' \cdot \hat{n}$ is the viscous normal stress (which is only the viscous component of the stress tensor). A negative sign is given before the viscous normal stress because pressure by definition is compressive while stress by definition is tensile. $\frac{\sigma'}{R'}$ in two-dimensional system is equal to $\sigma'\left(\frac{1}{R_1'} + \frac{1}{R_2'}\right)$ and the term $\frac{1}{R_1'} + \frac{1}{R_2'}$ is replaced by $\frac{1}{R'}$ in the present scenario. p_{ex}' is the disjoining pressure due to Van der Waals forces of interaction if the film thickness becomes very less. Now we will write the expressions of the individual terms of equation (1) out of which the major activity will be to expand the viscous normal stress term $\vec{\mathrm{T}}' \cdot \hat{n}$. In the previous chapter we have written the expression of $\vec{\mathrm{T}}'$ which is given by $\vec{\mathrm{T}}' = \begin{bmatrix} \tau_{xx}' & \tau_{xy}' \\ \tau_{xy}' & \tau_{yy}' \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix} = \left[\tau_{xx}' n_x + \tau_{xy}' n_y\right]\hat{i} + \left[\tau_{xy}' n_x + \tau_{yy}' n_y\right]\hat{j}$. We have to use this expression of the traction vector $\vec{\mathrm{T}}'$ in the dot product of $\vec{\mathrm{T}}'$ with the unit normal vector $\hat{n}$. The unit vector $\hat{n}$ is given by $\hat{n} = n_x \hat{i} + n_y \hat{j}$ and therefore, the dot product becomes

$$\vec{\mathrm{T}}' \cdot \hat{n} = \left[\left(\tau_{xx}' n_x + \tau_{xy}' n_y\right)\hat{i} + \left(\tau_{xy}' n_x + \tau_{yy}' n_y\right)\hat{j}\right] \cdot \left(n_x \hat{i} + n_y \hat{j}\right) = \tau_{xx}' n_x^2 + 2\tau_{xy}' n_x n_y + \tau_{yy}' n_y^2$$

. We substitute the expressions of $\tau_{xx}{}' = 2\mu \frac{\partial u'}{\partial x'}$, $\tau_{xy}{}' = \mu\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right)$ and $\tau_{yy}{}' = 2\mu \frac{\partial v'}{\partial y'}$ in the expression of $\vec{\mathrm{T}}' \cdot \hat{n}$ and we get $\vec{\mathrm{T}}' \cdot \hat{n} = 2\mu \frac{\partial u'}{\partial x'} n_x{}^2 + 2\mu\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right) n_x n_y + 2\mu \frac{\partial v'}{\partial y'} n_y{}^2$. Now we write the orders of the various terms instead of further simplification because from it we will have an explicit inference. $\frac{\partial u'}{\partial x'}$ is of the order of $\frac{u_c}{l_c}$, $\frac{\partial u'}{\partial y'}$ is of the order of $\frac{u_c}{h_0}$, $\frac{\partial v'}{\partial x'}$ is of the order of $\frac{v_c}{l_c}$ and $\frac{\partial v'}{\partial y'}$ is of the order of $\frac{v_c}{h_0}$. Now we look into expression of $\hat{n}$ which is given by $\hat{n} = -\varepsilon \frac{\partial h}{\partial x}\hat{i} + \hat{j}$, so, $n_x = -\varepsilon \frac{\partial h}{\partial x}$. So, $n_x{}^2$ is of the order of ε^2. So in the term $n_x{}^2$, one ε^2 term will appear. In the term $n_x n_y$, one ε will appear. Although the other term $n_y{}^2$ is equal to 1, it is multiplied by the term v_c which itself contains one ε term because v_c is of the order of εu_c. We should replace all v_c terms by εu_c. Substituting the orders of magnitudes of the different terms we get, $\vec{\mathrm{T}}' \cdot \hat{n} = 2\mu \frac{u_c}{l_c}\frac{\partial u}{\partial x}\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2 - 2\mu\left(\frac{u_c}{h_0}\frac{\partial u}{\partial y} + \frac{\varepsilon u_c}{l_c}\frac{\partial v}{\partial x}\right)\varepsilon\frac{\partial h}{\partial x} + 2\mu\frac{\varepsilon u_c}{h_0}\frac{\partial v}{\partial y}$. Now we replace the term l_c by $l_c = \frac{h_0}{\varepsilon}$ and the modified form of the viscous normal stress $\vec{\mathrm{T}}' \cdot \hat{n}$ becomes $\vec{\mathrm{T}}' \cdot \hat{n} = 2\mu \frac{\varepsilon u_c}{h_0}\frac{\partial u}{\partial x}\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2 - 2\mu\left(\frac{u_c}{h_0}\frac{\partial u}{\partial y} + \frac{\varepsilon^2 u_c}{h_0}\frac{\partial v}{\partial x}\right)\varepsilon\frac{\partial h}{\partial x} + 2\mu\frac{\varepsilon u_c}{h_0}\frac{\partial v}{\partial y}$. The first term on the right hand side of this expression $2\mu \frac{\varepsilon u_c}{h_0}\frac{\partial u}{\partial x}\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2$ is of the order of ε^3. The second term $-2\mu\frac{u_c}{h_0}\frac{\partial u}{\partial y}\varepsilon\frac{\partial h}{\partial x}$ is of the order of ε. The third term $-2\mu\frac{\varepsilon^2 u_c}{h_0}\frac{\partial v}{\partial x}\varepsilon\frac{\partial h}{\partial x}$ is of the order of ε^3. The final term $2\mu\frac{\varepsilon u_c}{h_0}\frac{\partial v}{\partial y}$ is of the order of ε. So, overall, one can argue that all these terms are either of the order of ε or even less and therefore, in the leading order the viscous normal stress will not matter. This is a very important conclusion. So when we work out the term $\vec{\mathrm{T}}' \cdot \hat{n}$, we will get term either of the order of ε or of the order which is even smaller than ε. This makes the term $\vec{\mathrm{T}}' \cdot \hat{n}$ negligible in the leading order as compared to the terms p_s' and p_0' which are of the order of 1. Very commonly in fluid dynamics problem we can see that when people write the normal stress balance at the interface, many times people do not write the viscous normal stress at all. Irrespective of whether it is a purposeful mistake or a mistake without a purpose, eventually that mistake will not matter because at the leading order we will find that the viscous normal stress will not have a role to play for writing the boundary condition for thin film equations. This is justified by the order of magnitude analysis of the viscous normal stress term. So, form now onwards, we will not consider the term $\vec{\mathrm{T}}'\cdot\hat{n}$ in the analysis. We need to remember that when we learn a subject, it is very important to know what is fundamentally correct. The presence of the term $\vec{\mathrm{T}}'\cdot\hat{n}$ in the normal stress balance equation $p_s' - p_0' - \left(\vec{\mathrm{T}}'\cdot\hat{n}\right) = \dfrac{\sigma'}{R'} + p_{ex}'$ is fundamentally correct. But when we incorporate the necessary assumptions, we have found that the term $\vec{\mathrm{T}}'\cdot\hat{n}$ turns out to be not as dominant as compared to the other terms. So we have to keep in mind this aspect and $\vec{\mathrm{T}}'\cdot\hat{n}$ is not exactly equal to zero. With this understanding, we write the normal stress balance equation as $p_s' - p_0' = \dfrac{\sigma'}{R'} + p_{ex}'$. The only thing that needs to be calculated is the term $\dfrac{1}{R'}$ before we can proceed further. From co-ordinate geometry, $\dfrac{1}{R'}$ is nothing but equal to $\nabla'\cdot\hat{n}$, i.e. $\dfrac{1}{R'} = \nabla'\cdot\hat{n}$. So, $\dfrac{1}{R'}$ is equal to the divergence of the unit normal vector $\hat{n}$ which establishes a relationship between $\dfrac{1}{R'}$ and $\hat{n}$ (from vector analysis with co-ordinate geometry we get this in elementary mathematics course). Now we evaluate the expression $\nabla'\cdot\hat{n}$ using the definition of the divergence operator ∇' and the expression of the unit vector $\hat{n}$, i.e. $\nabla'\cdot\hat{n} = \left(\hat{i}\dfrac{\partial}{\partial x'} + \hat{j}\dfrac{\partial}{\partial y'}\right)\cdot\left(-\varepsilon\dfrac{\partial h}{\partial x}\,\hat{i} + \hat{j}\right)$. $\nabla'\cdot\hat{n} = \left(\hat{i}\dfrac{\partial}{\partial x'} + \hat{j}\dfrac{\partial}{\partial y'}\right)\cdot\left(-\varepsilon\dfrac{\partial h}{\partial x}\,\hat{i} + \hat{j}\right)$ will be equal to $\dfrac{\partial}{\partial x'}\left(-\varepsilon\dfrac{\partial h}{\partial x}\right)$ and it will not have any derivative with respect to y'. Using $x' = x\,l_c$, $\nabla'\cdot\hat{n}$ can be written as $\nabla'\cdot\hat{n} = \dfrac{\partial}{\partial x'}\left(-\varepsilon\dfrac{\partial h}{\partial x}\right) = -\dfrac{\varepsilon}{l_c}\dfrac{\partial^2 h}{\partial x^2} = -\dfrac{\varepsilon^2}{h_0}\dfrac{\partial^2 h}{\partial x^2}$.

Now we will write the dimensional stress balance equation $p_s' - p_0' = \frac{\sigma'}{R'} + p_{ex}'$ in the corresponding non-dimensional form. p_s' can be written as $p_s' = p_s\, p_c$ where p_c is the characteristic pressure which is related to the characteristic velocity u_c. p_0' can be written as $p_0' = p_0\, p_c$; σ' can be written as $\sigma' = \sigma\sigma_0$ and p_{ex}' as $p_{ex}' = p_{ex}\, p_c$. With these considerations and using the substitution of $\frac{1}{R'} = \nabla' \cdot \hat{n} = -\frac{\varepsilon^2}{h_0}\frac{\partial^2 h}{\partial x^2}$ we will be able to write the normal stress balance equation in the dimensionless form $p_s - p_0 = -\frac{\sigma\sigma_0\,\varepsilon^2}{h_0\, p_c}\frac{\partial^2 h}{\partial x^2} + p_{ex}$. For the term p_{ex}, let us consider one example in which its dimensional form p_{ex}' can be written in terms of the film thickness as $p_{ex}' = \frac{A}{6\pi h'^3}$ where A is called as the Hamaker constant. This Hamaker constant indicates the strength of the Van der Waals force. We can clearly see that as the film thickness becomes smaller and smaller, p_{ex}' becomes very large and it can dominate over all other terms. So, p_{ex} is a non-dimensional representation of p_{ex}'. Now we substitute the relationship between p_c and u_c, i.e. $p_c = \frac{\mu u_c\, l_c}{h_0^2} = \frac{\mu u_c}{\varepsilon^2 l_c}$ in the expression $p_s - p_0 = -\frac{\sigma\sigma_0\,\varepsilon^2}{h_0\, p_c}\frac{\partial^2 h}{\partial x^2} + p_{ex}$ and we get $p_s - p_0 = -\frac{\sigma\sigma_0\,\varepsilon^2\varepsilon^2 l_c}{h_0\,\mu u_c}\frac{\partial^2 h}{\partial x^2} + p_{ex}$. Now $\frac{h_0}{l_c}$ is equal to ε, so one ε term gets cancelled from the numerator and the denominator and the resulting expression of the stress balance equation becomes $p_s - p_0 = -\frac{\sigma\sigma_0\,\varepsilon^3}{\mu u_c}\frac{\partial^2 h}{\partial x^2} + p_{ex} = -\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2} + p_{ex}$. In this expression, Ca is known as the Capillary number defined as $Ca = \frac{\mu u_c}{\sigma_0}$. So the final form of the non-dimensional normal stress balance equation is given by $p_s - p_0 = -\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2} + p_{ex}$. In this context one question may arise about the presence of the ε^3 term on the right hand side of the normal stress balance equation since we have neglected earlier ε and ε^2 terms in the derivation. In the term $\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}$, although ε^3 is present in the numerator we also have another term *Ca* (i.e. the Capillary number). Therefore, the strength of the term $\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}$ will be governed by both ε^3 as well as the value of *Ca*. The value of *Ca* is not like O(1), O(10), O(10^2), O(0.1) etc. We do not have any information about the value of *Ca*. From the definition of Capillary number $Ca = \frac{\mu\, u_c}{\sigma_0}$, it shows the dependence on three parameters μ, u_c and σ_0. We do not have any information about this parameters μ, u_c and σ_0. Hence, it is better to keep the term $\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}$ (written in terms of Capillary number). If the Capillary number is very large, then $\frac{\varepsilon^3}{Ca}$ will be negligible and the term $\frac{\varepsilon^3}{Ca}\sigma\frac{\partial^2 h}{\partial x^2}$ will not be important. But in the earlier cases, the orders of magnitudes of the multipliers were known to us and therefore, we neglected the order ε and the order ε^2 terms.

So, overall, we have got all the governing equations and the boundary conditions in a non-dimensional form. We will now write a summary of all the governing equations and the boundary conditions in a non-dimensional form such that it will be easier to follow.

Summary of governing equations and boundary conditions in dimensionless form:

Continuity equation:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2)$$

x-momentum equation:
$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \frac{\varepsilon^2 l_c^{\,2}}{\mu u_c} f_x' \qquad (3)$$

y-momentum equation:
$$0 = -\frac{\partial p}{\partial y} - \frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} f_y' \qquad (4)$$

These three equations (2)-(4) are the governing equations in the dimensionless form. Now we will write the three boundary conditions.

Kinematic boundary condition: $$v_i = \left(\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} \right)_i$$ ('*i*' indicates interface) (5)

Tangential force balance: $$\frac{\partial u}{\partial y} = \frac{\varepsilon}{Ca} \tilde{\nabla} \sigma \qquad (6)$$

Normal force balance: $$p_s - p_0 = -\frac{\varepsilon^3}{Ca} \sigma \frac{\partial^2 h}{\partial x^2} + p_{ex} \qquad (7)$$

In these equations (2)-(7), we do not really care about the derivative terms because the derivative terms are already constrained to be of the order of 1. Except for the derivative terms everything else will decide the characteristic velocity scale u_c and the characteristic pressure scale p_c. We can ascertain the characteristic velocity scale by looking into the governing equations and appealing to the physics of the problem which will be discussed in the next chapter.