Advanced Concepts In Fluid Mechanics
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Lecture - 43
Thin Film Dynamics (Contd.)

In the earlier chapters we have discussed about the thin film dynamics and in the context of thin film dynamics we have discussed about the simplified governing equations in the non-dimensional form. We have also started looking into the simplified boundary conditions and the first boundary condition, i.e. the kinematic boundary condition in the non-dimensional form was discussed in the previous chapter. The next logical extension of this will be to look into the tangential force balance and the normal force balance in the non-dimensional form.

First we will write the tangential force balance in the dimensional form and then we will convert it in the non-dimensional form. The tangential force balance in the dimensional form is given by $\vec{\mathrm{T}}'\cdot\hat{s}=\left(\nabla_s{}'\sigma'\right)\cdot\hat{s}$. Now we will write an expression for the vector $\hat{s}$. To do that first we need to known about the vector $\hat{n}$. This $\hat{n}$ vector we have already written implicitly in terms of the function F'. We recall the definition of the function F' which is given by $F'=y'-h'\left(x',t'\right)$ where h' is a function of both x' and t'. Now we calculate the gradient of the function F' which results $\nabla F'=\hat{i}\dfrac{\partial F'}{\partial x'}+\hat{j}\dfrac{\partial F'}{\partial y'}$. Question may arise about why the calculation of gradient of the function F' in the context of the vectors $\hat{s}$ and $\hat{n}$. $\nabla F'$ is calculated in the context of $\hat{n}$ because by definition, the gradient of a function is a vector which is normal to the graphical representation of the function. So, if F' is a function, the gradient of F' is a vector which is normal to the function F' drawn in a suitable plane. So, $\nabla F'$ will be in the direction of the vector $\hat{n}$. The form of gradient $\nabla F'$ is rewritten as $\nabla F'=\hat{i}\dfrac{\partial F'}{\partial x'}+\hat{j}\dfrac{\partial F'}{\partial y'}=-\hat{i}\dfrac{\partial h'}{\partial x'}+\hat{j}$. Then we write the expression of $\nabla F'$ in terms of the corresponding dimensionless parameters. h' has a characteristic dimension h_0 and x' has a characteristic dimension l_c. Using this, $\nabla F'$ can be rewritten as $\nabla F'==-\hat{i}\dfrac{h_0}{l_c}\dfrac{\partial h}{\partial x}+\hat{j}$; using $\varepsilon=\dfrac{h_0}{l_c}$, $\nabla F'$ becomes

$\nabla F' = -\varepsilon \frac{\partial h}{\partial x}\hat{i} + \hat{j}$. Using this express of $\nabla F'$, one can obtain the expression of the unit normal vector $\hat{n}$. By definite, the unit normal vector $\hat{n}$ is related to $\nabla F'$ as $\hat{n} = \frac{\nabla F'}{|\nabla F'|}$.

So, $\hat{n}$ is the unit vector in the direction of the gradient $\nabla F'$ which is also in the direction normal to the interface. The function F' is along the interface, so the gradient of the function $\nabla F'$ is normal to the interface and therefore, $\hat{n}$ is normal to the interface. Using the expression $\nabla F' = -\varepsilon \frac{\partial h}{\partial x}\hat{i} + \hat{j}$, one can tell that that the modulus $|\nabla F'|$ will be equal to $\sqrt{\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2 + 1}$. Using this, the expression of $\hat{n}$ becomes $\hat{n} = \frac{\nabla F'}{|\nabla F'|} = \frac{-\varepsilon \frac{\partial h}{\partial x}\hat{i} + \hat{j}}{\sqrt{\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2 + 1}}$.

Clearly, at the leading order, the denominator will be equal to 0. We are writing everything in terms of the leading order, so, at the leading order, $\hat{n} = \frac{-\varepsilon \frac{\partial h}{\partial x}\hat{i} + \hat{j}}{\sqrt{\varepsilon^2\left(\frac{\partial h}{\partial x}\right)^2 + 1}}$ is as good as $\hat{n} = -\varepsilon \frac{\partial h}{\partial x}\hat{i} + \hat{j}$.

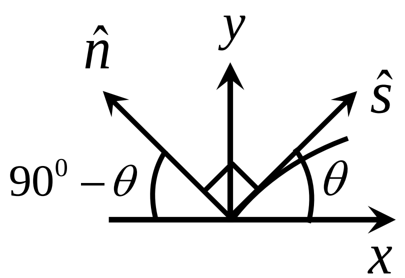

Figure 1 shows a curve where the unit vector $\hat{s}$ is acting along the tangent to the curve. The unit vector $\hat{n}$ is acting normal to the curve. The angle between $\hat{s}$ and $\hat{n}$ is 90^0.

Now to obtain the unit vector $\hat{s}$, we draw a curve as shown in figure 1. The unit vector $\hat{s}$ is acting along the tangent to the curve and the unit vector $\hat{n}$ is acting normal to the curve. The angle between $\hat{s}$ and $\hat{n}$ is equal to 90^0. Since $\hat{s}$ is a unit vector, its length is equal to 1. The corresponding *x*-direction and *y*-directions are shown in the figure. θ is the angle between the vector $\hat{s}$ and the *x*-direction while the angle between the negative

x-direction and the vector $\hat{n}$ is equal to $90^0 - \theta$. Resolving the unit vector $\hat{s}$ along the x-direction and y-direction we can write $\hat{s} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$. In the similar way, the unit vector $\hat{n}$ can be resolved as $\hat{n} = -\cos\left(90^0 - \theta\right)\hat{i} + \sin\left(90^0 - \theta\right)\hat{j} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$. Comparing this form of $\hat{n} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ with the form $\hat{n} = -\varepsilon\dfrac{\partial h}{\partial x}\,\hat{i} + \hat{j}$, we get $\sin\theta = \varepsilon\dfrac{\partial h}{\partial x}$ and $\cos\theta = 1$. So the unit vector $\hat{s}$ is given by $\hat{s} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j} = \hat{i} + \varepsilon\dfrac{\partial h}{\partial x}\,\hat{j}$. So, after some exercise, we have obtained the expressions of the unit vectors $\hat{s}$ and $\hat{n}$. This is very important because the knowledge of $\hat{s}$ and $\hat{n}$ is necessary in evaluating the dot products in the expression $\vec{\mathrm{T}}' \cdot \hat{s} = \left(\nabla_s{}'\sigma'\right) \cdot \hat{s}$. Our next endeavor is to calculate these dot products. To know about the traction vector $\vec{\mathrm{T}}'$, we need to remember the Cauchy's theorem, i.e. $\mathrm{T}_i^n = \tau_{ij}\, n_j$. We will apply the Cauchy's theorem; but in a two-dimensional framework. Apply Cauchy's theorem we get, $\vec{\mathrm{T}}' = \begin{bmatrix} \tau_{xx}{}' & \tau_{xy}{}' \\ \tau_{xy}{}' & \tau_{yy}{}' \end{bmatrix} \begin{bmatrix} n_x \\ n_y \end{bmatrix}$ where $n_x = -\sin\theta$ and $n_y = \cos\theta$. So, the product (i.e. the matrix multiplication) will be a vector with two components, i.e. $\vec{\mathrm{T}}' = \left[\tau_{xx}{}'\, n_x + \tau_{xy}{}'\, n_y\right]\hat{i} + \left[\tau_{xy}{}'\, n_x + \tau_{yy}{}'\, n_y\right]\hat{j}$ where $\tau_{xx}{}'\, n_x + \tau_{xy}{}'\, n_y$ is the x-component of the vector and $\tau_{xy}{}'\, n_x + \tau_{yy}{}'\, n_y$ is the y-component of the vector. Before just blindly writing individual terms we can clearly assess the terms and compare the relative strengths of the terms. The expressions of $\tau_{xx}{}'$, $\tau_{xy}{}'$ and $\tau_{yy}{}'$ are given by $\tau_{xx}{}' = 2\mu\dfrac{\partial u'}{\partial x'}$, $\tau_{xy}{}' = \mu\left(\dfrac{\partial u'}{\partial y'} + \dfrac{\partial v'}{\partial x'}\right)$ and $\tau_{yy}{}' = 2\mu\dfrac{\partial v'}{\partial y'}$. In the expression of $\vec{\mathrm{T}}'$, $\tau_{xx}{}'$ is multiplied by the term n_x. Now, n_x by definition is equal to $n_x = -\sin\theta = -\varepsilon\dfrac{\partial h}{\partial x}$ while n_y is equal to 1. So, if we multiply $\tau_{xx}{}'$ with n_x, the term $-\varepsilon\dfrac{\partial h}{\partial x}$ is already of the order of $\sim\varepsilon$. But we cannot conclude about the term $\tau_{xx}{}'$ because to conclude about this, we have to transfer the dimensional form into non-dimensional form. Then we can see the order of this term.

Using the order of magnitude analysis, we can write $\tau_{xx}{}' = 2\frac{\mu u_c}{l_c}\frac{\partial u}{\partial x}$, $\tau_{xy}{}' = \mu\left(\frac{u_c}{h_0}\frac{\partial u}{\partial y} + \frac{v_c}{l_c}\frac{\partial v}{\partial x}\right)$ and $\tau_{yy}{}' = 2\mu\frac{v_c}{h_0}\frac{\partial v}{\partial y}$. So, clearly, all these three terms have a coefficient which is a ratio of some velocity by some length. We do not exactly know about the coefficients but one thing we definitely know that v_c is of the order of εu_c. Now we look into all the multipliers $\tau_{xx}{}' n_x + \tau_{xy}{}' n_y$ and $\tau_{xy}{}' n_x + \tau_{yy}{}' n_y$. We do not write all these multiplications in detail, but by observing we can see which term is important and which term is not important. First we consider the multiplication of $\tau_{xx}{}'$ with n_x. So, when $\tau_{xx}{}'$ is multiplied with n_x, there is one $\mathrm{O}(\varepsilon)$ term and on top of that there is some multiplier. When $\tau_{xy}{}'$ is multiplied with n_y, in the expression $\tau_{xy}{}' = \mu\left(\frac{u_c}{h_0}\frac{\partial u}{\partial y} + \frac{v_c}{l_c}\frac{\partial v}{\partial x}\right)$, the term $\frac{v_c}{l_c}$ is of the order of ε and n_y is equal to 1. But the term $\frac{u_c}{h_0}$ is clearly not of the order of ε. It is the ratio of a velocity scale and a small length scale; u_c is the dominant velocity scale and h_0 is the smaller length scale. If we compare the term $\frac{u_c}{h_0}$ with the term $\frac{v_c}{l_c} = \frac{\varepsilon u_c}{l_c}$; out of h_0 and l_c, h_0 is smaller. So the term $\frac{u_c}{h_0}$ will clearly dominate the term $\frac{v_c}{l_c} = \frac{\varepsilon u_c}{l_c}$ because $\frac{u_c}{h_0}$ will be greater than $\frac{u_c}{l_c}$ and a multiplication with the small quantity ε makes it further less. So, the term $\mu\frac{u_c}{h_0}\frac{\partial u}{\partial y}$ will clearly dominate. When we multiply the term $\tau_{xy}{}'$ with n_x, $n_x = -\varepsilon\frac{\partial h}{\partial x}$ is already of the order of ε. So, whatever be the order of $\tau_{xy}{}' = \mu\left(\frac{u_c}{h_0}\frac{\partial u}{\partial y} + \frac{v_c}{l_c}\frac{\partial v}{\partial x}\right)$, when we multiply it with n_x, it will be of the order of ε; so, the term $\tau_{xy}{}' n_x$ will be less as compared to the term $\tau_{xy}{}' n_y$. In the expression of $\tau_{yy}{}' = 2\mu\frac{v_c}{h_0}\frac{\partial v}{\partial y} = 2\mu\frac{\varepsilon u_c}{h_0}\frac{\partial v}{\partial y}$ we can clearly see an order ε term. So, out of the four terms $\tau_{xx}{}' n_x$, $\tau_{xy}{}' n_y$, $\tau_{xy}{}' n_x$ and $\tau_{yy}{}' n_y$, except for the term $\tau_{xy}{}' n_y$ the other three terms are either of the order of ε or ε^2. So, the term $\tau_{xy}' n_y$ is therefore the leading order term. So in the leading order $\vec{\mathrm{T}}'=\mu \frac{u_c}{h_0}\frac{\partial u}{\partial y}\hat{i}$. We have to make a dot product of the traction vector $\vec{\mathrm{T}}'$ with the unit vector $\hat{s}$. When we make the dot product, $\vec{\mathrm{T}}'\cdot\hat{s}$ results $\vec{\mathrm{T}}'\cdot\hat{s}=\left(\mu\frac{u_c}{h_0}\frac{\partial u}{\partial y}\hat{i}\right)\cdot\left(\hat{i}+\varepsilon\frac{\partial h}{\partial x}\hat{j}\right)=\mu\frac{u_c}{h_0}\frac{\partial u}{\partial y}$ ($\hat{i}\cdot\hat{i}$ becomes equal to 1 and $\hat{j}$ does not feature). We have done this derivation rigorously but we can also do it from an intuition. Since $\vec{\mathrm{T}}'\cdot\hat{s}$ is the tangential stress it should in the form similar to $\mu\frac{du}{dy}$. Now question may arise about the term $\frac{du}{dy}$ because we have now assumed the interface to be curved instead of a flat interface. However, although the interface is curved, in the leading order the deviation from its flatness is not that significant and therefore, in the leading order, it is effectively like $\mu\frac{du}{dy}$.

Next we will calculate the right hand side of the tangential force balance equation $\vec{\mathrm{T}}'\cdot\hat{s}=\left(\nabla_s'\sigma'\right)\cdot\hat{s}$. The expression of the surface gradient operator ∇_s' is given by $\nabla_s'=\nabla'-\hat{n}\left(\hat{n}\cdot\nabla'\right)$. First of all, we will operate $\nabla_s'\sigma'$ on the unit vector $\hat{s}$. This dot product $\left(\nabla_s'\sigma'\right)\cdot\hat{s}$ is as good as $\hat{s}\cdot\left(\nabla_s'\sigma'\right)$. Using the form $\nabla_s'=\nabla'-\hat{n}\left(\hat{n}\cdot\nabla'\right)$, when we use the dot product of $\hat{s}$ with $\nabla_s'\sigma'$, for the first term ∇' it will be $\hat{s}\cdot\nabla_s'$ while the dot product for the second term $\hat{n}\left(\hat{n}\cdot\nabla'\right)$ will be equal to zero because the unit vectors $\hat{n}$ and $\hat{s}$ are orthogonal to each other. These unit vectors are perpendicular to each other according to the way they are defined. So, $\hat{s}\cdot\left(\nabla_s'\sigma'\right)$ will boil down to $\left(\hat{s}\cdot\nabla'\right)\sigma'$. So, out of $\nabla_s'=\nabla'-\hat{n}\left(\hat{n}\cdot\nabla'\right)$, we have only retained the term ∇' because $\hat{n}\cdot\hat{s}$ is equal to zero and we eventually get $\hat{s}\cdot\left(\nabla_s'\sigma'\right)=\left(\hat{s}\cdot\nabla'\right)\sigma'$. So, $\left(\hat{s}\cdot\nabla'\right)\sigma'=\left[\left(\hat{i}+\varepsilon\frac{\partial h}{\partial x}\hat{j}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x'}+\hat{j}\frac{\partial}{\partial y'}\right)\right]\sigma'$. σ' can be written as $\sigma'=\sigma\sigma_0$ where σ_0 is the reference surface tension coefficient. So, σ_0 is the dimensional reference surface tension and therefore, σ is the non-dimensional surface tension. In this way we are transforming a dimensional surface tension σ' to a non-dimensional surface tension σ via the reference σ_0. So we get $(\hat{s}\cdot\nabla')\sigma' = \left[\left(\hat{i} + \varepsilon\frac{\partial h}{\partial x}\hat{j}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x'} + \hat{j}\frac{\partial}{\partial y'}\right)\right]\sigma\sigma_0$ where the dot product $\left(\hat{i} + \varepsilon\frac{\partial h}{\partial x}\hat{j}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x'} + \hat{j}\frac{\partial}{\partial y'}\right)$ is simplified to the form $\left(\frac{\partial}{\partial x'} + \varepsilon\frac{\partial}{\partial y'}\frac{\partial h}{\partial x}\right)$. So we get the final form of $(\hat{s}\cdot\nabla')\sigma'$ as $(\hat{s}\cdot\nabla')\sigma' = \left(\frac{\partial}{\partial x'} + \varepsilon\frac{\partial}{\partial y'}\frac{\partial h}{\partial x}\right)\sigma\sigma_0$. Now x' can be written as $x' = x\, l_c$ and y' can be written as $y' = y\, h_0$. Using these expressions of x' and y' in the form of $(\hat{s}\cdot\nabla')\sigma'$, we can write $\left(\frac{\partial}{\partial x'} + \varepsilon\frac{\partial}{\partial y'}\frac{\partial h}{\partial x}\right)\sigma\sigma_0 = \left(\frac{1}{l_c}\frac{\partial}{\partial x} + \frac{\varepsilon}{h_0}\frac{\partial}{\partial y}\frac{\partial h}{\partial x}\right)\sigma\sigma_0 = \frac{1}{l_c}\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial h}{\partial x}\right)\sigma\sigma_0$ where we have used the definition $\varepsilon = \frac{h_0}{l_c}$. Now we define a new operator $\tilde{\nabla}$ as $\tilde{\nabla} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial h}{\partial x}\right)$ which is operating on the surface tension σ and we get $(\hat{s}\cdot\nabla')\sigma' = \frac{\sigma_0}{l_c}\tilde{\nabla}\sigma$. Now we equate the expression of $\vec{T}'\cdot\hat{s}$ and $(\hat{s}\cdot\nabla')\sigma'$ which reads as $\mu\frac{u_c}{h_0}\frac{\partial u}{\partial y} = \frac{\sigma_0}{l_c}\tilde{\nabla}\sigma$. We divide both sides of this equation by $\mu\frac{u_c}{h_0}$ to get $\frac{\partial u}{\partial y} = \frac{\sigma_0}{\mu u_c}\frac{h_0}{l_c}\tilde{\nabla}\sigma = \varepsilon\frac{\sigma_0}{\mu u_c}\tilde{\nabla}\sigma$. This $\frac{\mu u_c}{\sigma_0}$ is a very important non-dimensional parameter. $\frac{\mu u_c}{\sigma_0}$ can be written as $\frac{\mu u_c}{\sigma_0} = \frac{\frac{\mu u_c}{h_0}A}{\sigma_0\frac{A}{h_0}}$ where $\frac{\mu u_c}{h_0}$ is the shear stress and when it is multiplied by the area A it represents the viscous force. In the denominator, $\frac{A}{h_0}$ is a characteristic length. So in the denominator there is surface tension coefficient multiplied by the characteristic length which represents the surface tension force. So $\frac{\mu u_c}{\sigma_0} = \frac{\text{viscous force}}{\text{surface tension force}}$, which is called as the Capillary number (*Ca*) representing the ratio of the viscous force and the surface tension force.

Overall, we arrive at the final non-dimensional form of the tangential force balance boundary condition. In the next chapter we will discuss about the procedure to write the normal force balance boundary condition in the dimensionless form.