Advanced Concepts In Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 42
Thin Film Dynamics (Contd.)

In the previous chapter, we have discussed about the thin film boundary conditions at the interface in terms of the tangential force balance and the normal force balance. We have not discussed about the kinematics constraints at the interface. These kinematic constraints will be reflected through a boundary condition called as kinematic boundary condition. The initial form of the interface is shown in figure 1(i) at a certain time and then after some time the interface evolves like what is drawn in figure 1(ii). We have to decide that what remains unaltered when we move from the first configuration to the second configuration.

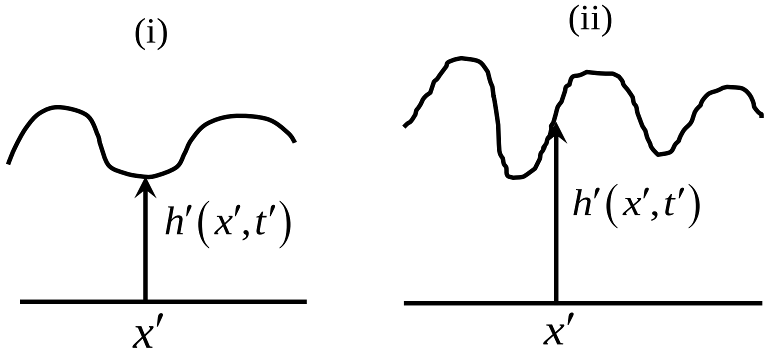

Figure 1(i) shows the initial configuration of the interface which after some time takes the configuration of what is drawn in figure 1(ii).

Let us assume that at a given x', the interface thickness is $h'(x',t)$. After sometime, at the same x', $h'(x',t)$ becomes different. This is a combined spatiotemporal change in the film thickness. Over a period of time, the interface topology changes altogether and that makes h' a variable which not only is a function of x' because of the variation along the x'-direction but h' itself becomes a function of time. That makes h' a combined function of position and time. To assess the combined function of position and time, we define a function F' as $F' = y' - h'(x',\mathrm{t}')$. So, clearly, if the interface is flat, at the interface y' and $h'(x',\mathrm{t}')$ are equal to each other. So, if we are located at the interface, for a flat interface $y' = h'(x', \mathrm{t}')$. If we are located on the interface where h' dynamically evolves we can say that the absolute location will change. But what will happen is if there is a particle located on the interface, the particle will still be located on the interface forever. It may go to a different position but if it is located at the interface, it will be constrained to be located at the interface forever. This of course gets violated if there is an exchange of mass between the two phases. But that is not the case what we are considering here. So, if we are not considering that particular case, then it says that if a particle is located on the interface it will be located on the interface forever. This is the physics of the kinematic boundary condition. That we can write equivalently as $\frac{DF'}{Dt'} = 0$ which means that the total derivative of the function F' is equal to zero. We can expand the total derivative $\frac{DF'}{Dt'}$ in terms of the unsteady term $\frac{\partial F'}{\partial t'}$ and the advective term $\vec{V}' \cdot \nabla F'$ and we can write $\frac{\partial F'}{\partial t'} + \vec{V}' \cdot \nabla F' = 0$. Now we determine the individual terms of the constraint $\frac{\partial F'}{\partial t'} + \vec{V}' \cdot \nabla F' = 0$. Using the expression $F' = y' - h'(x', \mathrm{t}')$, we can write $\frac{\partial F'}{\partial t'} = -\frac{\partial h'}{\partial t'}$, the vector $\vec{V}'$ can be written as $\vec{V}' = u'\,\hat{i} + v'\,\hat{j}$ where $\hat{i}$ and $\hat{j}$ are the unit vectors along the x'-direction and the y'-direction. The gradient of the function F' becomes $\nabla F' = \hat{i}\frac{\partial F'}{\partial x'} + \hat{j}\frac{\partial F'}{\partial y'} = -\hat{i}\frac{\partial h'}{\partial x'} + \hat{j}$. Using the expressions of $\vec{V}'$ and $\nabla F'$ we get $\vec{V}' \cdot \nabla F' = \left(u'\,\hat{i} + v'\,\hat{j}\right) \cdot \left(-\hat{i}\frac{\partial h'}{\partial x'} + \hat{j}\right) = -u'\frac{\partial h'}{\partial x'} + v'$. This expression of $\vec{V}' \cdot \nabla F'$ along with $\frac{\partial F'}{\partial t'} = -\frac{\partial h'}{\partial t'}$ will lead to $-\frac{\partial h'}{\partial t'} - u'\frac{\partial h'}{\partial x'} + v' = 0$, or, $v' = \frac{\partial h'}{\partial t'} + u'\frac{\partial h'}{\partial x'}$. So, $v' = \frac{\partial h'}{\partial t'} + u'\frac{\partial h'}{\partial x'}$ is the constraint that must be satisfied at the interface. This boundary condition must be satisfied at the interface. Let us take an example of the kinematic boundary condition. Let us consider that we have a rigid interface as the bottom wall. If we have a rigid interface as the bottom wall, $F' = y'$ because h' is equal to zero. Now we look into the constraint $v' = \frac{\partial h'}{\partial t'} + u'\frac{\partial h'}{\partial x'}$. In the present example, $\frac{\partial h'}{\partial t'}$ is equal to zero because h' itself is equal to zero. In the other term $u'\frac{\partial h'}{\partial x'}$, first of all $\frac{\partial h'}{\partial x'}$ is equal to zero and on the top that u' is also equal to zero because of the no-slip boundary condition. So the constraint $v' = \frac{\partial h'}{\partial t'} + u' \frac{\partial h'}{\partial x'}$ gets simplified to the form $v' = 0$. This is known as the no-penetration boundary condition. So the no-penetration boundary condition can be perceived as a special case of the kinematic boundary condition. It is not anything different; it is just a special case of the kinematic boundary condition. It tells that if a particle is located on the wall, it will be located on the wall forever because it cannot penetrate through the wall.

Let us now assess the things what we have learnt so far in the context of thin film dynamics. We have learnt about how to write the simplified form of the governing continuity and momentum equations. We have also learnt about the force balance equations as well as the kinematic boundary condition. We arrive at a pressure scale which is the function of the velocity scale. But we do not yet know about the velocity scale. So the key question arises about this velocity scale. To understand the velocity scale we should ideally non-dimensionalize all the governing equations, not only the governing equations but also the boundary conditions. Then we must figure out the physical force that is dominating. It can exist either in the governing equations or in the boundary conditions. Form that, we can figure out the velocity scale. So to figure out the velocity scale we need to have either a scaling analysis or a non-dimensionalization. Non-dimensionalization appears to be algebraically a little bit more convenient because we can use the non-dimensional equations further to solve for the film thickness.

Now we concentrate on the non-dimensionalization of the thin film equations and the boundary conditions. To do that, we should start with the continuity equation $\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$. The length scale of the problem is l_c, so, x' is of the order of ~ l_c. y' is of the order of ~ h_0; u' is of the order of ~ u_c and p' is of the order of ~ p_c. The scale p_c can be obtained from the momentum equation which tells that $p_c = \frac{\mu u_c l_c}{h_0^2} = \frac{\mu u_c}{\varepsilon^2 l_c}$, so, $p' \sim \frac{\mu u_c}{\varepsilon^2 l_c}$. Not only y' is of the order of $\sim h_0$ but also h' is of the order of $\sim h_0$; σ' is of the order of ~ σ_0. Additionally, of course, we have the disjoining pressure term but it involves its own variables. Now we will convert the dimensional variables into dimensionless variables using these scales. We define, $x=\frac{x'}{l_c}$, $y=\frac{y'}{h_0}$, $h=\frac{h'}{h_0}$, $u=\frac{u'}{u_c}$, $p=\frac{p'}{p_c}$ and $\sigma=\frac{\sigma'}{\sigma_0}$. We have missed the order of magnitude for the v' component which is of the order ~ v_c. Using the continuity equation, v_c becomes equal to εu_c; so, v' becomes of the order of $\sim\varepsilon u_c$. We define, $v=\frac{v'}{v_c}=\frac{v'}{\varepsilon u_c}$ where $\varepsilon=\frac{h_0}{l_c}$. Using these dimensionless variables we will write the governing equations and the boundary conditions. To do this, there is a simple way and there is a little bit more methodical but time consuming way. The time consuming way is that we can write the derivative $\frac{\partial u'}{\partial x'}$ as $\frac{\partial u'}{\partial x'}=\frac{\partial u'}{\partial u}\frac{\partial u}{\partial x}\frac{\partial x}{\partial x'}$ using the chain rule of differentiation. Using the orders of magnitudes stated before, $\frac{\partial u'}{\partial u}$ is equal to u_c and $\frac{\partial x}{\partial x'}$ is equal to $\frac{1}{l_c}$, so the dimensionless from of $\frac{\partial u'}{\partial x'}$ becomes $\frac{u_c}{l_c}\frac{\partial u}{\partial x}$. Simple way is that, $\frac{\partial u'}{\partial x'}$ can be non-dimensionalized by multiplying velocity scale u_c in the numerator and length scale l_c in the denominator. Using the dimensional forms of the variables and some algebraic rearrangement, we will get the same dimensionless form $\frac{u_c}{l_c}\frac{\partial u}{\partial x}$. In the similar way, the dimensionless form of the term $\frac{\partial v'}{\partial y'}$ is given by $\frac{v_c}{h_0}\frac{\partial v}{\partial y}$. Now we have already seen from the order of magnitude analysis that $\frac{u_c}{l_c}=\frac{v_c}{h_0}$ because v_c is defined as $v_c=\frac{h_0}{l_c}u_c=\varepsilon u_c$. Using this, the dimensionless form of the continuity equation becomes $\frac{u_c}{l_c}\frac{\partial u}{\partial x}+\frac{u_c}{l_c}\frac{\partial v}{\partial y}=0$ or, $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. So, the dimensional and the non-dimensional continuity equations are in the same form. Next we will consider the x'-momentum equation.

x'-momentum equation: $$0=-\frac{\partial p'}{\partial x'}+\mu\frac{\partial^2 u'}{\partial y'^2}+\rho g\sin\theta \qquad (1)$$

By following the same style that was done for the continuity equation, the dimensionless form of the x'-momentum equation is given by

$$0 = -\frac{p_c}{l_c}\frac{\partial p}{\partial x} + \frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2} + \rho g \sin\theta \qquad (2)$$

When we write $\rho g \sin\theta$ as a body force in the x'-momentum equation, it becomes a little bit restrictive because we are considering gravity as the only body force. So instead of that we write a general body force f_x' which can be gravity or something else. In the body force, we can have electric field, magnetic field or so many other effects. So, if we write a general formulation for f_x' we can use that formulation also for solving our specific problem. So, the dimensionless form of the x'-momentum equation is rewritten as $0 = -\frac{p_c}{l_c}\frac{\partial p}{\partial x} + \frac{\mu u_c}{h_0^2}\frac{\partial^2 u}{\partial y^2} + f_x'$. Using the order of magnitude analysis stated previously, $\frac{p_c}{l_c}$ is equal to $\frac{\mu u_c}{h_0^2}$. Actually by equating the expressions $\frac{p_c}{l_c}$ and $\frac{\mu u_c}{h_0^2}$, the relationship between p_c and u_c was established. So, $\frac{p_c}{l_c} = \frac{\mu u_c}{h_0^2}$ and therefore, $0 = -\frac{p_c}{l_c}\frac{\partial p}{\partial x} + \frac{p_c}{l_c}\frac{\partial^2 u}{\partial y^2} + f_x'$. We multiply both sides of this dimensionless momentum equation by $\frac{l_c}{p_c}$ and the momentum equation becomes $0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + f_x' \frac{l_c}{p_c}$. Now we substitute the expression of p_c here, i.e. $p_c = \frac{\mu u_c}{\varepsilon^2 l_c}$ and the final form of the momentum equation becomes

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + f_x' \frac{l_c^2 \varepsilon^2}{\mu u_c} \qquad (3)$$

If θ is equal to zero and there is no other body force, then the term $f_x' \frac{l_c^2 \varepsilon^2}{\mu u_c}$ will be equal to zero and the x-momentum equation will be $0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}$. Now we focus on the y'-momentum equation. The dimensional form of the y'-momentum equation is given below

y'-momentum equation: $$0 = -\frac{\partial p'}{\partial y'} - \rho g \cos\theta \tag{4}$$

Here also we can write the body force term $\rho g \cos\theta$ (the gravity force) in a general form f_y' and the y'-momentum equation can be rewritten as $0 = -\frac{\partial p'}{\partial y'} - f_y'$. Now we non-dimensionalize this momentum equation and the dimensionless form of this momentum equation becomes

$$0 = -\frac{p_c}{h_0}\frac{\partial p}{\partial y} - f_y' \tag{5}$$

We multiply both sides of equation (5) by $\frac{h_0}{p_c}$ and it becomes $0 = \frac{\partial p}{\partial y} + f_y' \frac{h_0}{p_c}$. Using the expression of $p_c = \frac{\mu u_c}{\varepsilon^2 l_c}$, we get $\frac{h_0}{p_c} = \frac{\varepsilon l_c \varepsilon^2 l_c}{\mu u_c} = \frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}$. Using this, the final form of the *y*-momentum equation is given by $0 = \frac{\partial p}{\partial y} + f_y' \frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}$. Regarding the body force terms, we should note that in many of the scenarios, the body forces along the *x*-direction and the *y*-direction govern the physics of the problem (perhaps in most of the practical scenarios either body force along the *x*-direction or body force along the *y*-direction is present). In order to see the importance of body force term, we need to look into the momentum equation $0 = \frac{\partial p}{\partial y} + f_y' \frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}$ where the term $\frac{\partial p}{\partial y}$ is already of the order of 1 because pressure (p) is normalized with respect to its characteristic scale and y is normalized with respect to its characteristic scale. If $\frac{\partial p}{\partial y}$ is of the order of 1, in order to satisfy the equation $0 = \frac{\partial p}{\partial y} + f_y' \frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}$, the term $f_y' \frac{\varepsilon^3 l_c^{\,2}}{\mu u_c}$ has to be of the order of 1. So, $f_y' \frac{\varepsilon^3 l_c^{\,2}}{\mu u_c} \sim$ O(1) from which one can get the velocity scale u_c. In this way, depending on the physics of problem one can actually arrive at the relevant velocity scale for the problem. So we have arrived at the various governing equations. Regarding the boundary conditions, we first consider the simplest one to algebraically deal which is the kinematic boundary condition.

The kinematic boundary condition tells that at the interface, $v' = \frac{\partial h'}{\partial t'} + u' \frac{\partial h'}{\partial x'}$. Using the non-dimensionalization scheme described earlier, we can write the kinematic boundary condition as $v v_c = \frac{h_0}{t_c}\frac{\partial h}{\partial t} + u_c \frac{h_0}{l_c}\frac{\partial h}{\partial x}$. The time scale t_c is given by $t_c = \frac{l_c}{u_c}$. Substituting this expression of t_c we get $v v_c = u_c \frac{h_0}{l_c}\frac{\partial h}{\partial t} + u_c \frac{h_0}{l_c}\frac{\partial h}{\partial x}$. The velocity scale v_c is given by $v_c = \varepsilon u_c$ where $\varepsilon = \frac{h_0}{l_c}$. Using this expression of v_c and the definition of ε, the modified form of the kinematic boundary condition becomes $v \varepsilon u_c = \varepsilon u_c \frac{\partial h}{\partial t} + \varepsilon u_c \frac{\partial h}{\partial x}$. So, εu_c gets cancelled from both sides and we finally get $v = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}$. This form $v = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}$ is not the most formal way of writing but it is a very practical way of writing. This form is not the most formal way of writing because the smallness of the parameter ε is inbuilt in this writing. So a more formal way is to write u as $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + ...$, v as $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + ...$ in the limit of $\varepsilon \to 0$. This is called as asymptotic expansion. In the limit of $\varepsilon \to 0$, al the terms u_1, u_2, v_1 and v_2 are gone. So, u in the leading order will be represented by u_0 and v in the leading order will be represented by v_0. So the form $v = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}$ is actually in terms of the leading order terms (so by presuming $\varepsilon \to 0$ we are actually writing the leading order terms). The expansion in the asymptotic series is a formal way of writing but the form $v = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}$ is little bit informal. Sometimes we have to use an informal way of writing which can alert us about the physics of the problem. The alertness that should come about the physics of the problem is that we have already considered ε as a small quantity. So, $v = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}$ is the kinematic boundary condition at the interface.

Overall in the present chapter we have discussed about how to non-dimensionalize the thin film governing equations and how to non-dimensionalize the kinematic boundary condition. Of course, there are tangential stress balance and normal stress balances which need to be non-dimensionalized before we take up the final formulation of the thin film problem. That will be discussed in the next chapter.