Advanced Concepts In Fluid Mechanics
Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture - 41
Thin Film Dynamics

(Contd.)

In the present chapter we will continue with the thin film equation which was discussed in the previous chapter. The physical situation that we are considering is just an example and it should not be considered as a general situation.

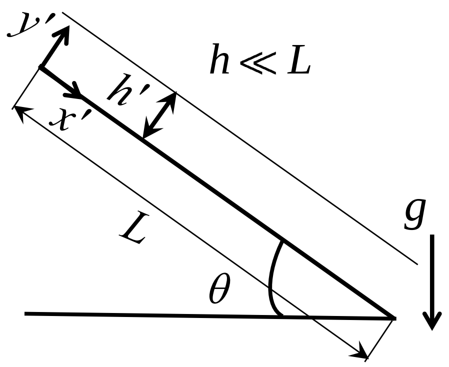

Figure 1. Schematic of a thin film flow down an inclined plane of inclination angle *θ*.

We have an inclined plane with an angle of inclination *θ*. There is a thin liquid film on the top of the inclined plane as shown in figure 1. The film thickness in general is not a constant, it is a variable. The film thickness h' is a function of *x* and time, i.e. $h'(x,t)$. The dimensional *x*-direction and the dimensional *y*-direction are shown in the figure. We assume two-dimensional and incompressible flow. The length scale h_0 (corresponding to the film thickness h') is considered to be very small as compared to the length scale L_c (corresponding to the length *L*). So, the ratio $\varepsilon = \frac{h_0}{L_c}$ is much less than 1. This is the most important consideration. Another consideration is that the product of ε^2 and Reynolds number (Re_L) is not large. So Reynolds number is not abnormally large otherwise even if ε is small, the product of ε^2 and Reynolds number can be moderate. So we are ruling out that situation. Our hope is that even if Reynolds number is moderately large, ε is so small that ε^2 is very small and the product of ε^2 and Reynolds number will be negligible (this is our intuitive hope). With these considerations, the continuity equation is given by $\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0$. Continuity equation will be always important. The two components of the momentum equation are given below

$$x'-\text{momentum}:\ \ 0=-\frac{\partial p'}{\partial x'}+\mu\frac{\partial^2 u'}{\partial y'^2}+\rho g\sin\theta \tag{1}$$

and

$$y'-\text{momentum}:\ \ 0=-\frac{\partial p'}{\partial y'}-\rho g\cos\theta \tag{2}$$

So we have a very interesting thing along the y'-direction which is across the film thickness. Here the pressure variation is like a hydrostatic pressure variation which is purely gravity driven pressure variation. We have already given detail consideration in the previous chapter on how different terms of the two momentum equations can be dropped to arrive at these two simplified equations. So we are not going to repeat those things in the present chapter. Here we concentrate on the requirement to solve these simplified equations. Out of these equations we really do not know about three variables u', v' and p'. We also do not know another thing, it is hidden within these equations because everything is expresses in terms of the film thickness h'. But the film thickness is not known to us; so h' is also unknown. So the film thickness is also not known. We have earlier discussed the thin film problem as a special problem in the context of exact solution of Navier-Stokes equation. But there is a difference between that problem and the present generalized thin film theory. At the thin film problem in the context of exact solution of Navier-Stokes equation we considered the film thickness to be constant, so it was not a variable for that problem. But in the present problem it is now a variable for the problem. So we require enough boundary conditions to solve this problem. Now we focus on the boundary conditions.

First of all, we will consider the boundary condition or the boundary conditions at the interface. It is important to highlight that although the interface is drawn as flat, in general, we should consider it as undulated. One should never keep that prejudice in mind that interface is flat. Under certain condition it may be treated as flat but in general it cannot be treated as flat. So the question arises about what are the boundary conditions at the interface. The boundary conditions at the interface are typically as follows. One type of boundary condition refers to force balance at the interface and therefore they are kinetic in nature; they refer to force balance. Another type of boundary condition at the interface purely follows from kinematic constraint and that is called as kinematic boundary condition. First we will start with the force balance boundary conditions and then we will arrive at the kinematic boundary condition.

When we have an interface like what is drawn in figure 2, there are two important directions, '*s*' and '*n*' directions in terms of the curvilinear co-ordinates. These '*s*' and '*n*' directions are very important. Now we will write one of the force balance equations along the '*s*' direction and then we will write the other force balance equation along the '*n*' direction. The force balance along the '*s*' direction is called as the tangential force balance.

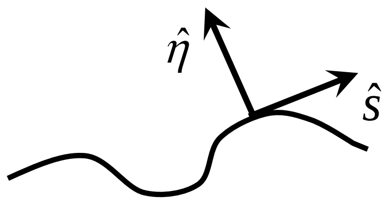

Figure 2. Schematic of an interface where the directions '*s*' and '*n*' are shown.

To understand clearly about the tangential force balance boundary condition, we will first consider a flat interface which will be further extended to a curved interface.

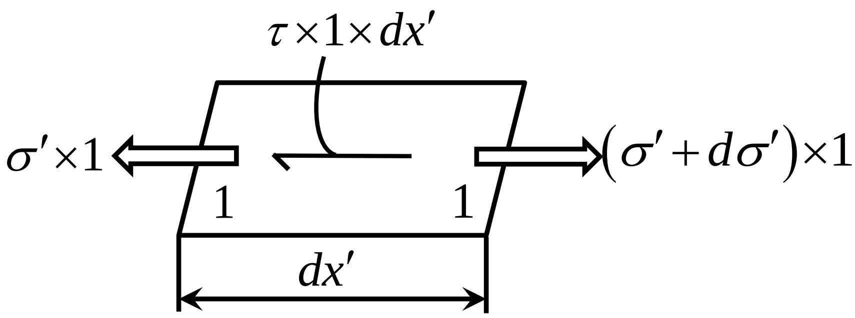

Figure 3. Schematic of the tangential force balance.

We choose the flat interface as an example. Figure 3 represents an edge view of the interface. In reality it is something like a flat membrane. We have to do force balance at the interface. On the top of the interface there is gas, i.e. there is a phase which is having much less viscosity and density as compared to the phase which is on the bottom side. This is our assumption and we do not really bother about what is there on the top side. On the bottom side there is some liquid. Now, at the interface, there can be a variation of the temperature along the tangential direction. If there is a variation of the temperature or composition along the tangential direction, there will be a variation in the surface tension force because surface tension is a strong function of temperature and composition. To do the tangential force balance, we consider a differential element of thickness dx' while the width of the element is chosen as unity. On the left hand side, the acting surface tension force is $\sigma' \times 1$ while on the right hand side, the acting surface tension force is $\left(\sigma' + d\sigma'\right) \times 1$. The other force which is acting on the free surface is the viscous force which is given by $\tau' \times 1 \times dx'$. We need to remember that the surface tension is force per unit length, so to obtain the force we need to multiply surface tension by the length. However, shear stress τ' is force per unit area, so to obtain the force we need to multiply stress by the area which is equal to the length multiplied by unit width. The free diagram of the element at the surface is shown in figure 3. But this is a fluid membrane not a solid membrane. For equilibrium, the resultant force along the *x'*-direction will be equal to zero. So, at equilibrium, we can write $-\sigma' + \left(\sigma' + d\sigma'\right) - \tau' dx' = 0$ from which we get $\tau' = \frac{d\sigma'}{dx'}$. So, for a flat interface we get $\tau' = \frac{d\sigma'}{dx'}$. Now the sanctity of the *x'*-direction gets completely lost if the interface is undulated. Then what matters is the local tangential direction and the local normal direction. Here, instead of writing the shear stress τ' along the *x'*-direction we should have a resolved component of τ' along the '*s*'-direction. To resolve τ' we need to remember the relationship between the traction vector and the stress tensor. In the expression $\tau' = \frac{d\sigma'}{dx'}$, τ' is nothing but a component of the stresses coming from the stress tensor along the *x'*-direction.

Now we write the traction vector in the index form $T_i^{n'} = \tau_{ij}' n_j'$ where *i* = 1 indicates the *x'*-direction, *i* = 2 indicates the *y'*-direction and *i* = 3 indicates the *z'*-direction. Since the problem under consideration is a two-dimensional problem, we will have only two values of *i*, *i* = 1 will be the *x'*-direction and *i* = 2 will be the *y'*-direction. If we get the components of $T_i^{n'}$, we can constitute the vector $\vec{T}^{n'}$ for which *i* = 1 will be the *x'*-

component and i = 2 will be the y-component. The vector $\vec{\mathrm{T}}^{n'}$ will have a component along the 's'-direction and $\vec{\mathrm{T}}^{n'} \cdot \hat{s}$ will be equal to τ' if the interface is flat. So the left hand side of the tangential force balance equation is a little bit more straightforward even for the case of a curved interface. On the other hand, in the right hand side, instead of finding the derivative of surface tension σ' along the x'-direction we need to find out the variation of σ' along the 's'-direction. To do this, first we write the expression $\vec{\mathrm{T}}^{n'} \cdot \hat{s}$ in the vector form which will be of the form $\left(\vec{\mathrm{T}}^{n'} \cdot \hat{s}\right)\hat{s}$ where $\hat{s}$ is the unit vector along the 's'-direction. Now in case of a curved interface, this vector form of stress $\left(\vec{\mathrm{T}}^{n'} \cdot \hat{s}\right)\hat{s}$ will not be equal to $\dfrac{d\sigma}{dx}$ simply, instead it will be equal to $\nabla_s{}'\sigma'$, so, $\left(\vec{\mathrm{T}}^{n'} \cdot \hat{s}\right)\hat{s} = \nabla_s{}'\sigma'$. Since the vectors in both the left hand side and the right hand side are in the same direction, we can eliminate the directional part. In the left hand side, the form $\vec{\mathrm{T}}^{n'} \cdot \hat{s}$ is already in resolved form, so we do not need to bother about this (and we do not need to unnecessarily write it in vector form, so, the unit vector $\hat{s}$ is dropped). Similarly, we have to write the right hand side in the scalar form and the scalar form is the resolved component of the vector along the tangential direction. So, we get, $\left(\vec{\mathrm{T}}^{n'} \cdot \hat{s}\right) = \left(\nabla_s \sigma'\right) \cdot \hat{s}$.

The operator $\nabla_s{}'$ is called as the surface gradient operator. From the vector calculus angle, we have to make sure that when we write various terms, the left hand side and the right hand side should be consistent. Since $\vec{\mathrm{T}}^{n'}$ is a vector and $\hat{s}$ is also a vector, the dot product $\vec{\mathrm{T}}^{n'} \cdot \hat{s}$ will be a scalar. On the right hand side, the operator is sort of a gradient operator. It is not the traditional gradient operator but the surface gradient operator. But the gradient operator acting on a scalar σ' will make $\nabla_s \sigma'$ a vector. To we have to convert it into a scalar so that both the left hand side and the right hand side remain consistent. To make the vector $\nabla_s \sigma'$ into a scalar, we use the vector dot product where the other vector is the unit vector $\hat{s}$ along the 's'-direction.

Now to understand about the surface gradient operator, let us take an example of the acceleration vector. Acceleration as a vector $\left(\vec{a}'\right)$ is the vector sum of the 's'-component

$\left(\vec{a}_s{}'\right)$ and the 'n'-component of the acceleration $\left(\vec{a}_n{}'\right)$, i.e. $\vec{a}' = \vec{a}_s{}' + \vec{a}_n{}'$. So the '$s$'-component $\left(\vec{a}_s{}'\right)$ can be written as $\vec{a}_s{}' = \vec{a}' - \vec{a}_n{}'$. The unit vector in the 'n'-direction is given by $\hat{n}$ which is a vector marker of $\vec{a}_n{}'$. The scalar component is nothing but $\hat{n}\cdot\vec{a}'$ which results $\vec{a}_s{}' = \vec{a}' - \hat{n}\left(\hat{n}\cdot\vec{a}'\right)$. In the similar way, the surface gradient operator can be written as $\nabla_s{}' = \nabla' - \hat{n}\left(\hat{n}\cdot\nabla'\right)$. This is not a procedure to derive the surface gradient operator. One should not take this as a derivation but it should be taken as an analogy. The analogy is that the component of ∇' along the 's'-direction is like the component of any vector along the 's'-direction, so, philosophically it is similar. The form $\vec{a}' - \hat{n}\left(\hat{n}\cdot\vec{a}'\right)$ is algebric quantity while the form $\nabla' - \hat{n}\left(\hat{n}\cdot\nabla'\right)$ is a vector operator, so although there are analogous they are not exactly the same. So we have defined $\nabla_s{}'$ through the expression $\nabla_s{}' = \nabla' - \hat{n}\left(\hat{n}\cdot\nabla'\right)$ and $\left(\vec{\mathrm{T}}^{n'}\cdot\hat{s}\right) = \left(\nabla_s\sigma'\right)\cdot\hat{s}$ represents the tangential force balance. We should check once all the expressions to see whether all terms are consistent or not. We have to make sure that all the variables used till now are in their dimensional form. All dimensional variables are represented by the '/' symbol while only for the unit vectors $\hat{s}$ and $\hat{n}$, we do not need to use the '/' symbol. So far we have discussed the tangential force balance. Next we will consider the normal force balance.

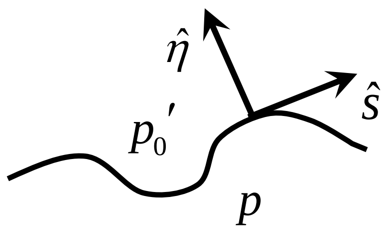

Figure 4. Schematic of an interface used for normal force balance.

Let us consider the interface (as shown in figure 4) which is in static equilibrium where on one side there is a pressure p_0' while on the other side there is there is a pressure p'. Then we can write $p_s{}' - p_0{}' = \dfrac{\sigma'}{R}$ where R is the radius of curvature of interface, $p_s{}'$ is the pressure on the surface and it is a planar problem (so there is no question of existence of the terms like R_1 and R_2 in the denominator). This is basically the Laplace pressure that we are talking about. Here, σ' is the surface tension locally and R is the radius of curvature of the interface locally.

So the question arises about whether the same equation will be valid under dynamic conditions or not. In general, the answer is that the same equation will not be valid under dynamic conditions. Under dynamic conditions there is also a viscous normal stress in addition to pressure. So the equation $p_s' - p_0' = \frac{\sigma'}{R}$ will remain valid only under static condition but in dynamic conditions we have the additional viscous normal stress

term. In order to find the viscous normal stress we have to make a dot product of the traction vector $\vec{\mathrm{T}}^{n'}$ with the unit vector in the normal direction $\hat{n}$. So, the normal force balance equation becomes $p_s' - p_0' - \left(\vec{\mathrm{T}}^{n'} \cdot \hat{n}\right) = \frac{\sigma'}{R}$ where $\vec{\mathrm{T}}^{n'} \cdot \hat{n}$ is the viscous normal stress. So, the tangential component was the dot product of the traction vector with the unit vector $\hat{s}$, here it is the dot product of the traction vector with unit vector $\hat{n}$. We have given a minus sign before the viscous normal stress because pressure by definition is compressive whereas the viscous normal stress is tensile and positive. This does not complete the discussion on the normal force balance because of a very interesting physical phenomenon. When the film thickness become smaller and smaller, there is a critical film thickness when the Van der Waals forces start becoming important. When the Van der Waals forces start becoming important, we represent the equivalent Van der Waals forces (in a pseudo continuum type of framework) by an augmented pressure called as disjoining pressure or excess pressure p_∞'. Even in the static condition we need to consider this additional term and the normal force balance equation in the static condition becomes $p_s' - p_0' = \frac{\sigma'}{R} + p_\infty'$. Similarly, considering this disjoining pressure or the excess pressure the normal force balance under dynamic conditions become $p_s' - p_0' - \left(\vec{\mathrm{T}}^{n'} \cdot \hat{n}\right) = \frac{\sigma'}{R} + p_\infty'$. Typically this excess pressure p_∞' is not a constant but it is a function of the local film thickness. It depends on the kind of the situation but typically it scales with $\frac{1}{h^3}$, so, $p_\infty' \propto \frac{1}{h^3}$. So, if the film thickness is smaller and smaller, the term p_∞' will becomes so much that it may overweigh all other terms in the normal force equation.

Overall, in the present chapter we have discussed the force balance boundary conditions at the interface, i.e. the normal force balance and tangential force balance. What we have not touched upon till now is the kinematic condition at the interface and it will be discussed in the next chapter.