Advanced Concepts In Fluid Mechanics

Prof. Suman Chakraborty
Department of Mechanical Engineering
Indian Institute of Technology, Kharagpur

Lecture – 40
Thin Film Dynamics

In the present chapter we start with a new topic which is thin film dynamics. Thin film dynamics refers to a general class of problems where we have a liquid film formed which is sufficiently thin as compared to other relevant length scales of the problem. For example, in micro-scale flows or even in nano-scale flows sometimes we may have a thin liquid film the thickness of which is much smaller than the characteristic geometric dimensions. So there could be many such situations. For example, one can think of a problem called as spin coating. In spin coating we have a liquid which needs to be coated on a surface. So the liquid is spun on a disc and then the thin liquid film forms on the surface. The film spreads radially outwards because of the rotational effects. So, in this way there can be many such situations where we can actually use the technology or the science of the thin film formation and dynamical evolution of the thin film to understand the motion or to characterize the motion or even to develop new applications. For example, one can use thin film dynamics to understand the application of the spreading of a thin liquid film under the application of an electric field. This is a very important consideration in modern science and technology where electric field combines with fluid dynamics to solve some outstanding problems. So it is therefore quite important to understand about how a thin liquid film forms and dynamically evolves on a surface.

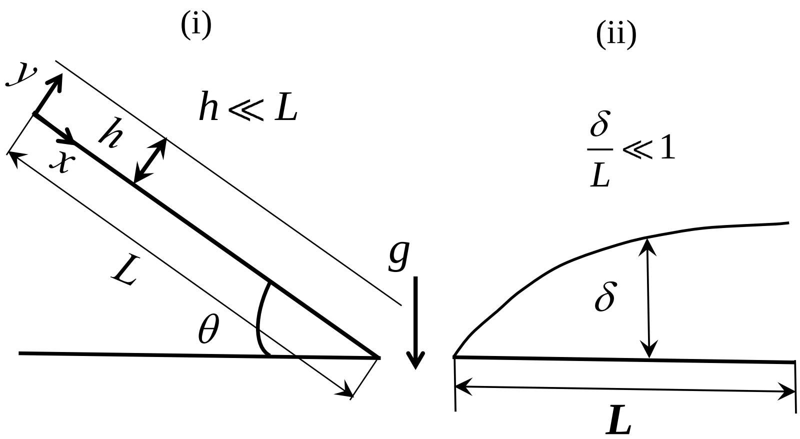

Figure 1. (i) Schematic of thin film flow down an inclined plane of inclination angle θ. (ii) corresponds to the situation for boundary layer of thickness δ over a flat plate.

Now we start with something which we have already discussed earlier, i.e. a special thin film dynamics problem. Here we have an inclined plate with an inclination angle θ as

shown in figure 1(i). We have a thin liquid film of height *h*. At this moment we are not commenting about whether *h* is a constant or a variable which is the key of our discussion. For the time being we are assuming this height as *h* and we are using only one constraint, i.e. the height *h* is much less than the length *L* which defines it as a thin film. Let us imagine a corresponding situation for boundary layer over a flat plate. We are just trying to draw an analogy. If we have a flat plate of length *L* and we have a boundary layer developing (as shown in figure 1(ii)), then the boundary layer theory remains valid if the thickness of the boundary layer δ is much less than the length *L*, i.e. $\frac{\delta}{L} \ll 1$. So in some sense, the thickness δ of the boundary layer theory is equivalent to the length scale *h* of thin film but there are interesting physical dissimilarities. In the boundary layer theory we are talking about a very high Reynolds number flow because the boundary layer is thin only when the Reynolds number is large. So we are talking about a high Reynolds number flow in the boundary layer theory. On the other hand, we are talking about typically sluggish or low Reynolds number flow in thin film dynamics. Although thin film dynamics really does not bother so much about the Reynolds number there are other forces which are equally or even more important for thin film dynamics. Not just viscous force and inertia force, surface tension force, gravity forces are also very important for thin film dynamics. So Reynolds number alone cannot decide anything. But in most of the situations in thin film dynamics we are actually concerned about flow at low Reynolds number. So what makes the thin film dynamics problem akin or similar to the boundary layer problem is not the Reynolds number, they are very disparate. Despite the difference in the magnitude of the Reynolds number in these two cases, the similarity lies in the separation of the length scales (this is the key). In thin film dynamics, the length scale *h* is considered much smaller than the axial length scale *L*. Similarly in the boundary layer problem, the boundary layer thickness δ is much smaller than the axial length scale *L*.

Here we have a thin film confined in an environment; we have considered similar type of problem in the context of lubrication theory. When we have considered this in the context of lubrication theory, the major consideration follows from the fact that the wall is rigid. The height *h* can be a function of the axial co-ordinate, it can also be a function of time but it is a rigid boundary; that is the consideration in the lubrication theory. However, in thin film dynamics, there is a boundary between two phases (let us say one is the liquid phase while the other is the gas phase). If there is a boundary between two phases, then it is actually defined as a free surface. So unlike the lubrication theory where there is a rigid surface, in case of thin film dynamics it is a free surface. The presence of free surface will bring in its additional issues or complications and we will discuss that as we move ahead with this.

Question may arise that why we select this particular problem to start since we have already solved this problem of constant film thickness in case of a low Reynolds number flow or fully developed flow in the context of exact solution of Navier-Stokes equation. Here we will start with the same problem; we will not really solve the problem again but we will formulate the problem. The reason of formulation of this particular problem is to see that how this formulation can be used to solve a problem which is significantly more complicated but essentially very similar. So we have a free surface with a height *h* and the key question arises about this height which may or may not be a constant. So we are talking about the height or the film thickness *h* (the height which we are talking about is actually the film thickness) which can be variable. In fact, in thin film dynamics, the general scenario talks about finding out *h* as a function *x* and time where *x* is the longitudinal co-ordinate.

Let us set up the *x*-axis and the *y*-axis like what is drawn in figure 1(i). Assuming a two-dimensional incompressible flow, the continuity equation is given by $\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0$.

Now, for the present problem, the *x*-momentum equation can be written as

$$\rho\left[\frac{\partial u'}{\partial t'}+u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}\right]=-\frac{\partial p'}{\partial x'}+\mu\left(\frac{\partial^2 u'}{\partial x'^2}+\frac{\partial^2 u'}{\partial y'^2}\right)+\rho g\sin\theta \qquad (1)$$

Before understanding the significance of *x* and *y*, it is important to recognize that gravity is acting vertically downwards which we need to consider in our problem. We now perform an order of magnitude analysis very similar to the order of magnitude analysis we have done for boundary layers.

Starting from the continuity equation $\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$, the order of magnitude of the term $\frac{\partial u'}{\partial x'}$ is given by $\sim \frac{u_c}{L_c}$. Here L_c is same as the length L; for any other problem L_c can be something else. But at least in the present example, L is the characteristic length scale. So question arises about the velocity scale u_c and about the characteristic velocity of the present problem. In a boundary layer over a flat plate we already have a predefined characteristic velocity u_∞ which is the velocity of the fluid coming from the free stream. But in the present case we do not know about the characteristic velocity or the scale u_c. Although u_c is known to us, the thing that we know is that u_c and v_c are related to each other through the continuity equation. Here in the term $\frac{\partial v'}{\partial y'}$ of the continuity equation, we have considered v_c as the order of magnitude of the velocity v'. Also, the order of magnitude of the characteristic length scale along the *y*-direction is equal to h_0. So, the term $\frac{\partial v'}{\partial y'}$ becomes of the order of $\sim \frac{v_c}{h_0}$. This h_0 will be a constant if the film thickness h is a constant. But there can be cases where h can be variable. For example, let us assume a typical situation where we have an undulated interface. In that case, the reference length scale can be chosen as the thickness at $x' = L$. Also, the average film thickness or the smallest film thickness can be chosen as the reference length scale h_0. In order to make the continuity equation to be satisfied, $\frac{u_c}{L_c}$ and $\frac{v_c}{h_0}$ must be of the same order. So, $\frac{u_c}{L_c} \sim \frac{v_c}{h_0}$ from which we get $v_c \sim u_c \frac{h_0}{L_c}$. Here, the ratio $\frac{h_0}{L_c}$ is a key parameter for thin film analysis, we define it to be equal to the parameter ε, so, $v_c \sim u_c \varepsilon$ where ε is a small quantity.

Actually the theory that we are going to follow in thin film dynamics is a specialization of the general asymptotic theory in mathematics. In asymptotic theory of mathematics, we require a small number based on which we make an asymptotic expansion. Here in the present problem, the small number is the quantity ε which is equal to $\frac{h_0}{L_c}$. The entire theory will fail if the parameter ε is not small. So we have to keep ε small. If the film becomes so thick that it becomes of the order of the length *L*, then we cannot use this theory.

Now we move into the *x*-momentum equation (i.e. equation (1)). The term $\frac{\partial u'}{\partial t'}$ is of the order ~ $\frac{u_c}{t_c}$; the term $u'\frac{\partial u'}{\partial x'}$ is of the order of ~ $\frac{u_c^2}{L_c}$ and $v'\frac{\partial u'}{\partial y'}$ is of the order of ~ $\frac{v_c\, u_c}{h_0}$. The term $\frac{\partial p'}{\partial x'}$ is of the order of ~ $\frac{p_c}{L_c}$ and we don't know about the pressure scale p_c. But we understand that p_c is important; in fact it is important in all problems. The reason is that the arbitrary shape of the interface can be sustained with the help of a characteristic pressure difference across two sides of the interface. That makes the characteristic pressure in this problem where we have an arbitrary evolution of the interfacial topology thereby making p_c important. It is the pressure difference across the two sides of the interface which allows the interface to sustain not only its static equilibrium shape but also its dynamical evolution. The term $\mu\frac{\partial^2 u'}{\partial x'^2}$ is of the order of ~ $\frac{\mu\, u_c}{L_c^2}$ and the term $\mu\frac{\partial^2 u'}{\partial y'^2}$ is of the order of ~ $\frac{\mu\, u_c}{h_0^2}$. We use a particular symbol or nomenclature for all the variables, i.e. we have used $'$ symbol for all variables in their dimensional form. Since eventually we will be dealing with the non-dimensional quantities, we represent these non-dimensional quantities without the $'$ symbol. Now we make an assessment of the individual terms of the momentum equation. The terms on the left hand side of the momentum equation are the inertial terms where question arises about the time scale t_c. There are different time scales possible but the natural choice of the time scale is the convective time scale, i.e. $t_c = \frac{L_c}{u_c}$. If we substitute this time scale we find that the two terms $\frac{\partial u'}{\partial t'}$ and $u'\frac{\partial u'}{\partial x'}$ are of the same order. Here, the order of magnitude of the third term $v'\frac{\partial u'}{\partial y'}$ can be rewritten as ~ $\frac{v_c\, u_c}{h_0} \sim \frac{u_c\, h_0\, u_c}{L_c\, h_0} \sim \frac{u_c^2}{L_c}$ if we substitute the order of magnitude of the velocity (v'), i.e. $v_c \sim u_c\frac{h_0}{L_c}$. Then, all three terms on the left hand side of the momentum equation becomes of the same order, i.e. of the order of ~ $\rho\frac{u_c^2}{L_c}$. In the right hand side we have two viscous terms $\mu\frac{\partial^2 u'}{\partial x'^2}$ and $\mu\frac{\partial^2 u'}{\partial y'^2}$;

and we have to decide that which term is important. The term $\mu\frac{\partial^2 u'}{\partial x'^2}$ is of the order of ~ $\frac{\mu u_c}{L_c^2}$ and the term $\mu\frac{\partial^2 u'}{\partial y'^2}$ is of the order of ~ $\frac{\mu u_c}{h_0^2}$. So, the term $\mu\frac{\partial^2 u'}{\partial x'^2}$ is not important because h_0 is much smaller than L_c. Therefore, the quantity $\frac{\mu u_c}{h_0^2}$ is much greater than the quantity $\frac{\mu u_c}{L_c^2}$. So we can see a striking similarity of this thin film dynamics with the boundary layer theory. In boundary layer theory, the boundary layer thickness δ is much less than the length L and here in thin film, h_0 is much less than L_c. Now we will arrive at a striking dissimilarity with the boundary layer theory. To show this, we compare the inertial and the viscous terms of the momentum equation. The inertial terms are present on the left hand side of the momentum equation while the viscous terms are present on the right hand side of the momentum equation. For comparing, one can take any one of the three inertial terms since they are of the same order. So, the ratio of the orders of the inertia term and the viscous term becomes ~ $\frac{\text{inertia}}{\text{viscous}} \sim \frac{\frac{\rho u_c^2}{L_c}}{\frac{\mu u_c}{h_0^2}}$. One of the u_c term gets cancelled from the numerator and the denominator, we multiply both the numerator and the denominator by L_c and we get, $\frac{\text{inertia}}{\text{viscous}} \sim \frac{\rho u_c L_c}{\mu}\cdot\left(\frac{h_0}{L_c}\right)^2 \sim \mathrm{Re}_L\cdot\left(\frac{h_0}{L_c}\right)^2$. Here, Re_L is the Reynolds number of the flow based on the characteristic length scale L_c. We always define Reynolds number based on the characteristic length scale, not on the length scale h_0 of the problem. So, the ratio of the inertia force and the viscous force becomes of the order of a product of Reynolds number and the quantity $\left(\frac{h_0}{L_c}\right)^2 = \varepsilon^2$. Since ε is a small quantity, ε^2 is also supposed to be very small, in fact smaller. So, when ε^2 is much less than 1, our intuitive expectation is that even for a moderate Reynolds number the product

$\mathrm{Re}_L\,\varepsilon^2$ will not be very large. Therefore, although it is not mandatory, it is customary that under all practical circumstances the inertial effects are not considered in the thin film problems until and unless the inertial effects become so important and dominating

that they have to be considered. So we have to keep in mind that the thin film dynamics does not tell us that the inertial effect has to be neglected. It has nothing to do with the formulation of the thin film dynamics. But under most practical circumstances the inertial effects are neglected when thin film dynamics is applied. So this is the reason why we will go ahead with the formulation with the inertial effects being neglected. Once we neglect the inertial effects, the governing equation boils down to a very simple scenario which is given by

$$0=-\frac{\partial p'}{\partial x'}+\mu\frac{\partial^2 u'}{\partial y'^2}+\rho g\sin\theta \tag{2}$$

Since the inertial effects are not important along the *x*-direction where the predominant motion is taking place, these terms will not appear in the *y*-momentum equation. We just write the *y*-momentum equation for the sake of completeness of the problem which is given by

$$\rho\left[\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right]=-\frac{\partial p'}{\partial y'}+\mu\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right)-\rho g\cos\theta \tag{3}$$

Now we perform the order of magnitude analysis. In the left side we write only one term because all the three terms will eventually have the same order of magnitude (as shown in the case of *x*-momentum equation). The inertial terms become of the order of ~ $\frac{\rho v_c u_c}{L_c}$. Now we have to decide that out of the two viscous terms $\mu\frac{\partial^2 v'}{\partial x'^2}$ and $\mu\frac{\partial^2 v'}{\partial y'^2}$ which one is important. Since the *x*-length scale (L_c) is much larger than the *y*-length scale, therefore, the term $\mu\frac{\partial^2 v'}{\partial y'^2}$ will be important. Now, this term $\mu\frac{\partial^2 v'}{\partial y'^2}$ is of the order of ~ $\frac{\mu v_c}{h_0^2}$. So, the ratio of the orders of the inertia term and the viscous term for the *y*-

momentum becomes $\dfrac{\text{inertia}}{\text{viscous}} \sim \dfrac{\dfrac{\rho u_c v_c}{L_c}}{\dfrac{\mu v_c}{h_0^2}} \sim \dfrac{\rho u_c L_c}{\mu} \cdot \left(\dfrac{h_0}{L_c}\right)^2$ where one term v_c gets cancelled from the numerator and the denominator. Since $h_0 \ll L_c$, in this case also, the inertial effects are negligible as compared to the viscous effect; so we can leave the

inertial effect part. Regarding the viscous effect part the big question arises about the viscous effect term $\mu \dfrac{\partial^2 v'}{\partial y'^2}$ since we have considered this term to be important as compared to the inertial term. $\mu \dfrac{\partial^2 v'}{\partial y'^2}$ is the term which highlights that inertial effect is negligible as compared to the viscous effect. The order of magnitude of this term $\mu \dfrac{\partial^2 v'}{\partial y'^2}$ is $\sim \dfrac{\mu v_c}{h_0^2}$ while the order of magnitude of its corresponding counterpart in the *x*-momentum equation is given by $\sim \dfrac{\mu u_c}{h_0^2}$. Hence, the ratio of these two contributions of viscous effect of the *y*-momentum and the *x*-momentum equation becomes of the order of the $\sim \dfrac{v_c}{u_c}$. As per the continuity equation, $\dfrac{v_c}{u_c}$ is of the order of $\sim \dfrac{h_0}{L_c}$ which is nothing but equal to the small quantity ε where ε is very small as compared to 1. Therefore, the moral of the story is that in y-momentum equation, of course the inertial effect much negligible as compared to the viscous effect but the viscous effect is itself negligible when compared with the *x*-momentum equation. So, in the *y*-momentum equation we are left only with the pressure variation $\dfrac{\partial p'}{\partial y'}$ and the body force term $-\rho g \cos\theta$. Finally we need to discuss one point before closing this particular chapter which is the scale of pressure p_c. We have seen that the scale u_c depends on the physics of the problem, it is not like that we have a fixed u_∞ for the boundary layer over a flat plate. But p_c is also another unknown variable like u_c. We have to establish a relationship between u_c and p_c. For that, we need to focus on the *x*-momentum equation

$0 = -\frac{\partial p'}{\partial x'} + \mu \frac{\partial^2 u'}{\partial y'^2} + \rho g \sin\theta$ where the inertial terms are already gone. There is a remaining body force term on the right hand side which becomes equal to zero if the inclination angle θ is equal to zero. But the two terms $\frac{\partial p'}{\partial x'}$ and $\mu \frac{\partial^2 u'}{\partial y'^2}$ will always remain. These two terms will always remain and the term $\rho g \sin\theta$ may or may not remain depending on the value of θ. We can arrive at a scale of p_c by equating the scale of the term $\frac{\partial p'}{\partial x'}$ with the scale of the term $\mu \frac{\partial^2 u'}{\partial y'^2}$. This means $\frac{p_c}{L_c}$ is of the order of ~ $\frac{\mu u_c}{h_0^2}$ from which we can write $p_c \sim \frac{\mu u_c}{h_0^2} \cdot L_c$. From this we can rewrite the scale of p_c as $p_c \sim \frac{\mu u_c}{h_0^2} \cdot L_c \sim \frac{\mu u_c}{\varepsilon^2 L_c}$ by substituting $h_0^2 = \varepsilon^2 L_c^2$ where one L_c term gets cancelled from the numerator and the denominator. From this scale $p_c \sim \frac{\mu u_c}{\varepsilon^2 L_c}$ we can see that pressure becomes very important because it scales with $\frac{1}{\varepsilon^2}$ where ε is a small quantity in this particular problem. So, overall, we have arrived at a pressure scale which depends on the velocity scale where the velocity scale depends on the physics of the problem. Also we have arrived at the governing equations of continuity and momentum and we will take it up from this point in the next chapter.