Advanced Concepts In Fluid Mechanics

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Lecture - 04

Linear and Volumetric Deformation; Perspectives from Mass Conservation

I. Linear and Volumetric Deformation of Fluid

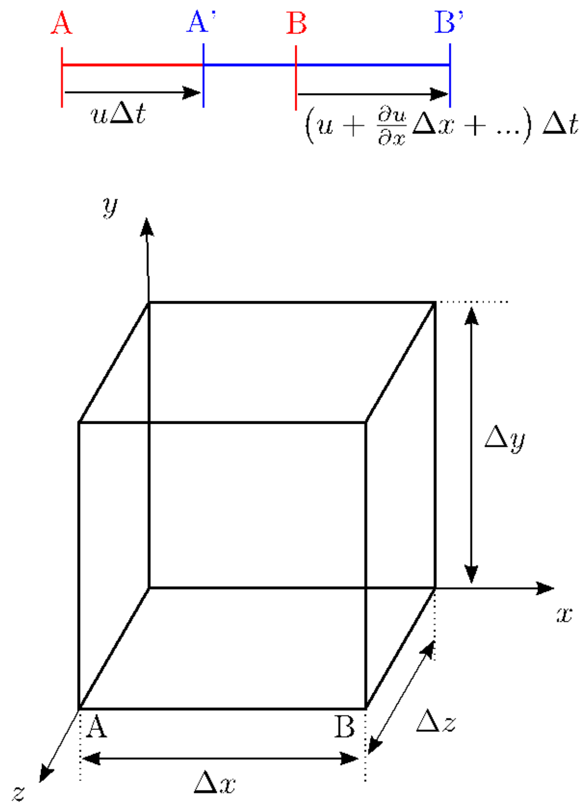

Fig 1: A rectangular element of fluid undergoing a generalized flow, the line segement AB deforms to A'B' in the time interval Δt.

After discussing Angular deformation in the previous lecture, we will discuss linear and volumetric deformation in this lecture. The deformations being discussed are small deformations occurring over infinitesimal interval of time. Therefore, it becomes possible as well as useful to decompose the deformation into angular and linear. For larger durations of time, the total deformation is simply the integral over its infinitesimally small splits.

To quantify the linear deformation of fluid, consider a rectangular volume of fluid as presented in Fig 1. The first step is to individually calculate the change in Δx, Δy and Δz. Once these are obtained, the change in volume of the fluid element can also be obtained.

To calculate the change in Δx, we consider the segment AB that deforms to A'B' as presented on the top of Fig 1. The lengths AA' and BB' are also presented. The ... represents higher order terms. The deformed length of AB is Δx plus the difference between AA' and BB' (more specifically, BB'-AA'), i.e. new-$\Delta x = \Delta x + \left(\frac{\partial u}{\partial x} \Delta x + ... \right) \Delta t$. The strain along x is defines as the change is length per unit length along x and the rate of strain along x is defines as the strain along x per unit time, i.e. change in Δx (i.e new-Δx minus Δx) divided by Δx divided by Δt with Δx and Δt being infinitesimally small. Denoting the strain along x by $\dot{\epsilon}_{xx}$ and utilizing the expression for new-Δx obtained above, we have

$$\dot{\epsilon}_{xx} = \lim_{\Delta t \to 0} \frac{\left(\frac{\partial u}{\partial x} \Delta x + ... \right) \Delta t}{\Delta x \Delta t} = \frac{\partial u}{\partial x}$$

Similarly, $\dot{\epsilon}_{yy} = \frac{\partial v}{\partial y}$ and $\dot{\epsilon}_{zz} = \frac{\partial w}{\partial z}$.

Here, $\dot{\epsilon}_{xx}$ is a component of the second order tensor $\dot{\epsilon}_{ij}$, which requires two indices i and j for its definition. A second order tensor requires two indices (or two directions) because one index represents the direction of the variable being represented (strain in this case), and the second index represents the (normal of the) plane that is used as the reference to calculate the variable. Stress is also a represented as a second order tensor, as we will see in later lectures. Formally, in terms of mathematics, a second order tensor maps a vector on to a vector.

For fluid the change in volume is a more important quantity that changes in length, as volume is typically the real practical quantity with which one deals with in the context of fluid mechancis. We therefore obtain the expression for the new volume of the fluid element considered in Fig 1. This new volume equals $\Delta x \left[1 + \frac{\partial u}{\partial x} \Delta t + ... \right] \cdot \Delta y \left[1 + \frac{\partial v}{\partial y} \Delta t + ... \right] \cdot \Delta z \left[1 + \frac{\partial w}{\partial z} \Delta t + ... \right]$, which simplifies to $\Delta x \Delta y \Delta z \left[1 + \frac{\partial u}{\partial x} \Delta t + \frac{\partial v}{\partial y} \Delta t + \frac{\partial w}{\partial z} \Delta t + ... \right]$. The rate of volumetric strain is the change in volume (i.e. new volume minus initial volume) divided by initial volume divided by time, i.e.,

Volumetric Strain = $\frac{\text{new}\Delta x \Delta y \Delta z - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z \Delta t}$ =

$$\frac{\Delta x \Delta y \Delta z \left[1 + \frac{\partial u}{\partial x} \Delta t + \frac{\partial v}{\partial y} \Delta t + \frac{\partial w}{\partial z} \Delta t + ... \right] - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z \Delta t} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \nabla \cdot \vec{v}.$$

Hence, from this expression, we deduce that volumetric strain of a fluid element is the divergence of the velocity vector.

A flow is said to be incompressible when there is no change in volume of a fluid element, implying that for an incompressible flow, divergence of the velocity vector is zero. It has been a long-standing misconception that divergence of velocity being zero is a consequence of conservation of mass for incompressible flow. However, as we can see here, it is purely kinematic constraints that $\nabla \cdot \vec{v} = 0$ for an incompressible flow.

<u>Strain Rate Tensor:</u> Having discussed linear, volumetric and angular deformations, we see that the common thing in all their expressions is that the involves terms are essentially a spatial partial derivative of some component of velocity. Therefore, the generalized rate of deformation can be expressed as $\frac{\partial u_i}{\partial x_j}$. This notation called index notation where i=1 means x, i=2 means y and i=3 means z, i.e. u_1=u, u_3=v and u_3=w. This $\frac{\partial u_i}{\partial x_j}$ is called the generalized rate of deformation tensor. This is a second order tensor it requires two components (i.e two indices) for its specification.

This tensor can be decomposed into two parts as shown below.

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right] + \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$$

The first term of this decomposition represents deformation. This deformation is linear if i=j and angular otherwise. The second terms represents rotation. Further, the first term is symmetric and the second term is skew symmetric.

II. Continuity Equation

Since continuity equation is strongly related to volumetric strain rate and thereby divergence of velocity vector, we discuss it at this stage. However, the readers are cautioned that Continuity equation is not a part of kinematics of fluids.

The holy grail of mass conservation in fluid mechanics is the continuity equation. Notwithstanding the complexity of the flow, if it does not satisfy continuity equation, one needs to examine whether such a flow will exist or not. We derive the continuity equation using two approaches – control mass approach and control volume approach, in this lecture.

<u>Control Mass Approach:</u> In this approach, consider a fixed element of fluid (also called as control mass). The mass of this element is density times the volume, i.e. $m = \rho \mathcal{V}$. Taking the log, we have $\ln m = \ln \rho + \ln \mathcal{V}$. Taking the time-derivative of this, we get the equation,

$$\frac{1}{m}\cancel{\frac{Dm}{Dt}} = \frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt}$$

Since the mass of the element of fluid must is finite and must be conserved, the LHS vanishes, as presented in the equation. Next, we convert the first term of RHS from the total derivative to spatio-temporal derivative, which effectively implies switch from Lagrangian paradigm (where control mass is specified) to Eulerial paradigm. We also substitute the second term $\frac{1}{\mathcal{V}}\frac{D\mathcal{V}}{Dt}$, which is the volumetric strain rate with $\nabla\cdot\vec{v}$ using the derivation in the previous section. This gives us the equation

$$\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t}+(\vec{v}\cdot\nabla)\rho\right)+\nabla\cdot\vec{v}=0\Rightarrow\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\vec{v})=0$$

The equation on the right is obtained by multiplying the equation on the left by ρ and then some straightforward algebra.

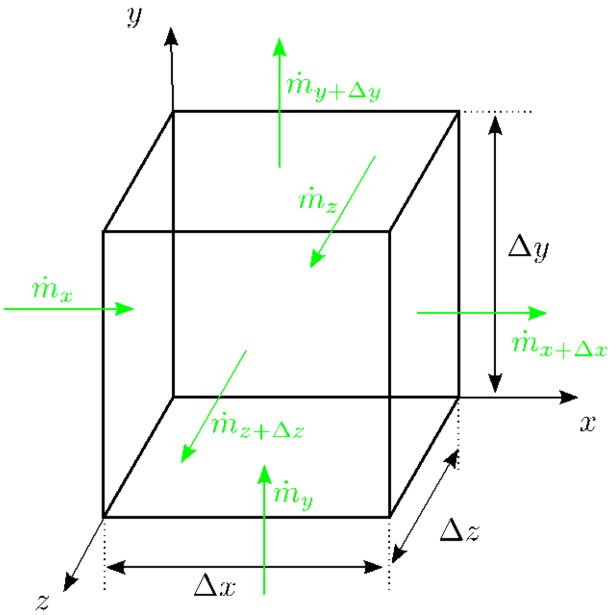

Fig 2: A cubical control volume in the a general flow-field, the mass fluxes at the boundaries are labelled

Control Volume Approach: This is the more traditional approach to deriving the continuity equation. In this approach, we consider the rectangular control volume presented in Fig 2. The mass fluxes on all the boundaries are presented in the figure. Applying mass balance to the control volume, i.e. net rate of mass coming into the control volume equals rate of increase of mass in the control volume, we have

$$\dot{m}_{\text{in}}-\dot{m}_{\text{out}}=\frac{\partial\dot{m}_{\text{C.V.}}}{\partial t}$$

Along the x-direction, the net rate of mass coming in equals $\dot{m}_x - \dot{m}_{x+\Delta x}$. Similar expressions apply for the other two directions as well. Therefore, the mass balance equation coverts to

$$(\dot{m}_x - \dot{m}_{x+\Delta x}) + (\dot{m}_y - \dot{m}_{y+\Delta y}) + (\dot{m}_z - \dot{m}_{z+\Delta z}) = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

Using the Taylor series expansion for $\dot{m}_{x+\Delta x}$, which is $\dot{m}_{x+\Delta x} = \dot{m}_x + \frac{\partial \dot{m}_x}{\partial x}\Delta x + \ldots$, the net rate of mass intake along x-direction is $\dot{m}_x - \dot{m}_{x+\Delta x} = -\frac{\partial \dot{m}_x}{\partial x}\Delta x + \ldots$. Further, $\frac{\partial \dot{m}_x}{\partial x}$ equals $\frac{\partial (u \Delta y \Delta z)}{\partial x} = \Delta y \Delta z \frac{\partial u}{\partial x}$. Substituting these into the mass balance equation, we get

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

This equation is the same as obtained using the Control Mass approach.

Incompressible Flow: We conclude this lecture with a short discussion on imcompresible flows. Oftentimes, fluid mechanists have the misconception that incompressible flow means density is constant. We examine this notion more carefully.

For any flow, continuity equation must be satisfied. This implies

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \vec{v}) = 0$$

As we have derived earlier in this lecture from pure kinematic constraints, for incompressible flow, divergence of the velocity vector is zero. Due to this, the continuity equation simplies as

$$\nabla \cdot \vec{v} = 0 \Rightarrow \frac{D\rho}{Dt} = 0 \Rightarrow \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = 0$$

While this equation is trivially satisfied by a constant density, it can also get satisfied by non-constant density. We illustrate this with an example. Consider a flow for which density varies as $\rho = kxy$, where k is a constant. For this variation of density, the simplified continuity equation for compressible flow takes the form $u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0$, which upon further substitution of the expression for ρ gives the constraint $\frac{u}{v} = -\frac{x}{y}$ which is fairly realistic. In summary, a flow-field that is kinematically constrained to have $\frac{u}{v} = -\frac{x}{y}$ can allow for a non-constant density of the form $\rho = kxy$ while still being incompressible.