Advanced Concepts In Fluid Mechanics
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Lecture - 39
Lubrication Theory (Contd.)

In the previous chapter we have discussed about the lubrication theory and we have come up with a set of simplified *x*-momentum and *y*-momentum equations where *x* and *y* are the two orthogonal directions in the plane of the flow. We have discarded some of the terms in the momentum equation keeping in view that they are not important because of the order of magnitude of these terms relative to the other more dominant terms.

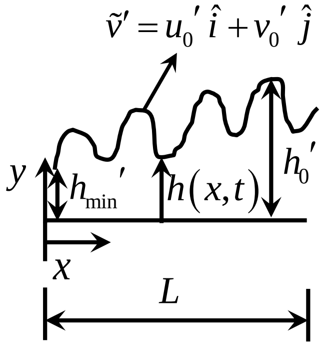

Figure 1. The confinement height *h* as a function of position and time *h*(*x*,*t*).

Now we just reconstruct the physical situation that we are considering. There is an arbitrary top surface for which the height *h* is a function of *x* and time, i.e. $h(x,t)$ where the two axes are *x*-axis and *y*-axis as shown in figure 1. For the gap height (*h*) much smaller than the length *L*, the simplified form of the *x*-momentum equation in the leading order is given by

$$x-\text{momentum}: 0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + \text{forcing term} \tag{1}$$

Similarly, the simplified *y*-momentum equation in the leading order is given by

$$y-\text{momentum}: 0 = -\frac{\partial p}{\partial y} + \text{forcing term} \tag{2}$$

We have discussed the exact mathematical descriptions of these forcing terms in the previous chapter. Now we consider an example where there is no body force. This is the reason we have generically included the body forces as the forcing terms which can be set equal to zero in absence of body force (which is a special case). As discussed earlier, we have made a co-ordinate transformation before solving these equations. Both these

two equations are non-dimensional (so in $h(x,t)$, *h* is non-dimensional height, *x* is non-dimensional axial co-ordinate and *t* is non-dimensional time). We have considered a reference frame which is having a non-dimensional velocity $\tilde{v}' = u_0'\,\hat{i} + v_0'\,\hat{j}$. With this we assume that there is a reference frame which moves towards the right with a velocity u_0'. Then after co-ordinate transformation, the first boundary condition in the dimensionless form becomes at *y* = 0, $u = -u_0$ while the second boundary condition is at $y = h(x,t)$, $u = 0$. These two boundary conditions correspond to the *x*-momentum equation (1), i.e. $0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2}$, or. $\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$. With this background, we solve the equation $\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$ to get the velocity profile. If we integrate the *x*-momentum equation with respect to *y*, we get $\frac{\partial u}{\partial y} = \frac{\partial p}{\partial x} y + c_1(x)$. Since we are integrating with respect to *y*, we can treat the right hand side as a constant, so c_1 will be a function of *x*. Integrating the expression of $\frac{\partial u}{\partial y}$ with respect to *y* we get $u = \frac{\partial p}{\partial x}\frac{y^2}{2} + c_1(x)y + c_2(x)$. Since there are two unknown factors c_1 and c_2, we need two boundary conditions. We apply the boundary condition at *y* = 0, $u = -u_0$; this is the first boundary condition. Using this boundary condition in the expression of the velocity $u = \frac{\partial p}{\partial x}\frac{y^2}{2} + c_1(x)y + c_2(x)$ we get $c_2 = -u_0$; so it becomes a constant. But we can see that this is a formulation which has a provision of accommodating a velocity which is a function of *x*. We have used u_0 where u_0 itself can be a function of *x* and in that case c_2 will become a function of *x*. The second boundary condition is $y = h(x,t)$, $u = 0$. Using this boundary condition in the expression of the velocity $u=\frac{\partial p}{\partial x}\frac{y^2}{2}+c_1(x)y+c_2(x)$, we get $0=\frac{\partial p}{\partial x}\frac{h^2}{2}+c_1 h-u_0$ or, $c_1=\frac{u_0}{h}-\frac{\partial p}{\partial x}\frac{h}{2}$. Substituting the expressions of c_1 and c_2 we get the final form of the velocity profile which is given by

$$u=\frac{1}{2}\frac{\partial p}{\partial x}\left(y^2-yh\right)+u_0\left(\frac{y}{h}-1\right) \tag{3}$$

Whenever we get a solution, not just for fluid mechanics but for any problem, the first thing that we should check is whether it is satisfying the boundary conditions or not. It is not that if it satisfies the boundary conditions, it is bound to be correct. But if it does not satisfy the boundary conditions, it cannot be correct, so it is the other way around. Looking into this final expression $u=\frac{1}{2}\frac{\partial p}{\partial x}\left(y^2-yh\right)+u_0\left(\frac{y}{h}-1\right)$ one can say that at *y* = 0, $u=-u_0$ and at $y=h(x,t)$, $u=0$. So the velocity profile at least satisfies the boundary conditions. The *y*-momentum equation does not come into picture here because it tells that the pressure gradient along the *y*-direction is equal to zero, i.e. $\frac{\partial p}{\partial y}=0$. So we do not have to bother about the pressure variation along the *y*-direction. However, we may have to bother if the body force along the *y*-direction is important. Here in the present example we are not considering any forcing term along the *x* and *y* directions. So we have to keep in mind that this is the example with the forcing term set equal to zero. Many nice problems can be generated from this general formulation by specifying different forcing terms. Now we will develop a strategy to fulfill our objective. Our objective is to find a governing equation for the pressure distribution. To highlight the importance of the pressure distribution let us recall the scale of the pressure. As discussed in the previous chapter, the scale of the pressure is $p_c\sim\frac{\mu u_c L}{h_0^2}\sim\frac{\mu u_c}{\varepsilon^2 L}$ where $\varepsilon=\frac{h_0}{L}$. So p_c scales with $\frac{1}{\varepsilon^2}$ or p_c varies with $\frac{1}{\varepsilon^2}$ (where we are keeping all remaining parameters fixed). But there is other form of stress which is the shear stress. The scale of shear stress is given by $\tau\sim\frac{\mu u_c}{h_0}$ (which is like $\tau=\mu\frac{du}{dy}$); in terms of ε it becomes $\tau \sim \frac{\mu u_c}{h_0} \sim \frac{\mu u_c}{\varepsilon L}$. So, τ scales with $\frac{1}{\varepsilon}$ or τ varies with $\frac{1}{\varepsilon}$. Since the parameter ε is small, pressure is much more significant in lubrication theory than the hydrodynamic shear itself. This is the reason why the pressure distribution itself dictates the force on the boundaries. This is the reason why we need to develop a governing equation for the pressure distribution (this is our objective). The strategy that we follow is very simple. We differentiate the expression of the velocity profile $u = \frac{1}{2}\frac{\partial p}{\partial x}\left(y^2 - yh\right) + u_0\left(\frac{y}{h} - 1\right)$ with respect to *x* and then equate the term $\frac{\partial u}{\partial x}$ with $-\frac{\partial v}{\partial y}$ using the continuity equation. Then we integrate that equation with respect to *y* to obtain *v*. Now *v* at $y = h$ is nothing but equal to $\frac{\partial h}{\partial t}$ which is the rate of change of *h* with respect to time; that will close the problem. Let us workout the steps such that we can understand it better. Differentiating the expression $u = \frac{1}{2}\frac{\partial p}{\partial x}\left(y^2 - yh\right) + u_0\left(\frac{y}{h} - 1\right)$ with respect to *x* results

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{\partial^2 p}{\partial x^2}\left(y^2 - yh\right) + \frac{\partial p}{\partial x}\left(-y\frac{\partial h}{\partial x}\right)\right] + u_0\, y\left(-\frac{1}{h^2}\frac{\partial h}{\partial x}\right) \tag{4}$$

where we have used the product rule of differentiation when we differentiate the expression $\frac{1}{2}\frac{\partial p}{\partial x}\left(y^2 - yh\right)$ with respect to *x*. Now according to the 2-D incompressibility condition we can write $\frac{\partial u}{\partial x}$ to be equal to $-\frac{\partial v}{\partial y}$, so,

$$\frac{\partial u}{\partial x} = \frac{1}{2}\left[\frac{\partial^2 p}{\partial x^2}\left(y^2 - yh\right) + \frac{\partial p}{\partial x}\left(-y\frac{\partial h}{\partial x}\right)\right] + u_0\, y\left(-\frac{1}{h^2}\frac{\partial h}{\partial x}\right) = -\frac{\partial v}{\partial y} \tag{5}$$

In principle, if we integrate this expression with respect to *y*, we will get the velocity *v*. Here we need to remember that all these are partial integration because all derivatives are partial derivatives. The 2-D incompressibility condition reads as $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ which is given the non-dimensional form. We will first derive this expression from the dimensional form of the incompressibility condition which is given by $\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$.

Now we use the non-dimensionalization scheme, i.e. $u = \frac{u'}{u_c}$, $v = \frac{v'}{v_c}$, $x = \frac{x'}{L}$ and $y = \frac{y'}{h_0}$.

The scales u_c and v_c are chosen to satisfy the equation $\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$. Since they are chosen to satisfy this equation, the relationship between u_c and v_c is such that when we substitute it, both dimensional and the non-dimensional forms will be the same. Now integrating the expression $-\frac{\partial v}{\partial y} = \frac{1}{2}\left[\frac{\partial^2 p}{\partial x^2}\left(y^2 - y\,h\right) + \frac{\partial p}{\partial x}\left(-y\frac{\partial h}{\partial x}\right)\right] + u_0\,y\left(-\frac{1}{h^2}\frac{\partial h}{\partial x}\right)$ with respect to *y* we get

$$v = -\frac{1}{2}\left[\frac{\partial^2 p}{\partial x^2}\left(\frac{y^3}{3} - \frac{y^2}{2}h\right) - \frac{\partial p}{\partial x}\left(\frac{y^2}{2}\frac{\partial h}{\partial x}\right)\right] - u_0\frac{y^2}{2}\left(-\frac{1}{h^2}\frac{\partial h}{\partial x}\right) + c_3\left(x\right) \tag{6}$$

In principle *v* can be a function of time *t* also. So we may extend the expression $c_3\left(x\right)$ as a function of both *x* and time $c_3\left(x,t\right)$ but eventually we will get a constant value of it by satisfying the boundary condition. The boundary condition is at $y = 0$, $v = 0$; this is the no-penetration boundary condition. If we use this boundary condition, then the factor c_3 will be equal to zero and we get an expression of *v*. We will get the governing equation for pressure if we substitute the condition at $y = h$, $v = \frac{\partial h}{\partial t}$. This is a kinematic constraint that the rate of change of *h* with respect to time is the velocity at $y = h$ which has to same as the velocity of the plate at $y = h$. Otherwise there will be a lack of contact between the fluid and the plate; plate and fluid have to move together (which is pure kinematics). Substituting this boundary condition in equation (6) we get

$$-\frac{1}{2}\left[\frac{\partial^2 p}{\partial x^2}\left(-\frac{h^3}{6}\right) - \frac{\partial p}{\partial x}\frac{h^2}{2}\frac{\partial h}{\partial x}\right] + \frac{u_0}{2}\frac{\partial h}{\partial x} = \frac{\partial h}{\partial t} \tag{7}$$

We can take the first two terms together and rewrite the equation as

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12}\frac{\partial p}{\partial x}\right) + \frac{u_0}{2}\frac{\partial h}{\partial x} = \frac{\partial h}{\partial t} \tag{8}$$

Here the term $\frac{\partial h}{\partial t}$ is equal to v_0; so the velocity at *y* = *h* is equal to v_0. With the co-ordinate transformation, the *x*-velocity has changed but the *y*-velocity has not changed. So at *y* = *h*, $v_0 = \frac{\partial h}{\partial t}$. If we have two plates and there is a fixed height between them, then the rate of change of *h* with respect to time is located from the bottom. Here we do not have any finite u_0 because the velocity u_0 has become equal to zero because of co-

ordinate transformation. This is the whole objective of the co-ordinate transformation because of which the entire plate is only having a vertical height and no horizontal velocity. So, $\frac{\partial}{\partial x}\left(\frac{h^3}{12}\frac{\partial p}{\partial x}\right)+\frac{u_0}{2}\frac{\partial h}{\partial x}=\frac{\partial h}{\partial t}$ is the equation which is called as the famous Reynolds equation in lubrication theory (a very famous equation). We will now work out a problem where we will use the Reynolds equation to solve the force between the two plates. So let us work out a problem.

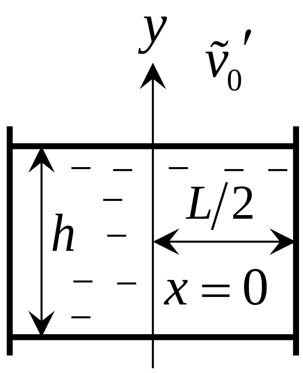

Figure 2. There are two plates where the top plate is moved towards the *y*-direction with a velocity $\tilde{v}_0{}'$.

Let us consider that there are two plates and there is a layer of viscous fluid in between the two plates. As shown in figure 2, the top plate is moved towards the *y*-direction with a dimensional velocity $\tilde{v}_0{}'$ so that the initial gap is increasing. The initial gap at time *t* = 0 is equal to h_0. Here *h* is not a function of *x*. Now the question arises about the force that is required to pull the plate. When we pull this up there will be a resistance in the fluid because of which we require a force to pull the plate otherwise it does not spontaneously happen. So this problem is a special case of the previous problem. First of all, here h is not a function of x, i.e. $\frac{\partial h}{\partial x}=0$. First we write the Reynolds equation in the dimensional form which is given by $\frac{\partial}{\partial x'}\left(\frac{h'^3}{12\mu}\frac{\partial p'}{\partial x'}\right)=\frac{\partial h'}{\partial t'}$ (so the only difference between the dimensional and the dimensionless form is the presence of the viscosity μ in the dimensional equation and nothing else will change). If we change the variables of the equation $\frac{\partial}{\partial x}\left(\frac{h^3}{12}\frac{\partial p}{\partial x}\right)=\frac{\partial h}{\partial t}$ from x to x', h to h', p to p' and t to t'; then only an additional μ term will appear and nothing else will change. So in the non-dimensional case, the μ term is absorbed in the velocity scale. So in the dimensional form we can write $\frac{\partial}{\partial x'}\left(\frac{h'^3}{12\mu}\frac{\partial p'}{\partial x'}\right)=\frac{\partial h'}{\partial t'}=\tilde{v}_0{}'$ where $'$ symbol corresponds to the dimensional variable everywhere and any variable without $'$ symbol means it is a dimensionless variable. Since in the present case h' is not a function of x', we can bring it out of the partial derivative and we get $\frac{h'^3}{12\mu}\frac{\partial^2 p'}{\partial x'^2}=\tilde{v}_0{}'$ where the velocity $\tilde{v}_0{}'$ is constant. It is given that the velocity $\tilde{v}_0{}'$ is constant because our objective is to pull the plate with this constant velocity $\tilde{v}_0{}'$. From the equation $\frac{h'^3}{12\mu}\frac{\partial^2 p'}{\partial x'^2}=\tilde{v}_0{}'$ we get $\frac{\partial^2 p'}{\partial x'^2}=\frac{12\mu\tilde{v}_0{}'}{h'^3}$ where we can substitute the term h' by $h_0{}'+\tilde{v}_0{}'t'$. So, h' is equal to the summation of the original height $h_0{}'$ and the change of the height with time because of the velocity $\tilde{v}_0{}'$. So this h' is an instantaneous height and $h_0{}'$ is the height at t = 0. Since all these terms are not a function of x, we can treat the right hand side of equation $\frac{\partial^2 p'}{\partial x'^2}=\frac{12\mu\tilde{v}_0{}'}{h'^3}$ as a constant. Let us assume this constant to be equal to k where k can be a function of time but not a function of x'; so, $\frac{\partial^2 p'}{\partial x'^2}=\frac{12\mu\tilde{v}_0{}'}{h'^3}=k(t)$. Integrating both sides of this equation with respect to x' we get $\frac{\partial p'}{\partial x'}=k\,x'+k_1$ and integrating again with respect to x results $p'=\frac{k\,x'^2}{2}+k_1\,x'+k_2$ where k_1 and k_2 are the integration constants. Before applying the boundary condition, we set up a central axis where the origin is located at the centre. The length L is divided into two parts with each being equal to $L/2$ (also shown in figure 2). The first boundary condition is at $x' = 0$, the pressure has to be symmetric with respect to the x' axis, so, $\frac{\partial p'}{\partial x'} = 0$ (at both sides there is atmosphere). Using this boundary condition we get $k_1 = 0$. Now at $x' = L/2$ or at $x' = -L/2$ (both are equivalent), $p' = 0$. We can set the pressure as the atmosphere pressure. If we set the pressure as the atmospheric pressure, then the remaining pressure will actually contribute to the force because the atmospheric pressure acts equally from all sides. So it does not contribute to the force. Using the condition at $x' = L/2$, $p' = 0$; we get $0 = \frac{k}{2}\left(\frac{L}{2}\right)^2 + k_2$ or, $k_2 = -\frac{k}{2}\left(\frac{L}{2}\right)^2$. Substituting the values of k_1 and k_2 we get the final form of the pressure distribution which is given by $p' = \frac{k}{2}\left[x'^2 - \left(\frac{L}{2}\right)^2\right]$. So we have only one step remaining, i.e. to calculate the force on the upper plate. The force on the upper plate is given by $F = \int_{-L/2}^{L/2} p'\,dx' = 2\int_0^{L/2} p'\,dx'$. Substituting the expression of pressure $p' = \frac{k}{2}\left[x'^2 - \left(\frac{L}{2}\right)^2\right]$, the integral becomes $F = 2 \cdot \frac{k}{2}\int_0^{L/2}\left[x'^2 - \left(\frac{L}{2}\right)^2\right]dx' = k\left[\frac{x'^3}{3} - \frac{L^2}{4}x'\right]_0^{L/2} = k\left[\frac{L^3}{24} - \frac{L^3}{8}\right] = -\frac{k\,L^3}{12}$. This force is calculated per unit width of the plate. So, one interesting thing to notice from this expression of the force is that the force comes out to be negative while all other parameters remain positive. It means that when the plate is moving up there is something which is actually dragging it down. This is the physical meaning of the negative force. Now question arises about whether we can calculate the force in this way unlimitedly or not. We cannot because if the plate goes up and up, there will be a situation when the gap height *h* will become comparable to the length *L* and in that case the lubrication theory will no longer remain valid. So for solving any problem, we have to keep in mind the assumptions behind the theory; these are the basic things which have to be kept in mind. Here the basic thing is that *h* is much less than *L*. So, when the plate is moving up and up, at some time *h* will become comparable with *L* and then lubrication theory fails and the prediction of the force does not work. On the other hand the other limit is a very interesting limit when the plate instead of moving up is coming down. So, when the plate is coming down, $\tilde{v}_0{}'$ is negative. If $\tilde{v}_0{}'$ is negative, the right hand side of the governing equation of the pressure distribution becomes negative which will result to a net positive force. It means that when the plate is coming down there is an upward force to resist that downward movement. In that case *h* will continuously decrease with time and the gap height h' will be represented by $h_0{}' - \tilde{v}_0{}' t'$ with $h_0{}'$ being the original height at $t = 0$. So there will be a time when the gap h' becomes very small such that the two plates almost touch each other. Since the term $\frac{\partial^2 p'}{\partial x'^2}$ varies with the reciprocal of the cube of that gap, i.e. $\frac{\partial^2 p'}{\partial x'^2} \propto \frac{1}{h'^3}$, the force will be tremendously high. So if we want to put an adhesive in between the two plates we will find a tremendously large resistance when we approach the plates (this is a very common day to day experience).

We take two glass plates and a thin film of water. If we try to bring one of the plates to the other we will face huge resistance when we come sufficiently close. Not only this, but on the top of that intermolecular forces will also start acting. Van der Waals force and other forces will come into play when the plates are sufficiently close. So this is actually a very interesting problem which is called as adhesive problem. We put an adhesive and try to bring the two surfaces very close to each other. The limiting condition will be the case when the plates almost touch each other. But we will observe that it is virtually impossible to make these two plates touch because of the dependence of the force with $\frac{1}{h'^3}$.

Overall, in the present chapter, we have discussed significantly about the lubrication theory. We have also understood about the pressure distribution in a confined system where the confinement height is a function of *x* and time. So far we have considered that any source of the acting force comes from either the pressure distribution or from the shear. But there are many interesting problems in micro-scale where the pressure distribution itself is generated because of the remarkable physics of surface tension. Form the next chapter, we will start understanding the role of surface tension in micro-scale flows.