Advanced Concepts In Fluid Mechanics
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Lecture - 38
Lubrication Theory (Contd.)

In the previous chapter we have discussed about the basic premises of the lubrication theory and we have looked into the x-momentum equation consistent with the schematic as shown in figure 1. We start with the simplified from of the x-momentum equation which was derived in the previous chapter. The x-momentum equation is given below

$$0=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2}+F_x\,F_{xc}\cdot\frac{h_0^2}{\mu u_c} \tag{1}$$

In equation (1), $\frac{\partial p}{\partial x}$ and $\frac{\partial^2 u}{\partial y^2}$ are definitely the leading order terms but we do not have any idea about the third term $F_x\,F_{xc}\cdot\frac{h_0^2}{\mu u_c}$. This term will remain here the strength of which depends on how strong the scale of the body force is. If there is no body force or the term $F_x\,F_{xc}\cdot\frac{h_0^2}{\mu u_c}$ becomes negligible then it will not be present in the simplified momentum equation.

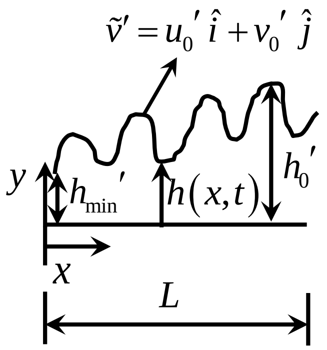

Figure 1. The confinement height h as a function of position and time $h(x,t)$.

Now we will consider the y-momentum equation which is given below

$$\rho\left[\frac{\partial v'}{\partial t'}+u'\frac{\partial v'}{\partial x'}+v'\frac{\partial v'}{\partial y'}\right]=-\frac{\partial p'}{\partial y'}+\mu\left(\frac{\partial^2 v'}{\partial x'^2}+\frac{\partial^2 v'}{\partial y'^2}\right)+F_y' \tag{2}$$

So with an intuition that the velocity v is one order of magnitude less than the velocity u, one would have had a tendency of neglecting the contribution of the *y*-momentum equation altogether. But the subtle thing is that we do not know anything about the body force term F_y' and this F_y' itself could actually dictate the physics of the problem. The term F_y' could dictate the order of magnitude of the velocity scale u_c which will be shown later.

Now we non-dimensionalize the *y*-momentum equation (2) with the relevant parameters according to the non-dimensionalization scheme discussed in the previous chapter. We are not going into the details of each steps but one can easily check this. The non-dimensional form of the *y*-momentum equation (2) is given by

$$\frac{\rho u_c v_c}{L}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{p_c}{h_0}\frac{\partial p}{\partial y}+\mu\frac{v_c}{L^2}\frac{\partial^2 v}{\partial x^2}+\mu\frac{v_c}{h_0^2}\frac{\partial^2 v}{\partial y^2}+F_y F_{yc} \qquad (3)$$

All three terms on the left hand side of equation (3) are of the same order with the coefficient $\frac{\rho u_c v_c}{L}$. The interesting thing about pressure is that pressure being a scalar its scale does not change. If we move from the *x*-momentum to the *y*-momentum, the scale of pressure remains the same whereas the velocity scales are different because of the directionalities. So the term $\frac{\partial p'}{\partial y'}$ is of the order of ~ $\frac{p_c}{h_0}$. Now substituting the scale of the velocity $v_c \sim u_c \frac{h_0}{L}$, the coefficient of the terms on the left hand side of equation (3) becomes $\frac{\rho u_c v_c}{L} \sim \frac{\rho u_c}{L}\cdot u_c\frac{h_0}{L} \sim \rho u_c{}^2\frac{h_0}{L^2}$. In the next step, we multiply both sides of equation (3) by the factor $\frac{h_0}{p_c}$ and equation (3) becomes

$$\rho u_c{}^2\frac{h_0}{L^2}\cdot\frac{h_0}{p_c}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right]=-\frac{\partial p}{\partial y}+\mu\frac{v_c}{L^2}\cdot\frac{h_0}{p_c}\frac{\partial^2 v}{\partial x^2}+\mu\frac{v_c}{h_0^2}\cdot\frac{h_0}{p_c}\frac{\partial^2 v}{\partial y^2}+F_y F_{yc}\cdot\frac{h_0}{p_c} \qquad (4)$$

Using the scale of the pressure $p_c \sim \dfrac{\mu u_c L}{h_0^2}$, the coefficient $\rho u_c{}^2 \dfrac{h_0}{L^2} \cdot \dfrac{h_0}{p_c}$ becomes $\sim \rho u_c{}^2 \dfrac{h_0}{L^2} \cdot \dfrac{h_0 h_0^2}{\mu u_c L} \sim \dfrac{\rho u_c L}{\mu} \cdot \left(\dfrac{h_0}{L}\right)^4 \sim \varepsilon^4 \,\mathrm{Re}_L$. The coefficient $\mu \dfrac{v_c}{L^2} \cdot \dfrac{h_0}{p_c}$ becomes $\sim \mu \dfrac{v_c}{L^2} \cdot \dfrac{h_0 h_0^2}{\mu u_c L} \sim \dfrac{v_c}{u_c} \cdot \left(\dfrac{h_0}{L}\right)^3 \sim \varepsilon \left(\dfrac{h_0}{L}\right)^3 \sim \varepsilon^4$ where we have used the scale $v_c \sim u_c \dfrac{h_0}{L} \sim u_c\, \varepsilon$.

Now the coefficient of the third term $\mu \dfrac{v_c}{h_0^2} \cdot \dfrac{h_0}{p_c}$ on the right hand side of equation (4) becomes $\sim \mu \dfrac{v_c}{h_0^2} \cdot \dfrac{h_0 h_0^2}{\mu u_c L} \sim \dfrac{v_c}{u_c} \cdot \dfrac{h_0}{L} \sim \varepsilon^2$ where we use $v_c \sim u_c\, \varepsilon$. Earlier we have written the *x*-momentum equation in terms of the leading order solution. Similarly, we will write the *y*-momentum equation in terms of the leading order solution. If the Reynolds number $\left(\mathrm{Re}_L\right)$ is small, then the coefficient $\dfrac{\rho u_c v_c}{L}$ and thus the terms in the left hand side of the *y*-momentum equation (3) will be negligible. The pressure gradient term $\dfrac{\partial p}{\partial y}$ will be of the order of 1 and will be present in the leading order equation. But there are other terms like $\mu \dfrac{v_c}{L^2} \dfrac{\partial^2 v}{\partial x^2}$ and $\mu \dfrac{v_c}{h_0^2} \dfrac{\partial^2 v}{\partial y^2}$, which are of the orders of $\sim \varepsilon^4$ and $\sim \varepsilon^2$ respectively. Both of these two terms will not be important if the parameter ε is small. Thus these two terms can be neglected in the leading order of the *y*-momentum equation. But we are uncertain about the body force term. So, the *y*-momentum equation in the leading order is given by

$$0 = -\frac{\partial p}{\partial y} + F_{yc}\, F_y \frac{h_0}{p_c} \tag{5}$$

By leading order we mean that if the variable p is expanded as $p = p_0 + \varepsilon\, p_1 + \varepsilon^2\, p_2 + \cdots\cdots$, u as $u = u_0 + \varepsilon\, u_1 + \varepsilon^2\, u_2 + \cdots\cdots\cdots$ and v as $v = v_0 + \varepsilon\, v_1 + \varepsilon^2\, v_2 + \cdots\cdots$, then equation only leading order terms result in the equation (5) (so equation (5) is a shorthand way of doing the same). So, we have got all of our equations. But it is not a practical thing to solve this problem for a general case because for every problem there is a specific physics and that will drive the solution of the problem. So up to this step we can give a generic formulation. This generic formulation is very important because in the class we may do one representative problem to highlight about how this lubrication theory is used. But later in research, we can encounter an entirely different problem altogether where we need to apply this lubrication theory. If we have our fundamentals correct, then we can arrive up to this step and then according to the specific problem we can guide our solution. Therefore these fundamentals are very important (this is like the grammar of the subject).

Now the next question arises about the pressure scale p_c which depends on the velocity scale u_c as $p_c \sim \frac{\mu u_c L}{h_0^2}$. So we have to find the velocity scale u_c. One of the most straightforward considerations is that the *x*-motion of the upper boundary is governing the physics. This is possibility number 1 but there could be many possibilities. We will write all the possibilities. The physics of the problem only can tell us that out of these possibilities which one is correct; mathematics cannot tell us anything. We have to make a judgment about what is actually the relevant driving mechanism. Accordingly the scale has to be decided because the entire solution will be incorrect if we cannot choose the scale correctly. If the *x*-momentum of the upper boundary governs the physics then we can clearly write $u_c = u_0'$. If the *y*-momentum of the upper boundary governs the physics then we can write $v_c = v_0'$ which also means $u_c = \frac{v_0'}{\varepsilon}$. The two velocity scales u_c and v_c are always linked by the ε ratio. No matter how complex the flow is, the continuity equation has to be satisfied (that is the holistic idea about the continuity equation). Now let us assume that the *x*-component of the body force governs the physics. In that case we have to relate the velocity scale u_c with the *x*-component of the body force. To do this we look into the *x*-momentum equation at the leading order which reads as $0 = -\frac{\partial p}{\partial x} + \frac{\partial^2 u}{\partial y^2} + F_x F_{xc} \cdot \frac{h_0^2}{\mu u_c}$. The first term on the right hand side of this equation $\frac{\partial p}{\partial x}$ is of the order of 1 because in this way we have defined the scale. For the same reason, the term $\frac{\partial^2 u}{\partial y^2}$ is also of the order of 1. Now to make the term $F_x F_{xc} \cdot \frac{h_0^2}{\mu u_c}$ important, it also has to be of the order of 1. If $F_x F_{xc} \cdot \frac{h_0^2}{\mu u_c}$ becomes of the order of 1, then we get

$F_x\, F_{xc} \cdot \dfrac{h_0^2}{\mu\, u_c} \sim 1$, or, $u_c \sim \dfrac{F_x\, F_{xc}\, h_0^2}{\mu}$. Now every problem can be twisted to an extent that it can make our life a misery. It becomes a critical situation when the top plate is given a motion and also there is a body force (let us consider electrical force as a body force). In that case we have to decide between these two factors (that which one has to be chosen) to determine the velocity scale u_c. Since the term F_x is already non-dimensional, it is of the order of 1 and therefore, $u_c \sim \dfrac{F_{xc}\, h_0^2}{\mu}$. Now we again think of the interesting situation that there is a body force and the upper boundary is given a motion and we have to decide out of these two which factor governs the physics. In that case also we have an answer. To do that, we have to put the numerical values of both these factors. Then out of these two factors, the one of higher order will govern the physics but we cannot discard these factors individually. The final situation comes when the *y*-component of the body force governs the physics of the problem. Then we have to look into the *y*-momentum equation in the leading order which reads as $0 = -\dfrac{\partial p}{\partial y} + F_{yc}\, F_y\, \dfrac{h_0}{p_c}$. From this equation we can tell that if the *y*-component of the body force governs the physics, then $F_{yc}\, \dfrac{h_0}{p_c} \sim 1$ or, $F_{yc}\, \dfrac{h_0\, h_0^2}{\mu\, u_c L} \sim 1$, or, $u_c \sim \dfrac{F_{yc}\, h_0^3}{\mu\, L}$. Our solution of the problem is based on the logical decision making. Out of these four possibilities, we have to choose one appropriate scale and then we can proceed with the solution of the problem. For further illustration, we will first consider an example and then we will consider the problem.

Let us assume that there is no body force, so, $F_{xc} = F_{yc} = 0$. If there is no body force still we can have a development of the pressure gradient because of the geometry of the flow passage (the geometry can be seen from figure 1). If there is no *y*-body force, then from the *y*-momentum equation in the leading order we can write $\dfrac{\partial p}{\partial y} = 0$. If there is no *x*-body force, then from the *x*-momentum equation in the leading order we can write $\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial p}{\partial x}$. Since there is no pressure gradient in the *y*-direction (i.e. $\dfrac{\partial p}{\partial y} = 0$), the general tendency is to replace the partial derivative $\frac{\partial p}{\partial x}$ by $\frac{dp}{dx}$. But we need to keep in mind that here time is a very important variable. So, although pressure does not vary with *y*, it can be a function of time. So we will keep the term $\frac{\partial p}{\partial x}$ as a partial derivative. Eventually our objective will be to solve the equation $\frac{\partial^2 u}{\partial y^2} = \frac{\partial p}{\partial x}$ to get the velocity profile. The complexity of the problem is the boundary condition. Here the boundary has both *x*-

motion as well as *y*-motion (as shown in figure 1 by the velocity $\tilde{v}' = u_0{}'\,\hat{i} + v_0{}'\,\hat{j}$). It is very difficult to accommodate these two together. To solve this issue, we will make a clever transformation of the co-ordinate system. We will consider a reference frame which is moving towards the right with a velocity $u_0{}'$ and with respect to this reference frame we will write the governing equation. Let us imagine a reference frame which is moving towards the right with a velocity $u_0{}'$. Then, with respect to that reference frame, one of the boundary conditions will be at y = 0, $u = -u_0{}'$ which means that sitting on the reference frame it will appear that it is moving towards the left with a velocity $u_0{}'$. The non-dimensional form of the boundary condition can be written as at y = 0, $u = -u_0$ where $u_0 = \frac{u_0{}'}{u_c}$. The second boundary condition will be at $y = h(x,t)$, $u = 0$ where *h* is still function of *x* and time; we have not relaxed that. Now it becomes convenient to tackle the upper boundary. Although the upper boundary still has the velocity $v_0{}'$, but it does not have the velocity $u_0{}'$. With these two boundary conditions we can solve the governing equation. We will take that up in the next lecture to see that how the present formulation can be used to calculate the pressure distribution in the confined passage.